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DOUBLY EXPONENTIALLY MANY INGLETON MATROIDS∗

PETER NELSON† AND JORN VAN DER POL‡

Abstract. A matroid is Ingleton if all quadruples of subsets of its ground set satisfy Ingleton’s inequality. In particular, representable matroids are Ingleton. We show that the number of Ingleton matroids on ground set \([n]\) is doubly exponential in \(n\); it follows that almost all Ingleton matroids are nonrepresentable.

Key words. matroid, enumeration, representable

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1. Introduction. Ingleton’s condition [Ing71] is a well-known linear inequality that holds for representable matroids but not matroids in general; it states that for all \(A, B, C, D \subseteq E\) in a representable matroid \(M = (E, r)\),

\[
\begin{align*}
r(A \cup B) &+ r(A \cup C) + r(A \cup D) + r(B \cup C) + r(B \cup D) \\
&\geq r(A) + r(B) + r(A \cup B \cup C) + r(A \cup B \cup D) + r(C \cup D).
\end{align*}
\]

An arbitrary matroid is Ingleton if the above inequality is satisfied for all choices of \(A, B, C, D\). The class of Ingleton matroids is closed under minors and duality (see, for example, Lemmas 3.9 and 4.5 in [Cam13]) and clearly all representable matroids are Ingleton. A natural question is to what extent a converse of this last statement holds: that is, do Ingleton matroids tend to be representable? We prove here that they do not. For \(n \geq 12\), the number of representable matroids on \([n]\) is at most \(2^{0.25n^3}\) [Nel16]; our main result is the following.

Theorem 1.1. For sufficiently large \(n\) and all \(0 < r < n\), there are at least
\[
2^{0.486 \log \frac{r(n-r)}{r(n)} \binom{n}{r}} \frac{(r(n-r))}{r(n)^{r(r-1)}}
\]
rank-\(r\) Ingleton matroids with ground set \([n]\).

Even when \(r = 4\), this eclipses the upper bound on the number of representable matroids on \([n]\) with no restriction on rank; thus, almost all Ingleton matroids are nonrepresentable. When \(r = \lfloor n/2 \rfloor\), the above formula is around \(2^{3.88 \log n \frac{n}{n^2} \binom{n}{n/2}}\), which is doubly exponential in \(n\), and even somewhat resembles the number of all matroids on \([n]\), which is \(2^{\frac{1}{n} \binom{n}{n/2}}\) with a constant between 1 and 2 + \(o(1)\) [BPvdP15]. We conjecture, however, that general matroids tend not to be Ingleton.

Conjecture 1.2. There is a constant \(c\) such that the number of Ingleton matroids on \([n]\) is \(2^{(c+o(1)) \frac{\log n}{n^2} \binom{n}{n/2}}\).

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Conceivably, the constant $c$ could be equal to 4 or even be the one of around 3.88 obtained by our proof.

In what follows, we assume some familiarity with matroid theory; see [Oxl11]. A nonbasis of a rank-$r$ matroid $M$ is a set in $(E(M)/r)$ that is not a basis of $M$, and a matroid is paving if all its nonbases are circuits. We call $M$ sparse paving if $M$ and $M^*$ are both paving. Logarithms are all base-two.

As part of the proof of Theorem 1.1, we also characterize exactly which sparse paving matroids are Ingleton, and as a result we easily derive the following theorem.

**Theorem 1.3.** There are exactly 41 excluded minors for the class of Ingleton sparse paving matroids: the matroids $U_{0,2} \oplus U_{1,1}$ and $U_{2,2} \oplus U_{0,1}$, and the 39 rank-4 non-Ingleton sparse paving matroids on eight elements.

The fact that this set is even finite is curious; the class of all Ingleton matroids, by contrast, has an infinite set of excluded minors, constructed in [MNW09]. In fact, their techniques show that every $\mathbb{R}$-representable matroid is a minor of an excluded minor for the class of Ingleton matroids.

Theorems 1.1 and 1.3 together imply that the Ingleton matroids are a “large” minor-closed class of matroids (in the sense of numbering at least $2^{2^n}/\text{poly}(n)$) that has 39 different sparse paving matroids as excluded minors. It was conjectured in [MNWW11] that any minor-closed class not containing all sparse paving matroids is asymptotically vanishing; our result shows that such a class may still be “large.”

