A global optimal solution to the eco-driving problem

Citation for published version (APA):

Padilla Cazar, G. P., Weiland, S., & Donkers, M. C. F. (2018). A global optimal solution to the eco-driving problem. *IEEE Control Systems Letters*, 2(4), 599 - 604. <https://doi.org/10.1109/LCSYS.2018.2846182>

DOI:

[10.1109/LCSYS.2018.2846182](https://doi.org/10.1109/LCSYS.2018.2846182)

Document status and date:

Published: 01/10/2018

Document Version:

Accepted manuscript including changes made at the peer-review stage

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

A Global Optimal Solution to the Eco-Driving Problem

G.P. Padilla, S. Weiland, M.C.F. Donkers

Abstract—Eco-driving aims at minimizing the energy consumption of a vehicle by adjusting the vehicle's velocity. This can be formulated as an optimal control problem and this paper provides a detailed view on the global optimal solution to this problem. A method to reformulate and discretize the problem avoiding the introduction of additional nonconvex terms is presented. Furthermore, physically realistic conditions are given that guarantee the existence of the global optimal solution to the eco-driving problem. Subsequently, a sequential quadratic programming algorithm is provided that allows finding the global optimal solution. Finally, two numerical examples are used to illustrate how solutions of the eco-driving problem can be obtained.

Index Terms—Automotive control, Optimal control, Optimization.

I. INTRODUCTION

IMPROVING energy efficiency of vehicles is an important topic of research for the automotive industry. High energy efficiency is important for reducing fuel consumption and meeting emission legislations. Moreover, energy efficiency is also supported by the functional argument of mitigating range anxiety of electric vehicles, i.e., giving a sufficiently large driving range for an electrical vehicle. The problem of reducing the energy consumption of a vehicle over a certain drive cycle can be formulated as an optimal control problem and its solution is often referred to as an energy management strategy. Most of these energy management problems are focused on controlling the power split between the combustion engine and the electric machine of a hybrid electric vehicle [1], [2]. By storing regenerative braking energy and shifting the operating points of the combustion engine, a significant amount of fuel can be saved. A recent trend is to extend this energy management system to incorporate more subsystems of the vehicle [3] or to consider emission constraints in the optimal control problem [4].

In the aforementioned approaches, the vehicle velocity (and thereby the power needed to propel the vehicle) is assumed to be fixed. Nevertheless, the vehicle inertia, which is the largest energy buffer in the vehicle, can have a large impact in energy savings and consequently in the extension of the driving range. For instance, in [5] it has been reported that changes in driving behavior could improve the energetic performance of

the vehicle more than 30%. The promising improvements in energy efficiency have contributed to the emergence of the eco-driving concept, which aims to increase the energy efficiency of a vehicle by means of a convenient selection of driving strategies; i.e. legal regulations, technological implementations or simply changes in the driver behavior. Hence, it is clear that eco-driving is a broad concept where government, manufacturers and users participate [6].

To solve the eco-driving problem, standard techniques used in optimal control have been adopted, see [7] for a detailed overview of the recent literature. In [8], [9], dynamic programming (DP) has been used to find a global solution to this problem. Alternatively, Pontryagin's minimum principle (PMP) has been used in [10]–[13]. The main disadvantages PMP are that it only provides a necessary condition for optimality and that incorporating state constraints is not a simple task. Therefore, in [14]–[16] static nonlinear optimization techniques are used to solve the problem in the presence of state constraints. It remains unclear from the papers that use static optimization techniques or PMP whether the solutions are, in fact, globally optimal. Unfortunately, the literature related to this topic is scarce. The noticeable exception is [15], where the continuous-time optimal control problem is certified to be convex, which guarantees that the obtained solution is globally optimal. However, [15] does not discuss the possible loss of convexity due to the discretization process. This might occur, as it is demonstrated in this manuscript.

This paper aims to expose a detailed view of the global optimal solution to the eco-driving problem. The results of this paper can be used to certify optimality of the results presented in the existing literature and in future works. The main contributions presented in this paper are threefold: Firstly, a method to reformulate and discretize the problem is presented. This is initially done for a simplified case to illustrate the nonconvexity of the problem and subsequently extended to the complete eco-driving problem. Secondly, a detailed analysis of the uniqueness of the solution to the reformulated problem is used to obtain a set of mild conditions that guarantee the global optimality of the solution. Thirdly, a sequential quadratic programming (SQP) method is employed to efficiently solve the eco-driving problem.

II. CONTINUOUS-TIME PROBLEM FORMULATION

In this section, a continuous-time formulation of the eco-driving concept as an optimal control problem is provided. Eco-driving aims at obtaining an optimal control force $u(t)$ and velocity profile $v(t)$ that minimizes the integral of the power $P(v, u)$ consumed by a vehicle while traveling during a given time interval $[t_o, t_f]$ over a given trajectory $s(t) \in$

This work has received financial support from the Horizon 2020 programme of the European Union under the grant 'Electric Vehicle Enhanced Range, Lifetime And Safety Through INGenious battery management' (EVERLASTING-713771).

