First-order Closure Approximations for Middle-Sized Systems with Non-linear Rates

Mahmoud Talebi
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Abstract

In this paper we consider the problem of approximating the behaviour of middle-sized population models involving non-linear rates. By using a number of case studies, each with a very different non-linear behaviour, we show that binomial and Poisson moment closure approximations have the potential to accurately represent the expected behaviour of these models. We then compare the two approximation methods to the mean field and moment closure approximations in terms of applicability and accuracy, in addition to investigating the empirical distributions, to derive conclusions regarding their strengths and drawbacks.

1 Introduction

Fluid flow analysis is a promising method for the performance evaluation of population processes with a large number of individuals. Also known as the mean field approximation, the method provides asymptotically exact descriptions of the expected behaviour of the population process in the form of a simple deterministic process, which is the solution to a system of ordinary differential equations (ODEs). However, for the approximation theorems to apply, a few conditions should be satisfied.

In particular, if the population of a species is not large enough and the behaviour of the individuals is expressed as a non-linear function of the state of the population, the fluid analysis is not guaranteed to give accurate results. The presence of both of these properties in a model can lead to very inaccurate approximations. This has been shown, e.g., in [1] where the authors conclude that for a client-server model the fluid flow analysis is insufficient, and that identifying the class of systems with this problem is generally difficult.

To solve this problem, there have been attempts to improve the accuracy by introducing more robustness to the approximation. For example, solutions based on Kurtz’ diffusion approximations have been shown to have better convergence. However, they suffer from some technical difficulties when the model gets close to its boundaries, and although there have been attempts to overcome these problems, e.g., by jump-diffusion approximations [2], these methods do not seem to have been adopted by the performance modelling community, probably due to their general complexity. More recently, in [3], a method more suitable for automation was proposed which can be potentially used in analysis tools, although it targets the particular problem of finding stability points using the mean field method.

The more popular approach is to derive the exact system of ODEs from the Kolmogorov forward equations of the population process, and to approximate them with moment closures. Moment closures are used to control the dimensions of the model (number of ODEs), by expressing higher-order moments in terms of lower-order moments. These often rely on assumptions about the distribution of the population at each point in time. There are numerous studies to identify the best fitting assumptions for certain models [4, 5], as well as discussions of general approaches to generate the system of ODEs and their moment closures [6–9].

In this paper we aim to investigate the effectiveness of Poisson and binomial moment closure approximations, both of which provide an additional means of approximation by only referring to first-order moments. We show that despite their simplicity, in many cases they significantly outperform mean field approximation, while in some cases they are more accurate than second-order moment closure approximations such as the normal moment closure. By considering vastly different behavioural patterns in our experiments, we also show that the accuracy of the approximations heavily depends on the underlying properties of the population process.

The rest of this paper is organized as follows. In section 2 we formally introduce population processes, we describe the exact characterization of their transient behaviour, the concept of moment closures, and the Poisson and binomial moment closure approximations. Next in section 3, we give descriptions of the models that we use for investigating the effectiveness of the moment closure approximations. Then in section 4 we discuss the results of the evaluation, and try to identify patterns associated with improvements in approximation. We conclude this paper by giving a brief summary of the results in section 5.
2 Moment closure approximations

In this section, we describe the approximation methods which will be used in section 4.

2.1 Population processes and master equations

Population processes are specified as follows. Let \( N \in \mathbb{N} \) be the total number of individuals (components), and let \( S = \{s_1, \ldots, s_l\} \) be a finite set of species (component types) in a population. Define the population process as the continuous-time Markov process \( \{X^{(N)}(t) : t \in \mathbb{R}_{\geq 0}\} \) on space \( \Delta^{(N)} = \{\mathbf{i} \in \{0, N\}^l : \sum_i v_i = N\} \), and let \( X^{(N)}_0 \in \Delta^{(N)} \) be its initial state. Moreover, define the space \( \Delta \subset (\mathbb{R}_{\geq 0})^l \) such that for all \( N \in \mathbb{N} \), \( \Delta^{(N)} \subset \Delta \) holds.

The behaviour of the population process is given by a set of \( N \) transitions \( J = \{j_1, \ldots, j_K\} \). For each transition \( j \in J \), we define \( \mathbf{v}_j \in \mathbb{Z}^l \) as the change vector of transition \( j \), and \( \phi_j : \Delta \rightarrow \mathbb{R}_{\geq 0} \) as the transition rate. These specify the impact of the transition \( j \) on the population of the species and the speed of the transition respectively. In notations for mass-action kinetics in chemical reaction networks, transitions denote reactions, while entries of the change vector denote stoichiometric coefficients.