2. Representing matroids with few nonbases.

**Lemma 2.1.** Let $M$ be a rank-$r$ matroid in which each set $W$ of nonbases with $|W| > 1$ satisfies $|\bigcap W| \leq r - |W|$. Then $M$ is $\mathbb{R}$-representable.

**Proof.** Let $X = \{X_1, \ldots, X_t\}$ be the set of nonbases of $M$; note that $0 \leq |\bigcap X| \leq r - t$, so $t \leq r$. Let $A$ be an $[r] \times E$ real matrix so that the nonzero entries of $A$ are algebraically independent over $\mathbb{Q}$, and $A_{i,e} = 0$ if and only if $i \in [t]$ and $e \in X_i$. We prove that $M = M(A)$. It is clear that for every nonbasis $W$ of $M$, the matrix $A[W]$ has a zero row and so is singular; it remains to show that $A|B|$ is nonsingular for each basis $B$ of $M$.

Let $B$ be a basis of $M$. Consider the bipartite graph $G$ with bipartition $([r], B)$ for which $(i, e)$ is an edge if and only if $A_{i,e} \neq 0$. Note that each $i \in \{t + 1, \ldots, r\}$ has degree $r$ in $G$. For each $S \subseteq [r]$, let $N(S)$ denote the set of vertices in $B$ that are adjacent to a vertex in $S$. We argue that $|N(S)| \geq |S|$ for each $S \subseteq [r]$; it will follow from Hall’s theorem that $G$ has a perfect matching.

Let $S \subseteq [r]$. If $S \not\subseteq [t]$, then clearly $N(S) = B$ and so $|N(S)| = r \geq |S|$. If $S \subseteq [t]$, then by hypothesis the set $\bigcap_{x \in S} X_x$ has size at most $r - |S|$, so $B$ contains at least $|S|$ elements $e$ for which there is some $s \in S$ with $e \notin X_s$. Each such $e$ is adjacent to $s$, so $|N(S)| \geq |S|$ as required. Therefore $G$ has a perfect matching.

Let $B = \{b_1, \ldots, b_r\}$ and $S_\sigma$ denote the symmetric group on $[r]$; note that $A|B|$ is singular if and only if the determinant $\sum_{\sigma \in S_\sigma} (-1)^{\text{sign}(\sigma)} \prod_{i \in [r]} A_{i, b_{\sigma(i)}}$ is zero. This determinant is a polynomial in the entries of $A$, with integer coefficients, whose nonzero monomials are algebraically independent over $\mathbb{Q}$, and since $G$ has a perfect matching, some monomial is nonzero. It follows that the determinant is nonzero, so $A|B|$ is nonsingular, as required.

**Lemma 2.2.** Every matroid with at most four nonbases is $\mathbb{R}$-representable.

**Proof.** Let $M$ be a minor-minimal counterexample. Note that $M$ is simple, that $r(M) \geq 3$, and that $M^*$ is also a minor-minimal counterexample, so $M$ is cosimple
with $r^*(M) \geq 3$. If $M$ has an element $e$ in no nonbases, then since $e$ is not a coloop, $M$ is the free extension of $M \setminus e$ by $e$; since $M \setminus e$ is $\mathbb{R}$-representable, so is $M$, a contradiction. So every element is in a nonbasis of $M$. Dually, no element is in all nonbases of $M$.

If $M$ has a dependent set $Y$ of size $r(M) - 1$, then $Y \cup \{e\}$ is a nonbasis for each $e \in E(M) \setminus Y$. Since no element of $Y$ is in four nonbases, this gives $|E(M) \setminus Y| \leq 3$, so $|E(M)| \leq r + 2$, giving $r^*(M) \leq 2$, a contradiction. Therefore every circuit of $M$ is spanning, so $M$ is a paving matroid; dually, $M$ is a sparse paving matroid.

If $e \in E(M)$ is in exactly one nonbasis $X$, then $M$ is the principal extension of the flat $X - \{e\}$ in $M \setminus e$, so $M$ is $\mathbb{R}$-representable, a contradiction. Therefore every $e \in E(M)$ is in at least two nonbases. Dually, every element lies outside at least two nonbases. Therefore $M$ has exactly four nonbases, and every element is in exactly two of them.