Paul Padilla, Tijs Donkers and Siep Weiland are with the Department of Electrical Engineering of Eindhoven University of Technology, Netherlands. E-mail: {g.p.padilla.cazar, m.c.f.donkers, s.weiland}@tue.nl

$[s_o, s_f]$ with known geographical characteristics, i.e., with a given road grade $\alpha : [s_o, s_f] \rightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$, where $\alpha(s)$ is the grade at position s ; while being subject to longitudinal vehicle dynamics, non-negative velocity bounds $v(t) \in [\underline{v}, \bar{v}]$, and boundary conditions on position and velocity. This can be stated in the form of the following optimal control problem:

$$\begin{aligned} \min_{s(t), v(t), u(t)} \quad & \int_{t_o}^{t_f} P(v(t), u(t)) dt \quad (1a) \\ \text{subject to} \quad & m \frac{dv}{dt} = u(t) - f(v(t), s(t)), \quad (1b) \\ & \frac{ds}{dt} = v(t), \quad (1c) \\ & s(t_o) = s_o, \quad s(t_f) = s_f \quad (1d) \\ & v(t_o) = v_o, \quad v(t_f) = v_f \quad (1e) \\ & \underline{v} \leq v(t) \leq \bar{v}, \quad (1f) \end{aligned}$$

where (1b) represents the longitudinal vehicle dynamics, in which $u(t)$ is the control force and $f(v, s)$ describes the aerodynamic drag, rolling resistance and gravity forces as

$$f(v, s) = \sigma_d v^2 + c_r mg \cos(\alpha(s)) + mg \sin(\alpha(s)), \quad (2)$$

In (1b) and (2), m represents the combined mass of the vehicle and the inertia of the driveline, $g \approx 9.81[m/s^2]$ is the gravitational constant, $c_r > 0$ describes the rolling force coefficient and $\sigma_d = \frac{1}{2} c_d \rho_a A_f$ with $c_d > 0$ is the drag coefficient, in which ρ_a denotes the air density and A_f is the frontal area of the vehicle.

The consumed power $P(v, u)$ can be obtained from different modeling approaches that capture the energy consumption in the powertrain. In this paper, it is assumed to be a quadratic function of the form

$$P(v, u) = \beta_0 v^2 + \beta_1 v u + \beta_2 u^2, \quad (3)$$

for some non-negative parameters β_0 , β_1 and β_2 . Equation (3) is a physically realistic approximation, e.g., for electric motors due to the fact that the friction and Ohmic losses are captured by the terms $\beta_0 v^2$ and $\beta_2 u^2$, respectively. The optimal control problem (1) does not consider constraints on the control force $u(t)$ nor constraints on propulsion power $P(v, u)$, as in (3). However, the results of this paper can be extended to include such constraints. Alternatively, constraints on propulsion power can be incorporated by connecting the eco-driving to vehicle energy management, as was done in [17], [15], where power constraints of powertrain components are present.

In general, (1) is a nonlinear optimal control problem that might have multiple local solutions due to specific features of the vehicle model and road profile, see (2). This implies that direct optimization methods or methods based on PMP only provide candidate minima, which might not correspond to the global solution to problem (1).

III. CONVEXITY IN RELATION TO DISCRETIZATION

To illustrate that the solution methods for the control problem (1) can introduce nonconvexity, which might complicate finding the global minimum, a simplified version of problem (1) is considered in this section. Using this simplified example, we will illustrate a possible reformulation of the problem as a convex optimal control problem, which implies the existence of a unique solution. Because of the presence

of state constraints in (1), we will focus on discrete-time approximations of the optimal control, which might by itself introduce nonconvexity, even for specific cases where the continuous-time problem (1) is convex.

For a simplified version of problem (1) used in this section, consider $\beta_0 = 0$, $\beta_1 = 1$ and $\beta_2 = 0$ in (3), which corresponds to an electric motor with perfect energy conversion and no friction and Ohmic losses. Constant velocity bounds are also considered. Moreover, assume a flat road, meaning that $\alpha(s) = 0$ so that the rolling friction is constant, i.e. $\cos(\alpha(s)) = 1$, and the gravitational force has no effect on the longitudinal vehicle dynamics as $\sin(\alpha(s)) = 0$. Under these assumptions, (1)-(3) reduces to

$$\min_{s(t), v(t), u(t)} \int_{t_o}^{t_f} u(t) v(t) dt. \quad (4a)$$

subject to (1c)-(1f) and

$$m \frac{dv}{dt} = u(t) - \sigma_d v(t)^2 - c_r mg. \quad (4b)$$

The above optimization problem is nonconvex due to (4b), meaning that application of PMP or finite dimensional optimization methods cannot guarantee a global optimal solution. We will show how this problem can be reformulated as a convex optimization problem, where we focus on discrete-time approximations so that static optimization methods can be applied.

A. Direct Discretization

In order to illustrate that caution should be taken when discretizing the optimal control problem (4) with (1c)-(1f), we show that direct discretization leads to a nonconvex optimization problem. In an attempt to assess convexity of (4) with (1c)-(1f), we eliminate $u(t)$ by substituting the equality constraints (4b) into the objective function (4a), thereby producing an equivalent optimization problem. This procedure is a useful tool to analyse the convexity of the problem in the feasible set, see, e.g., [18], and leads to

$$\min_{s(t), v(t)} \int_{t_o}^{t_f} m \frac{dv}{dt} v(t) + c_r mg v(t) + \sigma_d v(t)^3 dt, \quad (5)$$

subject to (1c)-(1f).