The probability \( \mathcal{P}(\mathbf{x}, t) \) of the population process being in state \( \mathbf{x} \in \Delta^{(N)} \) at time \( t \) is given by the Kolmogorov forward equation (or master equation)

\[
\frac{\partial \mathcal{P}(\mathbf{x}, t)}{\partial t} = \sum_{j \in J} \phi_j(\mathbf{x} - \mathbf{v}_j)\mathcal{P}(\mathbf{x} - \mathbf{v}_j, t) - \phi_j(\mathbf{x})\mathcal{P}(\mathbf{x}, t).
\]

Note that the expected number of components in state \( i \in S \) at time \( t \), or \( \mathbb{E}[X^{(N)}_i(t)] \), can be calculated by

\[
\mathbb{E}[X^{(N)}_i(t)] = \mathbb{E}[X^{(N)}]\mathcal{P}(\mathbf{x}, t).
\]

This insight can be used to calculate equations for different aspects of the behaviour of the population, which for the first moment behaviour gives rise to equations [10]

\[
\frac{d\mathbb{E}[X^{(N)}_i(t)]}{dt} = \sum_{j \in J} v_{ji}\mathbb{E}[\phi_j(X^{(N)}(t))] \quad (i \in S).
\]

Equations (2) are exact, meaning that their solution exactly describes the expected behaviour of the population process \( X^{(N)}(t) \). However, in general the value of the terms \( \mathbb{E}[\phi_j(X^{(N)}(t))] \) cannot be determined without extra information regarding the distribution of \( X^{(N)}(t) \) or alternatively, regarding other moments of behaviour. One special case is when \( \phi_j \) is a linear function on \( \Delta \), in which case \( \mathbb{E}[\phi_j(X^{(N)}(t))] = \phi_j(\mathbb{E}[X^{(N)}(t)]) \). Then, equations (2) become the system of ODEs:

\[
\frac{d\mathbb{E}[X^{(N)}_i(t)]}{dt} = \sum_{j \in J} v_{ji}\phi_j(\mathbb{E}[X^{(N)}(t)]).
\]

Although the system of ODEs in (3) only gives exact solutions for linear transition rates, in many cases it can be still considered as a sufficient approximation of the first moment behaviour. Specifically, it has been shown that for a class of models called density dependent population processes the approximation results in asymptotically exact solutions over finite time horizons [11], with numerous experiments showing that it performs sufficiently well for finite \( N \).

2.2 Non-linear rates and moment closure approximations

The system of ODEs in (3) is often called the mean field approximation of the population process. This can be also referred to as a first-order moment closure [6, 9] of equation (2), since it only refers to first moments \( \mathbb{E}[X^{(N)}(t)] \). However, in case any of the functions \( \phi_j \) in (2) are non-linear, the term \( \mathbb{E}[\phi_j(X^{(N)}(t))] \) may also be approximated by referring to (multiple) other higher-order moments. For example for transition \( j \in J \), let the associated rate be \( \phi_j(\mathbf{x}) = x_i^2 \), where \( x_i \) is the population of species \( i \in S \), then

\[
\mathbb{E}[\phi_j(X^{(N)}(t))] = \mathbb{E}[X^{(N)}(t)^2],
\]

in which \( \mathbb{E}[X^{(N)}(t)^2] \) can be calculated by deriving its corresponding differential equations from the Kolmogorov forward equations in (1). However, the new equations may in turn refer to other (potentially higher) moments. In general, this process can lead to an infinite number of coupled ODEs.
In essence, moment closures cut the chain of dependency by rewriting higher-order moments in terms of lower-order moments, based on some assumption. In case of the first-order moment closure, the assumption is

\[ \mathbb{E}(\phi_j(X^{(N)}(t))) \approx \phi_j(\mathbb{E}(X^{(N)}(t))). \]

It is possible to employ other moment closure approximations, which usually rely on the assumption that the population follows a certain distribution at each point in time, e.g., normal [6], log-normal [12] or beta-binomial [5]. Here, we describe the normal moment closure approximation, together with its assumptions and its application.

The normal moment closure assumes that the population is multivariate normal at each point in time [6]. By the application of the following theorem, this assumption provides a way to find second-order moment closure approximations of (2).

**Theorem 1.** (Isserlis’ theorem cf. [13]). Let \( \tilde{X}(t) \) be multivariate normal with \( I \) dimensions, with mean \( \tilde{\mu} \) and covariance matrix \( \Sigma \). Let \( Y_1, \ldots, Y_n \) be zero-mean random variables selected (with possible duplicates) from \( \{X_1(t) - \mu_1, \ldots, X_t(t) - \mu_t\} \), then:

\[
\mathbb{E}[Y_1 \ldots Y_n] = \begin{cases} 0, & \text{if } n \text{ odd} \\ \sum \prod \mathbb{E}[Y_i Y_j], & \text{if } n \text{ even} 
\end{cases}
\]

where the sum-product refers to the sum over all possible partitionings of indices 1, \ldots, \( n \) into pairs \( i, j \), where each summand is the product of \( (n/2) \) joint second moment terms.