If $r(M) = 3$, then $M$ has four triangles, so there are 12 pairs $(e,T)$, where $T$ is a triangle containing $e$, and every element is in two triangles, so there are also $2|E(M)|$ such pairs $(e,T)$. Thus $|E(M)| = 6$ and so $M$ is $\mathbb{R}$-representable, a contradiction.

Suppose, therefore, that $r(M) \geq 4$. By Lemma 2.1, we may assume that there is some set $X$ of nonbases of $M$ with $|X| > 1$ such that $|\cap X| > r - |X|$. Since no element is in three nonbases, if $|X| > 2$, then $|\cap X| = 0 \leq r - |X|$, so we must have $|X| = 2$ and thus there are nonbases $X_1, X_2$ with $|X_1 \cap X_2| = r - 1$. This contradicts the fact that $M$ is a sparse paving matroid.

3. Ingleton matroids. In this section, we use the well-known fact that $H \subseteq \binom{E}{r}$ is the set of nonbases of a sparse paving matroid on $E$ if and only if no two elements of $H$ have intersection of size exactly $r - 1$. The following technical condition characterizes violations of the Ingleton inequality in sparse paving matroids. The “canonical” example where such a violation occurs is the Vámos matroid $V_8$, where $Y,Z_1,Z_2 = \emptyset$ and $X_1, X_2, X_3, X_4$ are the disjoint two-element sets for which the union of any pair except $X_3 \cup X_4$ is a circuit.

**Lemma 3.1.** Let $M$ be a rank-$r$ sparse paving matroid. Sets $A, B, C, D$ violate the Ingleton inequality in $M$ if and only if there are pairwise disjoint sets $X_1, X_2, X_3, X_4, Y, Z_1, Z_2 \subseteq E(M)$ such that

- $|X_i| = 2$ for each $i \in \{1, 2, 3, 4\}$ while $|Y \cup Z_1 \cup Z_2| = r - 4$,
- $A = X_1 \cup Y \cup Z_1 \cup Z_2$,
- $B = X_2 \cup Y \cup Z_1 \cup Z_2$,
- $C = X_3 \cup Y \cup Z_1$, and
- $D = X_4 \cup Y \cup Z_2$.

while each of $A \cup B, A \cup C, A \cup D, B \cup C, B \cup D$ is a circuit-hyperplane of $M$, and $C \cup D$ is a basis.

**Proof.** For each matroid $N$ with $A \cup B \cup C \cup D \subseteq E(N)$, let

$$h_1(N) = r_N(A) + r_N(B) + r_N(A \cup B \cup C) + r_N(A \cup B \cup D) + r_N(C \cup D)$$

and

$$h_2(N) = r_N(A \cup B) + r_N(A \cup C) + r_N(A \cup D) + r_N(B \cup C) + r_N(B \cup D),$$

so sets $A, B, C, D$ violate the Ingleton inequality if and only if $h_1 > h_2$. Suppose that sets $A, B, C, D$ as in the hypothesis exist for a matroid $M$. Since $A, B$ are proper $(r - 2)$-element subsets of circuit-hyperplanes, they both have rank $r - 2$, and since $A \cup B \cup C$ and $A \cup B \cup D$ are proper supersets of circuit-hyperplanes, they are spanning.
Since $C \cup D$ is a basis, this gives $h_1(M) = 5r - 4$. Clearly $h_2(M) = 5(r - 1) < h_1(M)$, so $M$ violates the Ingleton inequality.

Conversely, suppose that $A, B, C, D$ are sets violating the Ingleton inequality in a rank-$r$ matroid $M$, so $h_1(M) > h_2(M)$ by assumption.

3.1.1. $A \cup B, A \cup C, A \cup D, B \cup C, \text{ and } B \cup D$ are circuit-hyperplanes.

Proof of Claim. Suppose otherwise. Let $M'$ be obtained from $M$ by relaxing each circuit-hyperplane other than those among the five sets above, so $M'$ is sparse paving and has at most four circuit-hyperplanes. By Lemma 2.2, $M'$ is $\mathbb{R}$-representable, so $h_1(M') \leq h_2(M')$. By construction, we have $h_2(M') = h_2(M)$, and since $M'$ is freer than $M$, we have $h_1(M') \geq h_1(M)$. Therefore

$$h_2(M) < h_1(M) \leq h_1(M') \leq h_2(M') = h_2(M),$$

a contradiction. \hfill \Box

3.1.2. $|A| = |B| = r - 2$, and the sets $A \cup B \cup C, A \cup B \cup D, \text{ and } C \cup D$ are spanning in $M$.