In order to solve (5) subject to (1c)-(1f), using direct optimization methods, we can approximate the integral in the objective function using a forward Euler discretization method at $v_k = v(t_k)$ and $s_k = s(t_k)$, i.e., at instances t_k , $k \in \{0, \dots, N\}$, for some $N \in \mathbb{N}$, where $t_{k+1} > t_k$, $t_0 = t_o$, $t_N = t_f$ and step size $\tau_k = t_{k+1} - t_k > 0$. This leads to a forward Euler discretization of the objective function in (5), given by

$$\sum_{k=0}^{N-1} \tau_k \left(m \frac{(v_{k+1} - v_k)}{\tau_k} v_k + \sigma_d v_k^3 + c_r mg v_k \right) \quad (6)$$

which can be rewritten as

$$\frac{1}{2} m (v_f^2 - v_o^2) + \sum_{k=0}^{N-1} \tau_k (\sigma_d v_k^3 + c_r mg v_k) - \frac{1}{2} m (v_{k+1} - v_k)^2, \quad (7)$$

using (1e). The last term in this expression is nonconvex, which is a direct consequence of the forward Euler discretization. It should be noted that this result is independent of the step size τ_k and a similar conclusion can be drawn from the application of a backward Euler method. This shows the loss of

convexity when applying discrete-time approximations of the eco-driving problem (1). Higher-order discrete approximations of (5) possibly lead to a convex optimization problem, albeit at the cost of increased complexity. Instead, we will here reformulate the optimal control problem (5) subject to (1c)-(1f) in a way that introduction of nonconvex terms is avoided, as will be shown below.

B. Continuous-Time Reformulation

The reformulation of the simplified eco-driving problem is based on the observation that, instead of discretizing the objective function directly, the first two terms in (5) can be integrated over the boundary conditions (1d) and (1e) to obtain

$$\min_{s(t), v(t)} \frac{1}{2} m(v_f^2 - v_o^2) + c_r m g(s_f - s_o) + \int_{t_o}^{t_f} \sigma_d v(t)^3 dt, \quad (8)$$

subject to (1c)-(1f). The two terms in this objective function describe the change in kinetic energy of the vehicle over the complete trajectory, and the loss due to rolling resistance, respectively. Since v_o , s_o , v_f and s_f are known, these terms do not influence the optimal solution. Interestingly, the remaining term describes the energy losses due to aerodynamic drag. Thus, the reduced expression is discretized using a forward Euler approach, leading to

$$\min_{s_k, v_k} \sum_{k=0}^{N-1} \tau_k \sigma_d v_k^3, \quad (9)$$

subject to $s_{k+1} = s_k + \tau_k v_k$, (1d)-(1f). The resulting discrete objective function (9) is convex for $v_k > 0$, which corresponds to the vehicle driving in forward direction.

Hence, this simplified case shows that a direct discretization of the eco-driving can lead to a nonconvex optimization problem, while a reformulated problem yields a convex optimization problem after discretization. The ideas in this section will be extended in the next section towards the complete eco-driving problem (1) to prove the existence of a unique solution of the optimal control problem.

IV. A GLOBAL SOLUTION TO THE ECO-DRIVING PROBLEM

In this section, the ideas presented in Section III will be applied to problem (1), without making the simplifications considered in Section III. This allows an equivalent optimization problem of reduced complexity to be formulated, which can be discretized using a forward Euler method without introducing additional nonconvex terms. The structure of the reduced discrete-time optimal control problem will be exploited to prove that a unique global optimal solution exists under mild and realistic conditions.

A. Reduction of the Continuous-Time Problem

Nonconvex optimization problems can show multiple local minima, therefore finding a global solution of the problem could be a cumbersome task. Nonconvexity is not only a consequence of a discretization action, it can also be related to other parameters of the nonlinear optimal control problem (1). For instance, (1a) could be a nonconvex objective function, which occurs for particular values of β_0 , β_1 and β_2 that make (3) a nonconvex function. Moreover, a realistic road

Fig. 1: Geometry of the road profile.

grade $\alpha(s)$ might also introduce nonconvexity in the equality constraint (1b). As done in the previous section, an equivalent optimal control problem will be obtained by substituting the equality constraint (1b) into the objective function.

Adopting the basic ideas presented in Section III, the continuous-time optimal control problem (1) is reformulated into a convenient form given by

$$\min_{a(t), s(t), v(t)} \int_{t_o}^{t_f} P(v(t), ma(t) + f(v(t), s(t))) dt \quad (10a)$$

subject to (1c)-(1f) and

$$\frac{dv}{dt} = a(t), \quad (10b)$$

where $a(t)$ is a new decision variable, which represents the vehicle acceleration, and $f(v, s)$ is given by (2). It is important to remark that (10a) is obtained from the substitution of (1b) into (1a), and its integrand is given by

$$P(v, ma + f(v, s)) = \beta_2(mg\gamma_g(s) + c_d v^2 + c_r mg\gamma_r(s) + ma)^2 + \beta_1 v(ma + c_d v^2 + mg\gamma_g(s) + c_r mg\gamma_r(s)) + \beta_0 v^2 \quad (11)$$