For example, by assuming that \( X^{(N)}(t) \) is multivariate normal and by using the Isserlis’ theorem one can reason that:

\[
\mathbb{E}[X_1^{(N)}(t)^3] = 3\mathbb{E}[X_1^{(N)}(t)^2]\mathbb{E}[X_1^{(N)}(t)] - 2\mathbb{E}[X_1^{(N)}(t)]^3.
\]

or that:

\[
\mathbb{E}[X_1^{(N)}(t)^4] = 6\mathbb{E}[X_1^{(N)}(t)^2]\mathbb{E}[X_1^{(N)}(t)]^2 + 3\mathbb{E}[X_1^{(N)}(t)^2]^2 - 5\mathbb{E}[X_1^{(N)}(t)]^4 + 3\mathbb{E}[X_1^{(N)}(t)]^2.
\]

In the following sections, we apply the normal moment closure approximations to our case studies and compare their accuracy to Poisson and binomial moment closures.

## 2.3 Binomial and Poisson moment closures

In this paper we introduce the Poisson and binomial moment closures. The binomial moment closure has been considered before, and is inspired by the idea of asymptotic independence of individuals in equilibrium, in population models [14]. We also consider the Poisson moment closure, since the Poisson distribution is related to both normal and binomial distributions via well-known approximation theorems.

These two moment closures assume that at each point in time:

1. the populations of species (component types) are pairwise independent, and
2. the population of each species (component type) is Poisson / binomial distributed.

In other words the Poisson moment closure states that if at time \( t \), \( \mathbb{E}[X^{(N)}(t)] = \bar{x} \), the probability of having \( k \) individuals of species \( s \) is:

\[
\Pr \{X_s^{(N)}(t) = k\} = e^{-\bar{x}} \frac{\bar{x}^k}{k!},
\]

and similarly, the binomial moment closure states that at each time \( t \),

\[
\Pr \{X_s^{(N)}(t) = k\} = \binom{N}{k} \left( \frac{x_s}{N} \right)^k \left( 1 - \frac{x_s}{N} \right)^{N-k}.
\]

Moreover, the probability of having \( k_s \) individuals of species \( s \) and \( k_{s'} \) of species \( s' \) is

\[
\Pr \{X_s^{(N)}(t) = k_s \land X_{s'}^{(N)}(t) = k_{s'}\} = \Pr \{X_s^{(N)}(t) = k_s\} \Pr \{X_{s'}^{(N)}(t) = k_{s'}\}.
\]

Going back to equation (2), consider the transition \( j \in J \) with rate \( \phi_j \) which depends on species \( s \), and assume \( \mathbb{E}[\phi_j(X^{(N)}(t))] = x \). Then in case of Poisson moment closure we get the equations

\[
\frac{d\mathbb{E}[X_i^{(N)}(t)]}{dt} = \sum_{j \in J} v_{ij} \phi_j(\mathbb{E}[X^{(N)}(t)]) \quad (i \in S).
\]
where
\[
\hat{\phi}_j(X^{(N)}(t)) = \sum_{k \in \mathbb{N}} \phi_j(k) e^{-r_s} \frac{X^k}{k!}
\] (5)
this leads to a first-order moment closure approximation. This approach is different from the normal moment closure which thanks to Theorem 1 has an elegant way of relating higher-order moments to first and second-order moments. However, as we show in later sections, for specific communication patterns the Poisson moment closure approximations have a simple closed form and for the rest, they can still be used for numerical solutions.

3 Description of non-linear models

In this section we explain the details of the models that will be used for comparing the performance of the moment closure approximations. Each model contains a specific non-linear rate, with which we try to examine and demonstrate the applicability of the moment closure approximations.

3.1 The client-server model

The client-server model is widely discussed in the context of the PEPA process algebra (e.g., [8, 15]). In the literature there are various specifications of this model. The following is the simplest specification which exhibits the properties interesting to us. The model consists of \( N_c \) clients and \( N_s \) servers. Initially, the servers are idle and the clients do some local tasks, which they stop with rate \( r_p \). Once a client stops its task, it sends a request to one of the idle servers, to which the server responds. This transaction occurs with rate \( r_s \) on the client’s side and with rate \( r_s \) on the server’s side. Subsequently, the servers moves on and locally log the transaction with rate \( r_l \) and the client proceeds with its local tasks. For this model, a good performance indicator is the number of clients that are idle and server logging respectively.