Proof of Claim. The first claim gives $h_2(M) = 5r - 5$, so by assumption $h_1(M) \geq 5r - 4$. If $A \cup B \subseteq \text{cl}_M(A)$, then we have

$$h_2(M) = r_M(A) + r_M(A \cup B \cup C) + r_M(A \cup B \cup D) + r_M(B \cup C) + r_M(B \cup D),$$

and $h_2(M) - h_1(M) = r_M(B \cup C) + r_M(B \cup D) - r_M(B) - r_M(C \cup D) \geq 0$ by submodularity, a contradiction. So $A \cup B \not\subseteq \text{cl}_M(A)$; since $A \cup B$ is a circuit, it follows that $|A| = r_M(A) \leq r - 2$ and, symmetrically, that $|B| = r_M(B) \leq r - 2$. Therefore

$$3r \geq r_M(A \cup B \cup C) + r_M(A \cup B \cup D) + r_M(C \cup D)$$

$$= h_1(M) - r_M(A) - r_M(B)$$

$$\geq (5r - 4) - 2(r - 2) = 3r,$$

so we have equality throughout, and $r_M(A \cup B \cup C) = r_M(A \cup B \cup D) = r_M(C \cup D) = r$ while $|A| = |B| = r - 2$. \hfill \Box

For each nonempty subset $S$ of $\{A, B, C, D\}$, write $J_S$ for the collection of elements belonging to all sets in $S$ but no sets in $\{A, B, C, D\} - S$, and let $n_S = |J_S|$. For example, $n_{AB}$ denotes $|(A \cap B) - (C \cup D)|$ (we omit commas and braces). Since $A \cup C$ and $B \cup C$ are circuit-hyperplanes in a sparse paving matroid, and $|A - B| = |A \cup B| - |B| = r - (r - 2) = 2$, we have

$$2 \leq |(A \cup C) - (B \cup C)|$$

$$= |A - (B \cup C)|$$

$$= |A - B| - |(A \cap C) - B|$$

$$= 2 - |(A \cap C) - B|,$$

so $(A \cap C) - B = \emptyset$, giving $n_{AC} = n_{ACD} = 0$. Using the symmetry between $A$ and $B$ and between $C$ and $D$, we also have $n_{AD} = n_{BC} = n_{BD} = n_{BCD} = 0$. Therefore

$$n_C + n_{CD} = n_C + n_{CD} + n_{BCD} + n_{BC} = |C - A| = |C \cup A| - |A| = 2.$$
The four undetermined $n_S$ thus far are $n_{AB}, n_{ABCD}, n_{ABC}$, and $n_{ABD}$; all others have been shown to be zero except $n_A = n_B = n_C = n_D = 2$. Using the fact that $C \cup D$ is spanning, we thus have
\[
r \leq |C \cup D| = n_{ABCD} + n_{ABC} + n_{ABD} + n_C + n_D.
\]
On the other hand,
\[
r - 2 = |A| = n_{ABCD} + n_{ABC} + n_{ABD} + n_A;
\]
since $n_A = n_C = n_D = 2$, these together imply that $n_{AB} = 0$. The above also gives $n_{ABCD} + n_{ABC} + n_{ABD} = r - 4$. Now setting $(X_1, X_2, X_3, X_4, Y, Z_1, Z_2) = (J_A, J_B, J_C, J_D, J_{ABCD}, J_{ABC}, J_{ABD})$ gives the required structure. Finally, we see that $|C \cup D| = n_{ABCD} + n_{ABD} + n_{ABC} + n_C + n_D = (r - 4) + 4 = r$; since $C \cup D$ is spanning, it must be a basis.

A simpler characterization of these matroids below follows with $K = Z \cup Y_1 \cup Y_2$ and the $P_i$ equal to some ordering of the $X_i$ above.

**Corollary 3.2.** Let $M$ be a sparse paving matroid. Then $M$ is non-Ingleton if and only if there are pairwise disjoint sets $P_1, P_2, P_3, P_4, K$ so that $|K| = r - 4$ and $|P_i| = 2$ for each $i$, and exactly five of the six sets in $\{K \cup P_i \cup P_j : i \neq j\}$ are circuit-hyperplanes of $M$.