in which $\gamma_g(s) = \sin(\alpha(s))$ and $\gamma_r(s) = \cos(\alpha(s))$. The relevance of (11) is that it contains information of the longitudinal vehicle dynamics, where the majority of the nonlinearities of the problem are embedded. This information and the structure of the problem (10) with (1c)-(1f) can be exploited to obtain a reduced, yet equivalent, optimal control problem. The first step to achieve this goal is to observe that

$$\gamma_g(s) = \frac{dh}{ds} \quad \text{and} \quad \gamma_r(s) = \frac{ds'}{ds} \quad (12)$$

where $h(s)$ is the given elevation profile and $s'(t)$ is the horizontal projection of $s(t)$. The validity of these expressions can be explained using Fig. 1, where the geometry in this physical configuration shows that for any specific point in the road, the change in elevation and horizontal projection with respect to the displacement are given by $\frac{dh}{ds} = \sin(\alpha(s))$, and $\frac{ds'}{ds} = \cos(\alpha(s))$, respectively.

The next step is to remove the terms in the objective function (10a) that can be solved in advance and have no contribution to the optimal control problem. In particular, consider the following terms from (10a):

$$\begin{aligned} E_G &= \int_{t_o}^{t_f} \beta_1 m(a + g\gamma_g(s) + c_r g\gamma_r(s))v + 2\beta_2 m\sigma_d v^2 a dt \\ &= \int_{t_o}^{t_f} \beta_1 m \left(v \frac{dv}{dt} + g \frac{dh}{ds} \frac{ds}{dt} + g c_r \frac{ds'}{ds} \frac{ds}{dt} \right) + 2\beta_2 m\sigma_d v^2 \frac{dv}{dt} dt \\ &= \frac{1}{2} \beta_1 m(v_f^2 - v_o^2) + \beta_1 mg(h_f - h_o) + \beta_1 mgc_r(s'_f - s'_o) \\ &\quad + \frac{2}{3} \beta_2 m\sigma_d(v_f^3 - v_o^3). \end{aligned} \quad (13)$$

The first equality in (13) is obtained by substitution of (1c), (10b) and (12), and the second equality by solving the integral over the boundary conditions of the problem. The first two terms shows that the total kinetic and potential energy of the vehicle depend only on the velocities and elevations at the boundaries. In a similar way, the last two terms demonstrate that part of the energy consumed by the drag and rolling forces, respectively, are defined by the velocity and horizontal displacement at the boundaries.

Since the value of (13) is given by the boundary conditions of the optimization problem (10), it is possible to rewrite (10a) as

$$\int_{t_o}^{t_f} P(v, ma + f(v, s)) dt = \int_{t_o}^{t_f} P_R(a, v, s) dt + E_G, \quad (14)$$

where

$$P_R(a, s, v) = \beta_0 v^2 + \beta_1 \sigma_d v^3 + 2\beta_2 m^2 g a (\gamma_g(s) + c_r \gamma_r(s)) + \beta_2 (ma)^2 + \beta_2 (mg \gamma_g(s) + \sigma_d v^2 + c_r mg \gamma_r(s))^2. \quad (15)$$

Removing the constant term E_G from the objective function, thereby changing (11) to (15) in the optimal control problem, does not change its optimal solution.

B. Unique Solution to the Discrete-Time Optimal Control Problem

The reduced continuous-time optimal control problem obtained in the previous section can be discretized in order to make it solvable using static optimization methods. In this section, it will be shown that the discrete-time problem has a unique solution under realistic conditions and an efficient method to obtain this global minimum will be presented in Section V.

In order to discretize the problem, we consider again a forward Euler discretization method and define $a_k = a(t_k)$, $v_k = v(t_k)$ and $s_k = s(t_k)$ at instances $t_k = k\tau + t_o$, $k \in \mathcal{K} = \{0, \dots, N\}$, with fixed step size $\tau = \frac{t_f - t_o}{N}$, for some $N \in \mathbb{N}$. This leads to

$$\min_{a_k, s_k, v_k} \sum_{k=0}^{N-1} \tau P_R(a_k, s_k, v_k) \quad (16a)$$

$$\text{subject to} \quad s_{k+1} = s_k + \tau v_k, \quad (16b)$$

$$v_{k+1} = v_k + \tau a_k, \quad (16c)$$

$$s_0 = s_o, \quad s_N = s_f, \quad (16d)$$

$$v_0 = v_o, \quad v_N = v_f, \quad (16e)$$

$$\underline{v} \leq v_k \leq \bar{v} \quad (16f)$$

The theorem below is the main results of this section and provides conditions under which the optimal control problem (16) has a unique global minimum.

Theorem 1. Suppose optimization problem (16) is feasible. If $\beta_2 > 0$ and if $g\tau^2 \left(\frac{d\gamma_g(s_k)}{ds} + c_r \frac{d\gamma_r(s_k)}{ds} \right) \neq -1$ for all $k \in \{0, \dots, N-1\}$, optimization problem (16) has a unique global minimum.