Clients stop their local task:
\[ v_{j_c} = (-1, 1, 0, 0) \] with \( \phi_{j_c}(\vec{X}) = r_p X_c. \]

Clients and servers engage in request-response:
\[ v_{j_v} = (1, -1, -1, 1) \] with \( \phi_{j_v}(\vec{X}) = \min(r_c X_c, r_s X_l). \]

Servers finish logging:
\[ v_{j_s} = (0, 0, 1, -1) \] with \( \phi_{j_s}(\vec{X}) = r_s X_l. \]

The term \( \min(r_c X_c, r_s X_l) \) expresses the fact that in one-to-one interactions the speed of a transactions are controlled by the slowest component type involved, and are proportional to the number of participating components.

In the mean field approximation, the terms \( \mathbb{E}[\min(r_c X_c, r_s X_l)] \) appear according to the approximation:
\[
\mathbb{E}[\min(r_c X_c, r_s X_l)] \approx \min(r_c \mathbb{E}[X_c], r_s \mathbb{E}[X_l]).
\] (6)

As discussed in [1, 8] the accuracy of this approximation depends on the parameters of \( \min \). Given that the function \( \min(x, y) \) is a concave function and using Jensen’s inequality, for general random variables \( X \) and \( Y \) we have:
\[ \mathbb{E}[\min(X, Y)] \leq \min(\mathbb{E}X, \mathbb{E}Y). \]

However, the function \( \min \) is also piece-wise linear, and as long as the solution remains strictly on the linear parts, the approximation (6) performs well. In function \( \min(x, y) \), the point \( x = y \) at which the transition between behaviors occurs is appropriately called a switch point [8]. The underlying intuition is that for a function \( \min \) which operates close to switch points, by tiny changes of the parameters the “modes” of behaviour will switch and drastically change. Therefore, for a population process which operates close to this point, infinitesimal fluctuations caused by random individual events lead to divergent behaviour.

The normal moment closure tackles this problem by proposing the use of the so-called min-normal closure [6]. This uses a result which states that for bivariate normal random variables \( X \) and \( Y \),
\[ \mathbb{E}[\min(X, Y)] = \mathbb{E}X \Phi\left(\frac{\mathbb{E}Y - \mathbb{E}X}{\theta}\right) + \mathbb{E}Y \Phi\left(\frac{\mathbb{E}X - \mathbb{E}Y}{\theta}\right) - \theta \Phi\left(\frac{\mathbb{E}Y - \mathbb{E}X}{\theta}\right), \]
where \( \theta = (\mathbb{V}ar[X] - 2\text{Cov}[X, Y] + \mathbb{V}ar[Y])^{1/2} \), and \( \Phi \) and \( \phi \) are the CDF and PDF of the standard normal distribution [17].

As for the Poisson and binomial moment closures, to the best of our knowledge no closed form is known for the minimum function. However, directly plugging the sum (5) in the equations results in equations that can be numerically solved.
3.2 Multipacket reception model (MPR)

Multipacket reception (MPR) is a phenomenon which occurs in wireless communication. Consider a wireless receiver and multiple wireless transmitters in range of the receiver. If the transmitters transmit a packet at the same time, because of location diversity and power attenuation, there is a chance that one of the transmitters can capture the receiver by having a signal strong enough to stand out, with the capture probability becoming smaller as the number of transmitters increase.

The specification of the model we consider is as follows. Consider $N$ wireless nodes, each attempting to communicate to wireless receivers. We assume the number of receivers to remain constant. Initially the nodes are in the processing state, and they generate a new packet with probability $p_r$, and move to a transmit state. While in the transmit state, a node will succeed with probability $2^{-X}$, where $X_i$ is the number of transmitters, and move back to the processing state. If a transmitter fails, it will subsequently move to the backoff state. A node in backoff state will retry transmission with probability $p_v$. In this model, the number of nodes in the backlog state can be used to measure the performance of the network.

For the corresponding population model, let $S = \{p, t, b\}$ be the set of states, with $(X_p, X_t, X_b)$ the number of individuals in each state. The transitions of the model would be:

- Nodes generate packets: $v_{ji} = (-1, 1, 0)$ with $\phi_{ji}(\vec{X}) = p_g X_p$.
- Nodes succeed in transmitting packets: $v_{ji} = (1, -1, 0)$ with $\phi_{j1}(\vec{X}) = X_i 2^{-X_i}$.
- Nodes fail in transmitting packets: $v_{ji} = (0, -1, 1)$ with $\phi_{j3}(\vec{X}) = X_i \left(1 - 2^{-X_i}\right)$.
- Nodes retry transmission: $v_{ji} = (0, 1, -1)$ with $\phi_{j3}(\vec{X}) = p_r X_b$.