As observed in [Cam13], if $M$ is a matroid for which the above condition holds, then it also holds in the eight-element, rank-4 matroid $(M/K)/(P_1 \cup P_2 \cup P_3 \cup P_4)$; therefore, every non-Ingleton sparse paving matroid has an eight-element, rank-4 non-Ingleton sparse paving matroid as a minor. Mayhew and Royle [MR08] showed that there are precisely 39 such matroids; for every such matroid $N$, the Vámos matroid $V_N$ can be obtained from $N$ by a sequence of circuit-hyperplane relaxations. (We remark that [MR08] uses different terminology from ours, calling these matroids “Ingleton non-representable” rather than “non-Ingleton.”) The unique minor-minimal matroids that are not sparse paving are $U_{0,2} \oplus U_{1,1}$ and $U_{2,2} \oplus U_{0,1}$; together these facts imply Theorem 1.3.

### 4. Counting Ingleton matroids

The proof of the following theorem uses techniques from [CM14, Proposition 2.1]. For integers $r, n$ with $0 \leq r \leq n$, the **Johnson graph** $J(n, r)$ is the graph with vertex set $\binom{n}{r}$, in which two vertices $X, X'$ are adjacent if and only if $|X \cap X'| = r - 1$.

**Theorem 4.1.** There exists $n_0$ such that for all $n \geq n_0$ and all $0 < r < n$, the number of rank-$r$ Ingleton matroids with ground set $[n]$ is at least $2^{\frac{n}{2} \log \binom{n}{r}} \binom{n}{r}^{n-1}$.

**Proof.** We may assume that $2 \leq r \leq \frac{n}{2}$, since otherwise the theorem is trivial or follows by duality. Write $N = \binom{n}{r}$ and $d = r(n - r)$ for the number of vertices and the valency of $J(n, r)$.

For $x \in \mathbb{R}$, let $f(x) = 1 - \frac{1}{2}x - \frac{1}{64}x^4$. We show that if $c > 0$ is a real number and $\gamma < cf(c)$, then there are at least $2^{\frac{\gamma}{c} N}$ Ingleton sparse paving matroids of rank $r$ on $[n]$, provided $n$ is sufficiently large; the result as stated follows with $c = 0.95$ and $\gamma = 0.486$.

Given $c$ and $\gamma$, let $\alpha$ be such that $\gamma/c < \alpha < f(c)$ and let $\epsilon = f(c) - \alpha$, so $1 - f(c) + \epsilon = 1 - \alpha$. Set $k = \lceil r \frac{N}{2} \rceil$, so $(1 - o(1)) \frac{k}{N} \leq \frac{1}{2} \leq \frac{k}{N}$. Pick a $k$-set $H$ of vertices in $J(n, r)$ uniformly at random from among all $k$-subsets of vertices and write $E(H)$ for the set of unordered pairs of vertices in $H$ joined by an edge in $J(n, r)$, and $e_H$ for $|E(H)|$.
4.1.1. $\mathbf{E}(e_H) \leq \frac{ck}{d}$ and $\mathbf{Var}(e_H) = o(k^2)$.

Proof of Claim. We have

$$\mathbf{E}(e_H) = \frac{1}{2} d N \binom{N-2}{k-2} \leq \frac{d k^2}{2 N} \leq \frac{ck}{2}.$$ 

Let $\Theta$ denote the set of ordered pairs $(e,f)$ of edges of $J(n,r)$. Write $\Theta_j, j \in \{0,1,2\}$, for the set of pairs in $\Theta$ that span $4-j$ vertices. Note that $|\Theta| = \frac{1}{2} d^2 N^2$, while $|\Theta_1| = N d (d - 1) \leq N d^2$ and $|\Theta_2| = \frac{1}{2} d N$. Now, using the fact that $\binom{N-\ell}{k}/\binom{N}{k} = (1 + o(1))(k/N)\ell$ for each constant $\ell$, we have

$$\mathbf{Var}(e_H) = \sum_{(e,f) \in \Theta} \left[ \mathbf{Pr}(e,f \in E(H)) - \mathbf{Pr}(e \in E(H)) \mathbf{Pr}(f \in E(H)) \right]$$