Proof. The first-order necessary conditions for optimality of (16) are given by the so-called Karuhn-Kuhn-Tucker (KKT) conditions, see, e.g., [18]. Instrumental in these KKT condi-

tions is the Lagrangian

$$\begin{aligned} L(a_k, s_k, v_k, \kappa_k, \lambda_k, \mu_k, \nu_k) = & \sum_{k=0}^{N-1} \beta_0 \tau v_k^2 + \beta_1 \tau \sigma_d v_k^3 + \beta_2 \tau (ma_k)^2 \\ & + \beta_2 \tau (mg \gamma_g(s_k) + \sigma_d v_k^2 + c_r mg \gamma_r(s_k))^2 \\ & + 2\beta_2 \tau m^2 g a (\gamma_g(s_k) + c_r \gamma_r(s_k)) + \kappa_{k+1} (s_{k+1} - s_k - \tau v_k) \\ & + \lambda_{k+1} (v_{k+1} - v_k - \tau a_k) + \mu_k (v_k - \bar{v}) + \nu_k (\underline{v} - v_k). \end{aligned} \quad (17)$$

Since all constraints are linear, a critical point, or stationary point, of (16) is characterized by

$$\frac{\partial L}{\partial a_k} = 2\beta_2 m^2 \tau (a_k + g \gamma_g(s_k) + c_r g \gamma_r(s_k)) - \lambda_{k+1} \tau = 0 \quad (18a)$$

$$\begin{aligned} \frac{\partial L}{\partial s_k} = & 2\beta_2 \tau mg (ma_k + mg \gamma_g(s_k) + \sigma_d v_k^2 + c_r mg \gamma_r(s_k)) \\ & \times \left(\frac{d\gamma_g(s_k)}{ds} + c_r \frac{d\gamma_r(s_k)}{ds} \right) + \kappa_k - \kappa_{k+1} = 0 \end{aligned} \quad (18b)$$

$$\begin{aligned} \frac{\partial L}{\partial v_k} = & 2\beta_0 \tau v_k + 3\beta_1 \tau \sigma_d v_k^2 + 4\beta_2 \tau \sigma_d v_k (mg \gamma_g(s_k) + \sigma_d v_k^2 \\ & + c_r mg \gamma_r(s_k)) - \tau \kappa_{k+1} + \lambda_k - \lambda_{k+1} + (\mu_k - \nu_k) = 0 \end{aligned} \quad (18c)$$

and $\frac{\partial L}{\partial \kappa_k}$, $\frac{\partial L}{\partial \lambda_k}$ described by (16b) and (16c) respectively for some Lagrange multipliers κ_k , λ_k and $\mu_k \geq 0$, $\nu_k \geq 0$, $k \in \{0, \dots, N\}$, such that

$$\mu_k (v_k - \bar{v}) = 0 \quad \text{and} \quad \nu_k (\underline{v} - v_k) = 0, \quad (18d)$$

and κ_0 , λ_0 , κ_N , and λ_N chosen such a way that (16d) and (16e) are satisfied. Now rewriting (18a) as

$$a_k = \frac{1}{2\beta_2 m^2} \lambda_{k+1} - g \gamma_g(s_k) - c_r g \gamma_r(s_k), \quad (19)$$

and substituting this into (18b) yields

$$\kappa_k = \kappa_{k+1} - 2\beta_2 \tau mg \left(\frac{1}{2m\beta_2} \lambda_{k+1} + \sigma_d v_k^2 \right) \left(\frac{d\gamma_g(s_k)}{ds} + c_r \frac{d\gamma_r(s_k)}{ds} \right). \quad (20)$$

Substituting (19) into (16c) leads to a characterization of a critical point of (16), given by

$$\begin{bmatrix} s_{k+1} \\ v_{k+1} \end{bmatrix} = \begin{bmatrix} s_k + \tau v_k \\ v_k - \tau g \phi(s_k) + \frac{\tau}{2\beta_2 m^2} \lambda_{k+1} \end{bmatrix} \quad (21a)$$

and

$$\begin{bmatrix} \kappa_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} 1 & -\tau g \frac{d\phi}{ds_k} \\ \tau & 1 \end{bmatrix} \begin{bmatrix} \kappa_{k+1} \\ \lambda_{k+1} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (\nu_k - \mu_k) - \tau \Phi(s_k, v_k) \quad (21b)$$

with

$$\Phi(s, v) = \begin{bmatrix} 2\beta_2 m g^2 \sigma_d v^2 \frac{d\phi}{ds_k} \\ 2\beta_0 v + 3\beta_1 \sigma_d v^2 + 4\beta_2 \sigma_d v (\sigma_d v_k^2 + mg \phi(s_k)) \end{bmatrix} \quad (21c)$$

and $\phi(s) = \gamma_g(s) + c_r \gamma_r(s)$. For this characterization, $\mu_k \geq 0$ and $\nu_k \geq 0$ can always be uniquely chosen so that λ_k has the value needed to ensure $\underline{v} \leq v_k \leq \bar{v}$, while satisfying (18d).

Furthermore, since the matrix $\begin{bmatrix} 1 & -\tau g \frac{d\phi}{ds_k} \\ \tau & 1 \end{bmatrix}$ is full rank for every $k \in \{0, \dots, N-1\}$ by the hypothesis of the theorem, every pair (κ_N, λ_N) leads to a unique trajectory of κ_k and λ_k , meaning that they can be uniquely chosen such that (16d) and (16e) are satisfied if (16) is feasible.