In general, calculation of the success probability is much more complex [18]. However, for the purposes of this paper the simple model of an exponentially diminishing probability ($2^{-X}$) captures the essence of this phenomenon.

The transition rates $\phi_{ji}$ and $\phi_{ji}$, have simple closed forms with the Poisson moment closure approximation, since using (5), for large enough $N$

\[
\phi_{ji}(\vec{X}) = \sum_{k=0}^{\infty} k 2^{-k} e^{-X_i} \frac{X_i^k}{k!} = \sum_{k=0}^{\infty} k 2^{-k} e^{-X_i} \frac{X_i^k}{k!} = \frac{X_i}{2} e^{-\frac{X_i}{2}}.
\]

Similarly, for the binomial moment closure approximation we get

\[
\phi_{ji}(\vec{X}) = X_i \left(1 - X_i/2N\right)^N / 2 - X_i/N.
\]

For the normal moment closure, it is possible to use Taylor’s expansion and Isserlis’ theorem to derive

\[
\phi_{ji}(\vec{X}) = 2^{-EX_i} \left[ -\log \frac{2 Var[X_i]}{2} \right.
\]

\[
(2 \cosh(\log 2 \sqrt{Var[X_i]} - \log 2 \sqrt{Var[X_i]} \sinh(\log 2 \log 2 Var[X_i])))
\]

\[
+ EX_i \log^2 \sqrt{Var[X_i]} \cosh(\log 2 \log 2 Var[X_i]) \right],
\]

however as we will discuss in section 4, this term does not fully represent the behaviour.

3.3 Queue model

The next model we look at is a system of $N$ servicing queues, each with a one place buffer, as presented in [19]. Clients arrive according to a Poisson distribution, and we assume that the intensity of arrivals scales with the system size as $N \rho_a$. Upon arrival a client joins the shortest queue. At each point in time a queue can be in one of 3 states: idle (state 0), serving with an empty buffer (state 1), or serving with a waiting client (state 2). The service time of a queue is exponential, and regardless of the number of waiting clients always has rate $r_s$. The number of queues with a full buffer is a good performance indicator for this model.

Let $S = \{0, 1, 2\}$, and let $X_0$, $X_1$ and $X_2$ denote the number of nodes in each state, based on the above description define the transitions:
• Arriving clients go to idle servers if any exist: \( v_{j_b} = (-1, 1, 0) \) with \( \phi_{j_b}(\vec{X}) = \text{NRs} \cdot \mathbb{1}[X_0 > 0] \).

• If no idle servers exist, arriving clients go to servers with empty buffers if any exist: \( v_{j_1} = (0, -1, 1) \) with \( \phi_{j_1}(\vec{X}) = \text{NRs} \cdot \mathbb{1}[X_0 = 0 \wedge X_1 > 0] \).

• Servers with empty buffers serve clients: \( v_{j_2} = (1, -1, 0) \) with \( \phi_{j_2}(\vec{X}) = \text{rs} \cdot X_1 \).

• Servers with full buffers serve clients: \( v_{j_3} = (0, 1, -1) \) with \( \phi_{j_3}(\vec{X}) = \text{rs} \cdot X_2 \).

The terms \( \mathbb{1}[X_0 > 0] \) and \( \mathbb{1}[X_0 = 0 \wedge X_1 > 0] \) ensure that arriving clients always join the shortest queue available. However, these same terms lead to non-linearity in equation (2). Moreover, since these terms are not Lipschitz continuous\(^2\), the mean field approximation is not guaranteed to find accurate results. However, in [20], a family of such population processes are investigated and conditions under which the mean field approximation can be found are given. In our experiments, we chose the parameters such that these conditions are met so that the mean field approximations exist.

The communication model for the queue model results in relatively simple terms in case of using Poisson moment closures. For \( j_0 \) and \( j_1 \), this approximation will result in:

\[
\hat{\phi}_{j_0}(\vec{X}) = \text{NRs} (1 - e^{-X_0}), \quad \hat{\phi}_{j_1}(\vec{X}) = \text{NRs} e^{-X_0} (1 - e^{-X_1}),
\]

and with the binomial approximation, in:

\[
\hat{\phi}_{j_0}(\vec{X}) = \text{NRs} \left( 1 - (1 - X_0/N)^N \right), \quad \hat{\phi}_{j_1}(\vec{X}) = \text{NRs} \left( 1 - X_0/N \right)^N \left( 1 - (1 - X_1/N)^N \right).
\]

Note that for large \( N \) the Poisson and binomial closure terms approximately have the same values.