$$= |\Theta_0| \left[ \binom{N-4}{k-4} \binom{N-2}{k-2} \right] + |\Theta_1| \left[ \binom{N-3}{k-3} \binom{N-2}{k-2} \right]$$

$$+ |\Theta_2| \left[ \binom{N-2}{k-2} \binom{N-2}{k-2} \right]$$

$$\leq \frac{1}{4} N^2 d^2 o \left( \frac{k^4}{N^4} \right) + N d^2 \left( \frac{k}{N} \right)^3 + \frac{1}{2} d N \left( \frac{k}{N} \right)^2$$

$$= o(d^{-2} N^2) + O(d^{-1} N) + O(d^{-1} N) = o(k^2),$$

since $k = (1 + o(1))c N/d$.

Let $\Omega$ be the set of all pairs $\{P_1, P_2, P_3, P_4, K\}$, where $P_1, P_2, P_3, P_4, K$ are pairwise disjoint subsets of $[n]$ with $|P_1| = 2$ and $|K| = r - 4$ (note that the collection of $P_i$ is unordered). Now

$$|\Omega| = \frac{1}{4!} \binom{8}{2,2,2,2} \binom{n}{8} \binom{n-8}{r-4}$$

$$= \frac{8!}{2^4 \cdot 4!} \binom{n}{r} r! (n-r)! (n-r-4)!$$

$$\leq \frac{d^4}{2^7 \cdot 3} N.$$ 

For each $\omega \in \Omega$, let $U(\omega) = \{K \cup P_i \cup P_j : i,j \in \{1,2\}\}$, so $|U(\omega)| = 6$. For each $H$ and each $i \in \{0,\ldots,6\}$, let $b_{i,H}$ denote the number of $\omega$ in $\Omega$ for which $|H \cap U(\omega)| = i$.

4.1.2. $\mathbf{E}(b_{5,H}) \leq \frac{ck}{d^4}$ and $\mathbf{E}(b_{6,H}) = o(k)$ while $\mathbf{Var}(b_{5,H}) = o(k^2)$.

Proof of Claim. The claim is trivial when $r < 4$ since $\Omega$ is empty, so suppose that $r \geq 4$. Given $\omega \in \Omega$, the probability that $|H \cap U(\omega)| = i$ is $\binom{6}{i} \binom{N-6}{k-1}/\binom{N}{k} \leq \binom{6}{i} \binom{k}{N} \leq \binom{6}{i} c^d d^{-i}$, so

$$\mathbf{E}(b_{i,H}) \leq |\Omega| \binom{6}{i} c^d d^{-i} = \binom{6}{i} c^d i^d \cdot 3 N \leq \binom{6}{i} c^{i-1} d^{5-i} k$$

giving $\mathbf{E}(b_{5,H}) \leq \frac{ck}{d^4}$ and $\mathbf{E}(b_{6,H}) = o(k)$ as required.

Let $\Pi = \Omega^2$, so $|\Pi| = |\Omega|^2 \leq d^8 N^2$. Let $\Pi_0 := \{ (\omega, \omega) : \omega \in \Omega \} \subseteq \Pi$, let $\Pi_2$ be the set of all $\omega_1, \omega_2 \in \Pi$ for which $U(\omega_1) \cap U(\omega_2) = \emptyset$, and let $\Pi_1 = \Pi \setminus (\Pi_0 \cup \Pi_2)$.