The KKT conditions provide necessary conditions for optimality in the sense that they characterize points that satisfy

$$\nabla F(x) d \geq 0 \quad (22)$$

for all feasible directions d at x . In the above equation $\nabla F(x)$ is the gradient of the objective function (16a), where $x = [a_0, \dots, a_{N-1}, s_0, \dots, s_{N-1}, v_0, \dots, v_{N-1}]^T$, and the feasible directions d at $x \in \mathcal{F}$ are all vectors that satisfy $x + \alpha d \in \mathcal{F}$ for some $\alpha > 0$ and where $\mathcal{F} = \{x | (16b) - (16f)\}$, which is a compact set. Points satisfying (22) can in principle be minima, maxima or saddle points. Because of uniqueness

of the solutions to (21), the obtained critical point has to be a minimum. Indeed, suppose the critical point would be a maximum or a saddle point, there would exist at least one other x that would satisfy the necessary conditions. In other words, if x is a saddle point or a maximum, it is possible to move away from this x and lower the value of the objective function until the boundary of the feasible set $\mathcal{F}$ would be reached. At this point, we would find another point x that satisfies (22). This contradicts the fact that there is only one critical point. Therefore, this unique critical point has to be the global minimum, which completes the proof. $\square$

Remark 1. The conditions presented in Theorem 1 are satisfied for many realistic cases. For instance, $\beta_2 > 0$ is always satisfied if Ohmic losses of the electric motor are considered in the objective function. Moreover, standard road design guidelines, e.g., [19], suggest curvatures that yield $\left| \frac{d\gamma_g(s_k)}{ds} \right| \ll 1$ and $\left| \frac{d\gamma_r(s_k)}{ds} \right| \ll 1$.

V. SOLUTION TO THE ECO-DRIVING PROBLEM

In this section, we will propose an efficient numerical method for solving the optimal control problem (16). The solution method we propose uses Sequential Quadratic Programming (SQP). The effectiveness of SQP to solve constrained nonlinear optimization problems is the main reason for the large acceptance of this method. The scent of this approach is to iteratively solve linearly constrained quadratic (QP) subproblems, that are an approximation of the original problem evaluated in a previous solution, until some convergence criterion is achieved [20]. The uniqueness of the solution guarantees that the SQP algorithm finds the global minimum, provided that the algorithm converges.

Since the objective function in (16) is nonconvex even after the reformulation and reduction presneted in Section IV, we employ the results of [21] to formulate a convex QP subproblem by making an approximation of the Hessian matrix of (15), which is given by

$$H = \begin{bmatrix} \frac{\partial^2 P_R}{\partial a^2} & \frac{\partial^2 P_R}{\partial a \partial s} & \frac{\partial^2 P_R}{\partial a \partial v} \\ \frac{\partial^2 P_R}{\partial a \partial s} & \frac{\partial^2 P_R}{\partial s^2} & \frac{\partial^2 P_R}{\partial s \partial v} \\ \frac{\partial^2 P_R}{\partial a \partial v} & \frac{\partial^2 P_R}{\partial s \partial v} & \frac{\partial^2 P_R}{\partial v^2} \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12}(s) & 0 \\ h_{12}(s) & h_{22}(a, s, v) & h_{23}(s, v) \\ 0 & h_{23}(s, v) & h_{33}(s, v) \end{bmatrix}, \quad (23)$$

in which we have omitted the arguments of the function $P_R(a, s, v)$ for compactness of notation, and where

$$h_{11} = 2\beta_2 m^2, \quad (24a)$$

$$h_{12}(s) = 2\beta_2 m^2 g \left(\frac{d\gamma_g(s)}{ds} + c_r \frac{d\gamma_r(s)}{ds} \right), \quad (24b)$$

$$h_{22}(a, s, v) = 2\beta_2 m g \left(m g \left(\frac{d\gamma_g(s)}{ds} + c_r \frac{d\gamma_r(s)}{ds} \right)^2 + (ma + mg\gamma_g(s) + \sigma_d v^2 + mgc_r \gamma_r(s)) \times \left(\frac{d^2 \gamma_g(s)}{ds^2} + c_r \frac{d^2 \gamma_r(s)}{ds^2} \right) \right), \quad (24c)$$

$$h_{23}(s, v) = 4mg\beta_2 \sigma_d v \left(\frac{d\gamma_g(s)}{ds} + c_r \frac{d\gamma_r(s)}{ds} \right), \quad (24d)$$

$$h_{33}(s, v) = 4\beta_2 \sigma_d m g (\gamma_g(s) + c_r \gamma_r(s)) + 2\beta_0 + 6\beta_1 \sigma_d v + 12\beta_2 \sigma_d^2 v^2. \quad (24e)$$

Since $h_{22}(a, s, v)$ and $h_{33}(s, v)$ can become negative, we propose in this paper to use

$$\hat{H}(s, v) = \text{diag} (2\beta_2 m^2, \epsilon_{22}, \max(h_{33}(s, v), \epsilon_{33})), \quad (25)$$

as a positive definite approximate Hessian in the SQP algorithm presented below, in which ϵ_{22} , ϵ_{33} are small positive numbers and $h_{33}(s, v)$ is given by (24e).