### 4 Evaluation and comparison

In this section, for the client-server model and MPR model described in 3, we compare the results of analysis done by mean field approximation, normal moment closure, and Poisson and binomial moment closures. For the queue model, we will only compare the mean field approximation with the Poisson and binomial moment closures. Moreover, based on the simulation results for each case we test the accuracy of the assumptions regarding distributions. An overview of all the experiments and their corresponding parameters is given in Table 1.

<table>
<thead>
<tr>
<th>Experiment set</th>
<th>Parameters</th>
<th>Set size</th>
</tr>
</thead>
<tbody>
<tr>
<td>asymmetric client-server</td>
<td>( N_r = 10000, N_r \in [2, \ldots, 12], r_p = 0.06, r_s = 500, r_i = 2, r_f = 120 )</td>
<td>11</td>
</tr>
<tr>
<td>symmetric client-server</td>
<td>( N_r = {50, 100, 150}, N_r = 50, r_p = 0.5, r_s = 14, r_i = 10, r_f = 0.5, 0.7, 1.0, 1.2, 1.5 )</td>
<td>15</td>
</tr>
<tr>
<td>large network with MPR</td>
<td>( N = 500, p_e \in (0.005, 0.01, 0.02), p_s \in (0.005, 0.008), x_0(0) \in (0, 500), x_i(0) = 0 )</td>
<td>12</td>
</tr>
<tr>
<td>small network with MPR</td>
<td>( N = 50, p_e \in (0.005, 0.01, 0.02), p_s \in (0.005, 0.008), x_0(0) \in (0, 50), x_i(0) = 0 )</td>
<td>12</td>
</tr>
<tr>
<td>queues (saturated &amp; unsaturated)</td>
<td>( N = 100, r_s \in {1, 1.5, 2.25}, r_s \in {1.5, 2.25}, (x_0(0), x_1(0)) \in {(0, 1.0), (0.5, 0.3)} )</td>
<td>24</td>
</tr>
</tbody>
</table>

4.1 Error requirements and calculation

In the experiments described in the following sections, the discrete event simulations were all carried out with 50,000 replications. This was mostly due to the fact that in models with smaller populations, putting requirements on the relative error of samples of the performance measures quickly leads to a huge number of simulations.

In this paper, we only calculate the error of the performance evaluation indicators, i.e., the number of requesting clients in the client-server model, the number of nodes in backlog in the MPR model, and the number of full queues in the queue model. Let \( x(t) \) be the approximation of the performance indicator at time \( t \), and \( s(t) \) the estimated mean according to simulations. The error of the approximation \( x \) at time \( t \) is the relative error

\[
\text{Err}_x(t) = \left| \frac{x(t) - s(t)}{s(t)} \right|.
\]

For a set of experiments with various parameter settings leading to simulation results \( s_1, \ldots, s_k \) and approximations \( x_1, \ldots, x_k \) derived by the same approximation method, \( \text{Err}_x(t) \) indicates the average of errors \( \text{Err}_{s_1}, \ldots, \text{Err}_{s_k} \) at time \( t \).

---

1The indicator function \( I[c] \) evaluates to 0 if \( c \) is false, and 1 if \( c \) is true.

2A function \( f : \mathbb{R} \to \mathbb{R} \) is (universally) Lipschitz continuous, if for a constant \( L \in \mathbb{R} \), for all \( a, b \in \mathbb{R} \), \( |f(b) - f(a)| < L|b - a| \) holds.
Table 2: The average times for evaluating the performance of the client-server models (in seconds)

<table>
<thead>
<tr>
<th>experiment</th>
<th>mean field</th>
<th>normal closure</th>
<th>Poisson closure</th>
<th>binomial closure</th>
<th>simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>asymmetric case</td>
<td>64.2</td>
<td>981.6</td>
<td>756.4</td>
<td>305.9</td>
<td>48180</td>
</tr>
<tr>
<td>symmetric case</td>
<td>2</td>
<td>8</td>
<td>40.1</td>
<td>22.5</td>
<td>784</td>
</tr>
</tbody>
</table>

Table 3: The maximum and average errors of each approximation in the client-server models

<table>
<thead>
<tr>
<th>experiment</th>
<th>mean field</th>
<th>normal</th>
<th>Poisson closure</th>
<th>binomial closure</th>
</tr>
</thead>
<tbody>
<tr>
<td>max</td>
<td>avg</td>
<td>max</td>
<td>avg</td>
<td>max</td>
</tr>
<tr>
<td>asymmetric case</td>
<td>43.09%</td>
<td>5.59%</td>
<td>31.83%</td>
<td>4.68%</td>
</tr>
<tr>
<td>symmetric case</td>
<td>49.68%</td>
<td>6.76%</td>
<td>9.52%</td>
<td>0.75%</td>
</tr>
</tbody>
</table>

4.2 Client-server model

For the evaluation of the client-server model we run two experiments. The first set of experiments are inspired by [1] to specifically test the accuracy of the moment closure approximations around the switch points in the asymmetric case, i.e., when the number of clients is large \((N_c = 10000)\) and the number of servers is small \((N_s \in \{2, \ldots, 12\})\). In these experiments we only varied the number of servers.