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Since \( U(\omega) \) contains six vertices of \( J(n, r) \) for each \( \omega \in \Omega \), symmetry and a counting argument give that for each vertex \( v \) of \( J(n, r) \), we have
\[
|\{ \omega \in \Omega : v \in U(\omega) \}| = \frac{6|\Omega|}{N} = O(d^4).
\]
It follows that \( |\Pi_1| = O(d^8 N) \). Call an \( \omega \in \Omega \) bad for \( H \) if \( |H \cap U(\omega)| = 5 \). Recall that the probability that a given \( \omega \) is bad is \( 6(N_k - 6)/N_k = (6 + o(1))k^5/N^5 \). Note that \( \omega \) is determined uniquely by any four sets in \( U(\omega) \); it follows that if \( (\omega_1, \omega_2) \in \Pi_1 \), then \( U(\omega_1) \) and \( U(\omega_2) \) have at most three sets in common, so if both \( \omega_1 \) and \( \omega_2 \) are bad, then \( H \) contains at least seven of the sets in \( U(\omega_1) \) and \( U(\omega_2) \). Since \( |U(\omega_1) \cup U(\omega_2)| \leq 10 \), a pair \( (\omega_1, \omega_2) \in \Pi_1 \) is thus bad with probability at most \((10(N_k - 7)/N_k)^2 = O(d^{-7}) \). If \( (\omega_1, \omega_2) \in \Pi_0 \), then \( \omega_1 \) and \( \omega_2 \) are both bad with probability \( (6 + o(1))k^5/N^5 = O(d^{-5}) \). If \( (\omega_1, \omega_2) \in \Pi_2 \), then both are bad with probability \( 36(N_k - 12)/N_k \). Therefore
\[
\text{Var}(b_{5, H}) = \sum_{(\omega_1, \omega_2) \in \Pi} \left[ \Pr(\omega_1, \omega_2 \text{ bad}) - \Pr(\omega_1 \text{ bad}) \Pr(\omega_2 \text{ bad}) \right]
\leq |\Pi_2| \left( \frac{(36 + o(1))k^{10}}{N^{10}} - \left( \frac{(6 + o(1))k^5}{N^5} \right)^2 \right)
+ |\Pi_0|O(d^{-5}) + |\Pi_1|O(d^{-7})
\leq d^8 N^2 o(k^{10}/N^{10}) + O(d^{-1} N) + O(d N)
= o(d^{-2} N^2) + O(dN).
\]
Now \( d^{-2} N^2 = (1 + o(1))k^2 \), and, using \( 4 \leq r \leq \frac{N}{2} \), we have \( dN = r(n - r)(N/r) = o\left( \frac{1}{r^2(n - r)^2} \right) \). Therefore, it follows that \( \text{Var}(b_{5, H}) = o(k^2) \) as required.

By the two claims, the random variables \( e_H \) and \( b_{5, H} \) have means at most \( \frac{c}{2} k \) and \( \frac{c}{64} k \), respectively, and both have standard deviations in \( o(k) \); it follows by Chebyshev’s inequality that \( \Pr(e_H > \left( \frac{c}{2} + \frac{c}{64} \right) k) = o(1) \) and \( \Pr(b_{5, H} > \left( \frac{c}{2} + \frac{c}{64} \right) k) = o(1) \). Since \( \mathbf{E}(b_{6, H}) \in o(k) \), Markov’s inequality gives \( \Pr(b_{6, H} > \frac{c}{6} k) = o(1) \). Therefore, with probability \( 1 - o(1) \), we have
\[
e_H + b_{5, H} + 2b_{6, H} \leq \left( \frac{c}{2} + \frac{c}{64} + \epsilon \right) k = (1 - f(c) + \epsilon) k = (1 - \alpha) k.
\]
Call a set \( W \subseteq \binom{[n]}{r} \) good if \( e_W = b_{5, W} = b_{6, W} = 0 \). Each set \( H \subseteq \binom{[n]}{r} \) of size \( k \) contains a good set \( W \) of size \( |H| - e_H - b_{5, H} - 2b_{6, H} \). With probability \( 1 - o(1) \) we have \( e_H + b_{5, H} + 2b_{6, H} \leq (1 - \alpha) k \) and so \( |W| \geq k - (1 - \alpha) k = \alpha k \); thus there are at least \( (1 - o(1)) \binom{N}{k} \) different choices of \( H \) that contain a good set \( W \) of size at least \( \alpha k \). On the other hand, each good set \( W \) of size at least \( \alpha k \) is contained in at most \( \binom{N}{(1 - \alpha) k} \) different \( H \); therefore the number of good sets is at least \( \nu = (1 - o(1)) \binom{N}{k}/\binom{N}{(1 - \alpha) k} \). We have
\[
\log \nu = \log \binom{N}{k} - \log \binom{N}{(1 - \alpha) k} - o(1)
\geq k \log(N/k) - k \log(eN/(1 - \alpha) k) - o(1)
= (1 - o(1)) \alpha k \log(N/k)
= (1 - o(1)) \alpha c \frac{\log d}{d} N.
\]
But for large \( n \) we have \((1 - o(1))c_2 > \gamma\), so there are at least \(2\gamma^{\frac{\log N}{N}}\) good stable sets. By Corollary 3.2, each such set is the collection of circuit-hyperplanes of an Ingleton sparse paving matroid of rank \( r \) on ground set \([n]\); the theorem follows.