The approximated Hessian matrix allow us to solve the eco-driving problem by sequentially solving the following convex second-order approximation of (15) as:

$$\{a_k^{i+1}, s_k^{i+1}, v_k^{i+1}\}_{k \in \mathcal{K}} = \underset{a_k, s_k, v_k}{\text{argmin}} \sum_{k=0}^{N-1} \tau P_{QP}(a_k, s_k, v_k, a_k^i, s_k^i, v_k^i), \quad (26)$$

subject to (16b)-(16f), where

$$P_{QP}(a_k, s_k, v_k, a_k^i, s_k^i, v_k^i) = \frac{1}{2} \begin{bmatrix} a_k \\ s_k \\ v_k \end{bmatrix}^T \hat{H}(s_k^i, v_k^i) \begin{bmatrix} a_k \\ s_k \\ v_k \end{bmatrix} + \left(\nabla P_R(a_k^i, s_k^i, v_k^i) - \begin{bmatrix} a_k^i \\ s_k^i \\ v_k^i \end{bmatrix}^T \hat{H}(s_k^i, v_k^i) \begin{bmatrix} a_k^i \\ s_k^i \\ v_k^i \end{bmatrix} \right) \begin{bmatrix} a_k \\ s_k \\ v_k \end{bmatrix}. \quad (27)$$

In (27), $\nabla P_R(a_k^i, s_k^i, v_k^i)$ is the gradient of (15), and $\hat{H}(s_k^i, v_k^i)$ the positive definite approximated Hessian matrix given by (24), both evaluated at (a_k^i, s_k^i, v_k^i) , where i indicates the iteration of the SQP algorithm. Convergence of the proposed SQP approach is guaranteed due to the fact that the Hessian matrix (25) satisfies the conditions for convergence presented in [21, Section 3.2].

VI. NUMERICAL EXAMPLES

In this section, the SQP algorithm proposed in Section V is used to find the optimal solution of two numerical examples of the eco-driving problem.

A. Benchmark Problem for Electric Vehicles

In this example, we revisit the numerical benchmark problem for eco-driving that have been introduced in [10] and uses PMP to solve it. The energy consumption in this model considers a frictionless electric motor, i.e., $\beta_0 = 0$. The effects of the rolling force in the vehicle are assumed to be constant, i.e., $\gamma_r = 1$, while the effects of the gravitational force are described by

$$\gamma(s) = g\gamma_g(s) = p_0 + p_1 s + p_2 s^2 + p_3 s^3. \quad (28)$$

It is important to note that the polynomial function $\gamma(s)$ also embeds information related to the road profile used in this example. In Table I, the parameters presented in [10] are translated to the eco-driving formulation proposed in this paper. Since the example of [10] is a numerical example, the parameters in Table I and the optimal solution do not have a physical interpretation. The method proposed in Section V returns an optimal solution that is depicted in Fig. 2a. This solution shows the same features reported in [10]. In fact, the final cost obtained using SQP is 1.2295×10^6 , which differs only 0.0767% from the results reported in [10], while having a similar computation time. Considering that in this case $\beta_2 > 0$, from Theorem 1, it is possible to conclude that the

TABLE I: EV parameters in [10].

p_0 :	3	$v(t_o)$:	0	m :	1	t_o :	0
p_1 :	0.4	$v(t_f)$:	0	g :	1	t_f :	1
p_2 :	-1	$s(t_o)$:	0	c_r :	0.1	τ :	0.001
p_3 :	0.1	$s(t_f)$:	10	σ_d :	10^{-3}	$v_0(t)$:	0

Fig. 2: Global solution to the eco-driving problem [10].

Fig. 3: Global solution to the eco-driving problem.

solution is a global minimum of the problem. Using the SQP approach presented in this paper, velocity constraints can be easily added, as shown in Fig. 2b, which is not straightforward using the PMP-based approach of [10].

B. Hybrid Electric Heavy-Duty Vehicle

In this example, a hybrid electric heavy-duty vehicle driving at a constant speed is compared with an eco-driving strategy, where the velocity is allowed to change between given bounds. The settings of the eco-driving problem used in this example are summarized in Table II and the elevation profile is defined by

$$h(s) = 225 \cos\left(\frac{3\pi}{21000}s + \frac{\pi}{4}\right) + 225 \text{ [m]}, \quad (29)$$

which is depicted in Fig. 3 as a green surface.

The constant-speed driving profile is described by a solid line in Fig. 3. In this case, the total energy consumed by the vehicle is $3.0719 \times 10^4 \text{ [kJ]}$. On the other hand, the dashed line presented in Fig. 3 describes the optimal control force u and velocity profile v obtained as the global solution to the eco-driving optimal control problem studied in this case. The total energy consumed by the vehicle under this strategy is $2.843 \times 10^4 \text{ [kJ]}$, which is approximately 7.44% lower than the energy consumed by the vehicle driving at constant velocity.

TABLE II: Parameters for the heavy-duty vehicle example.