The second set of experiments refer to the symmetric case, in which the number of clients and servers have the same scale \((N_c \in \{50, 100, 150\}, N_s = 50)\). In these experiments we vary the number of clients and the parameter \(r_l\) (servers’ logging rate). In both sets of experiments, we chose the parameters in a way that for some configurations the models pass through or stay around the switch points.

As indicated earlier, the terms \(\mathbb{E}[\min(X, Y)]\) do not have a closed form in case of binomial and Poisson moment closures. For these cases, we solve the ODEs by truncating insignificant terms and employing a method of dynamic programming which substantially improved the performance of the numerical solver. For this study, we found the GPA tool [21] to be an efficient tool for the simulations. In Table 2 we compare the average run-time for solving the ODEs and the simulations.

The maximum errors in Table 3 and the curves for the average relative errors through time in Fig. 1 show that in general using one of the closure approximations improves the accuracy of the performance measures. Overall, the min-normal closure approximation results in more reliable results. As can be seen, the Poisson and binomial moment closure approximations tend to cluster together in terms of accuracy, with the Poisson approximation performing slightly better on average.

Apart from these general remarks, in the asymmetric case the Poisson closure approximation shows the best performance, while in the symmetric case the min-normal closure approximation gives the best results. It is worth noting that in the asymmetric case and for parameters \(N_c \in \{6, \ldots, 12\}\), the ODEs that we wrote led to negative second moments, and an analysis by the GPA tool [21] faced errors as well, which points to a limitation of the normal moment closure with these configurations.

In the asymmetric cases, the inaccuracy of the Poisson and binomial distributions can be attributed to the insufficient representation of the long tail that can be seen in, e.g., the histogram in Fig. 2(a). This spread of data can only be well captured by considering the second order moments, however, note that the moderate skeweness is captured by neither of the approximations. In contrast, and probably by chance, the histogram in the asymmetric cases (e.g., Fig. 2(b)) are better captured by the Poisson approximation.

4.3 MPR model

In the multipacket reception model the non-linear behaviour is highly dependent on the size of the system, therefore it would not be insightful to look at aggregate results coming from systems with different sizes. For the evaluation of the MPR model we run two series of experiments, one with a relatively small number of nodes \((N = 50)\), and another one with a larger number of nodes \((N = 500)\). In each case we vary the retry probability \(p_r\) and the initial number of nodes in the backoff and idle states.

For \(N = 500\) employing the normal moment closure approximation in formula (7) gave relatively acceptable results. However, for \(N = 50\), the formula performs poorly resulting in negative values for the occupancy of states. This could be due to the way formula (7) is calculated, which involves a Taylor’s expansion of term \(X_t 2^{-X_t}\) around point 0. However, a more careful definition of the function would be

\[
\begin{cases}
X_t 2^{-X_t} & X_t > 0, \\
0 & X_t \leq 0,
\end{cases}
\]
Figure 1: Average relative error in the number of waiting clients, in the (a) symmetric case, and the (b) asymmetric case, for different moment closure approximations.

Figure 2: Empirical distribution of the number of waiting clients (a) for $N_c = 100$, $N_s = 50$, $r_l = 1.0$. (b) for $N_c = 10000$, $N_s = 5$, and the assumed approximations in the equilibrium. The vertical dashed line marks the center of mass in the samples from simulations.
Table 4: The maximum and average errors of each approximation in the MPR models

<table>
<thead>
<tr>
<th>experiment</th>
<th>mean field</th>
<th>normal</th>
<th>Poisson closure</th>
<th>binomial closure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max</td>
<td>avg</td>
<td>max</td>
<td>avg</td>
</tr>
<tr>
<td>$N = 500$</td>
<td>8.22%</td>
<td>1.66%</td>
<td>10.04%</td>
<td>3.92%</td>
</tr>
<tr>
<td>$N = 50$</td>
<td>81.68%</td>
<td>63.95%</td>
<td>43.86%</td>
<td>28.78%</td>
</tr>
</tbody>
</table>

Figure 3: Average relative error in the number of nodes in backlog, for (a) $N = 500$, and (b) $N = 50$, for different moment closure approximations.

which does not have a second derivative at point 0, and cannot be represented accurately by a Taylor’s series. For this reason, for $N = 50$, we replaced formula (7) with an explicit calculation of the expected value of the terms $\mathbb{E}[X_t^2]$, assuming that $X_t$ is truncated normal with $0 \leq X_t \leq N$.