We have attempted to optimize the constant \(0.486\ldots\) in the exponent as much as possible within the constraints of our techniques; the proof can be simplified to use first rather than second moments, at the expense of a lowered constant of around 0.4. One case where the constant can certainly be improved is where the rank \( r \) (or, dually, the corank \( n - r \)) is constant, in which case the estimate on \([\Omega]\) can be improved by an asymptotically significant factor of \((1 - \frac{1}{2})(1 - \frac{2}{3})(1 - \frac{3}{4})\). Carrying through this better estimate has the effect of slightly increasing the constant in the exponent further toward 0.5, giving 0.5 exactly when \( r \leq 3 \), and roughly 0.498 when \( r = 4 \).

We complement the above enumeration result, which is based on the construction of a large family of sparse paving matroids each of which contain roughly \(\frac{1}{r(n-r)}{n \choose r}\) circuit-hyperplanes, by a construction that shows that sparse paving Ingleton matroids with many more circuit-hyperplanes exist.

We sketch the construction, which is originally due to Graham and Sloane [GS80]. Suppose that \(0 < r < n\). Identifying the elements of \([n]\) with the group \(\mathbb{Z}_n\), the function \(c: \binom{[n]}{r} \to \mathbb{Z}_n\) given by \(c(X) = \sum X\) is a proper vertex coloring of \(J(n, r)\), since any two adjacent vertices \(X, X'\) satisfy \(X' = (X - \{e\}) \cup \{f\}\) for distinct \(e, f \in \mathbb{Z}_n\), which gives \(c(X) - c(X') = e - f \neq 0\) and so \(c(X) \neq c(X')\). It follows that for each \(\gamma \in \mathbb{Z}_n\), the set \(\mathcal{N} = c^{-1}(\gamma) = \{X : c(X) = \gamma\}\) is a stable set in \(J(n, r)\) and hence \(B = \binom{[n]}{r} \setminus \mathcal{N}\) is the collection of bases of a rank-\(r\) sparse paving matroid \(S(n, r, \gamma)\) with ground set \([n]\).

**Lemma 4.2.** \(S(n, r, \gamma)\) is Ingleton.

**Proof.** For the sake of contradiction, suppose that \(A, B, C, D \subseteq [n]\) violate Ingleton’s inequality and obtain \(K, P_1, P_2, P_3, P_4\) as in Corollary 3.2. Write \(P_1 = \{p_1, p_1'\}\). We may assume that \(K \cup P_1 \cup P_2\) is a circuit-hyperplane for all \(\{i, j\} \in \binom{[4]}{2} \setminus \{\{3, 4\}\}\), while \(K \cup P_3 \cup P_4\) is a basis. Define \(\gamma' = \gamma - \sum_{x \in K} x \mod n\). It follows that

\[
P_1 + p_1 + p_2 + p_2' = p_1 + p_1' + p_3 + p_3' = p_2 + p_2' + p_4 + p_4' = \gamma',
\]

so in particular

\[
p_3 + p_3' + p_4 + p_4' = \gamma',
\]

which implies that \(c(K \cup P_3 \cup P_4) = \gamma\), contradicting that \(K \cup P_3 \cup P_4\) is a basis of \(S(n, r, \gamma)\).

**Corollary 4.3.** For all \(0 < r < n\), there exists a sparse paving Ingleton matroid of rank \(r\) on ground set \([n]\) that has at least \(\frac{1}{n} \binom{n}{r}\) circuit-hyperplanes.

**Proof.** As \(\{c^{-1}(\gamma) : \gamma \in \mathbb{Z}_n\}\) partitions \(V(J(n, r))\), there is \(\gamma_0 \in \mathbb{Z}_n\) such that \(|c^{-1}(\gamma_0)| \geq \frac{1}{n} \binom{n}{r}\). Consequently, the matroid \(S(n, r, \gamma_0)\), which is Ingleton by Lemma 4.2, has at least \(\frac{1}{n} \binom{n}{r}\) circuit-hyperplanes.

**References**