$\beta_0: 0.292$	$v(t_o): 70 \text{ [km/h]}$	$m: 15950 \text{ [kg]}$	$t_o: 0 \text{ [s]}$
$\beta_1: 1.005$	$v(t_f): 70 \text{ [km/h]}$	$g: 9.81 \text{ [m/s}^2\text{]}$	$t_f: 1080 \text{ [s]}$
$\beta_2: 2.652 \times 10^{-4}$	$s(t_o): 0 \text{ [km]}$	$c_r: 0.1$	$\bar{v}: 80 \text{ [km/h]}$
$\tau: 5 \text{ [s]}$	$s(t_f): 21 \text{ [km]}$	$\sigma_d: 3.1246$	$\bar{v}: 60 \text{ [km/h]}$

VII. CONCLUSIONS

In this paper, a detailed study has been conducted on the global optimality of the eco-driving optimal control problem. We have proposed to reformulate and discretize the problem and have subsequently derived conditions that guarantee the existence of the global solution to the eco-driving problem. Taking advantage of these results, a SQP algorithm that efficiently solves the eco-driving problem has been proposed. The methodologies and results were illustrated in two numerical examples.

REFERENCES

- [1] B. Egardt, N. Murgovski, M. Pourabdollah, and L. J. Mardh, "Electromobility Studies Based on Convex Optimization: Design and Control Issues Regarding Vehicle Electrification," *IEEE Control Sys Mag*, 2014.
- [2] J. Kessels, J. H. M. Martens, P. P. J. van den Bosch, and W. H. A. Hendrix, "Smart vehicle powernet enabling complete vehicle energy management," in *Proc. Vehicle Propulsion & Power Control*, 2012.
- [3] T. C. J. Romijn, M. C. F. Donkers, J. T. Kessels, and S. Weiland, "A dual decomposition approach to complete energy management for a heavy-duty vehicle," *IEEE Trans on Control Syst Technol*, 2018.
- [4] M. Donkers, J. van Schijndel, W. Heemels, and F. Willems, "Optimal control for integrated emission management in diesel engines," *Control Engineering Pract.*, 2017.
- [5] C. Bingham, C. Walsh, and S. Carroll, "Impact of driving characteristics on electric vehicle energy consumption and range," *IET Intell. Transport Syst.*, 2012.
- [6] B. Seraens, "Optimal Control Based Eco-Driving: Theoretical Approach and Practical Applications," Ph.D. dissertation, Katholieke Universiteit Leuven, 2012.
- [7] A. Sciarretta, G. D. Nunzio, and L. L. Ojeda, "Optimal Ecodriving Control: Energy-Efficient Driving of Road Vehicles as an Optimal Control Problem," *IEEE Control Systems*, 2015.
- [8] E. Hellström, J. Åslund, and L. Nielsen, "Management of kinetic and electric energy in heavy trucks," *SAE Journal of Engines*, 2010.
- [9] E. Ozatay, S. Onori, J. Wollaeger, U. Ozguner, G. Rizzoni, D. Filev, J. Michelini, and S. D. Cairano, "Cloud-Based Velocity Profile Optimization for Everyday Driving: A Dynamic-Programming-Based Solution," *IEEE Trans on Intell Transport Syst*, 2014.
- [10] N. Petit and A. Sciarretta, "Optimal drive of electric vehicles using an inversion-based trajectory generation approach," *Proc. IFAC World Congress*, 2011.
- [11] W. Dib, A. Chasse, P. Moulin, A. Sciarretta, and G. Corde, "Optimal energy management for an electric vehicle in eco-driving applications," *Control Eng Pract*, 2014.
- [12] E. Ozatay, U. Ozguner, and D. Filev, "Velocity profile optimization of on road vehicles: Pontryagin's Maximum Principle based approach," *Control Eng Pract*, 2017.
- [13] M. A. S. Kamal, M. Mukai, J. Murata, and T. Kawabe, "Ecological Vehicle Control on Roads With Up-Down Slopes," *IEEE Transactions on Intelligent Transportation Systems*, vol. 12, 2011.
- [14] N. J. Kohut, J. K. Hedrick, and F. Borrelli, "Integrating traffic data and model predictive control to improve fuel economy," *Proc. IFAC World Congress*, 2009.
- [15] N. Murgovski, L. Johannesson, X. Hu, B. Egardt, and J. Sjöberg, "Convex relaxations in the optimal control of electrified vehicles," in *Proc. American Control Conf*, 2015.
- [16] M. Vajedi and N. L. Azad, "Ecological Adaptive Cruise Controller for Plug-In Hybrid Electric Vehicles Using Nonlinear Model Predictive Control," *IEEE Trans on Intell Transport Syst*, 2016.
- [17] Z. Khalik, G. P. Padilla, T. C. J. Romijn, and M. C. F. Donkers, "Vehicle Energy Management with Ecodriving: A Sequential Quadratic Programming Approach with Dual Decomposition," in *Proc American Control Conference*, 2018.
- [18] S. P. Boyd and L. Vandenberghe, *Convex optimization*, Cambridge University Press New York, 2004.
- [19] A. A. of State Highway and T. O. (AASHTO), "A policy on geometric design of highways and streets," Tech. Rep.
- [20] J. Nocedal and S. J. Wright, *Numerical Optimization*. Springer, 2006.
- [21] P. T. Boggs and J. W. Tolle, "Sequential quadratic programming," *Acta numerica*, 1995.