Both experiments support the choice of binomial distribution as the most reliable moment closure approximation, as supported by Fig. 3 and the histograms in Fig. 6. The histograms in Fig. 6(a) also shows that the term (7) underapproximates the capture probability of the transmitter, which could be due to the fact that for a small number of expected transmitters the left tail of the normal distribution goes into negative values of $X_t$. With the use of truncated normal approximation in Fig. 6(b), the accuracy improves.

Note that as the model gets large, the mean field equations also become an accurate method of approximation. However, one could argue that an infinitely large MPR model is not interesting; as it is easy to see that the non-linear term $X_t^2 - X_t$ tends to zero as $N$ gets large, and most of the nodes will be eventually trapped in backlog.

4.4 Queue model

Moment closure approximations are not only applied to improve the accuracy of the approximation, but also to make it possible to produce some results in case the mean field ODEs do not satisfy the necessary assumptions for the existence and uniqueness of a solution. The transitions of the queue model as described in section 3, lead to mean field models which generally belong to this class of differential equations.

In [20], conditions under which the ODEs of the queue model satisfy a unique reliable solution are analysed. However, we find that even in situations where unique reliable solutions theoretically exist, numerical solvers tend to be unstable. For the queue models, we created two class of parameters, one with numerically stable mean field equations, which were solved by either VODE or lsoda libraries. The other class were the models with numerically unstable mean field equations, which were solved by the StiffnessSwitching methods of the Mathematica tool. By examining the simulation results (for $N = 100$ queues), we observed that the numerically stable models correspond to unsaturated queues, while for the numerically unstable models, the queues tend to be saturated (have full buffers most of the time).

For numerically stable mean field approximations, we found all the three approximation methods: mean field, Poisson and binomial to be equally accurate. Despite initial fluctuations, all the methods tend to rest below 0.1% error as the simulations reach a seeming equilibrium state. These confirm the conclusions of [20]. In most of the time points, we found that the simulations show a tendency to exhibit two peaks, as exhibited by the histogram in Fig. 5(a).

For numerically unstable mean field approximations, none of the other approximation methods can be trusted.
Figure 4: Empirical distribution of the number of nodes in backlog (a) for $N = 500$, (b) for $N = 50$, and the assumed approximations in the equilibrium.

Table 5: The maximum and average errors of each approximation in the queue models

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Mean Field</th>
<th>Poisson Closure</th>
<th>Binomial Closure</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>max</td>
<td>avg</td>
<td>max</td>
</tr>
<tr>
<td>Unsaturated queues</td>
<td>0.63%</td>
<td>0.8%</td>
<td>0.63%</td>
</tr>
<tr>
<td>Saturated queues</td>
<td>178681%</td>
<td>1809%</td>
<td>294%</td>
</tr>
</tbody>
</table>

In long-term where the simulations reach an equilibrium state, the error still remains very large (around 50%). By looking at the histogram in Fig. 5(b), the inaccuracy could be attributed to under-representation of the many rare events in which the queues can empty their buffers, momentarily leading to an unsaturated system. Here, choosing a suitable fat-tailed distribution would have led to better approximations.

5 Conclusions

In this paper we analysed the accuracy of four different methods of approximation for middle-sized systems with non-linear rates. We chose three models with completely different non-linear behavioural patterns, and in addition, for each model we chose two groups of parameters which correspond to different modes of behaviour which arise from the interaction of individuals in the models.

We can summarize our general observations as follows:

1. In most cases applying any moment closure approximation other than the mean field approximation gives more accurate results. This is more specifically the case when the system size is small.
2. Even for the same system, the choice of parameters significantly affects the accuracy of a moment closure approximation.
3. In the absence of information regarding the actual distribution of populations at a point in time, adding higher moments to the equations does not lead to a progressive improvement of results.
4. All approximations are at their poorest around the boundaries of the system.

In the models with parameters considered in this paper, the binomial moment closure had the best overall reliability. Despite the promising appearance of Isserlis’ theorem, in practice we found that it is difficult to automate the normal moment closure for arbitrary non-linear models, e.g., in the MPR model (section 3.2) the resulting ODEs led to big errors, and in the queue model (section 3.3) the method is awkward to apply. In contrast, the Poisson and binomial moment closures can be effectively employed in numerical calculations.
Figure 5: Average relative error in the number of queues with full buffers, for (a) unsaturated queues (numerically stable), and (b) saturated queues (numerically unstable), for Poisson and binomial closure approximations.

Figure 6: Empirical distribution of the number of idle queues (a) for the unsaturated system (b) for the saturated system, and the assumed approximations in the equilibrium.
References


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<th>Title</th>
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