Locally correct Fréchet matchings

Locally Correct Fréchet Matchings

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Abstract

The Fréchet distance is a metric to compare two curves, which is based on monotonous matchings between these curves. We call a matching that results in the Fréchet distance a Fréchet matching. There are often many different Fréchet matchings and not all of these capture the similarity between the curves well. We propose to restrict the set of Fréchet matchings to "natural" matchings and to this end introduce *locally correct* Fréchet matchings. We prove that at least one such a matching exists for two polygonal curves and give an algorithm to compute it.

1 Introduction

Many problems ask for the comparison of two curves. Consequently, several distance measures have been proposed for the similarity of two curves P and Q, for example, the Hausdorff and the Fréchet distance. Such a distance measure simply returns a number indicating the (dis)similarity. However, the Hausdorff and the Fréchet distance are both based on matchings of the points on the curves. The distance returned is the maximum distance between any two matched points. The Fréchet distance uses monotonous matchings (and limits hereof): if point p on P and q on Q are matched, then any point on P after p must be matched to q or a point on Q after q. The Fréchet distance is the maximal distance between two matched points minimized over all monotonous matchings of the curves. We call a matching resulting in the Fréchet distance a Fréchet matching.

There are often many different Fréchet matchings for two curves. However, as the Fréchet distance is determined only by the maximal distance, not all of these matchings capture the similarity between the curves well (see Fig. 1). There are applications that directly use a matching, for example, to compute the average distance [5] or to morph between the curves [3]. In such situations it is particularly important to be able to compute a "good" matching.

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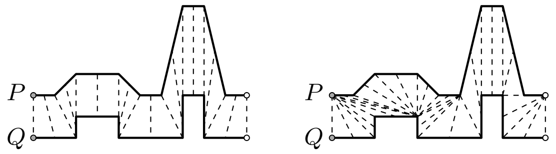

Figure 1: Two Fréchet matchings for curves P and Q.

Results. We propose to restrict the set of Fréchet matchings to "natural" matchings and to this end introduce *locally correct* Fréchet matchings: matchings that are Fréchet matchings for any two matched subcurves. We prove the following theorem.

Theorem 1 *For any two polygonal curves P and Q, there exists a locally correct Fréchet matching.*

This theorem follows directly from Lemma 3 in Section 3. The proof of this lemma results in a recursive algorithm to compute a locally correct matching.

Related work. The first algorithm to compute the Fréchet distance was given by Alt and Godau [1]. Since then, the Fréchet distance has received significant attention. Here we focus on approaches that restrict the allowed matchings. Efrat *et al.* [3] introduced Fréchet-like metrics, the geodesic width and link width, to restrict to matchings suitable for curve morphing. Their method is suitable only for non-intersecting polylines. Moreover, geodesic width and link width do not resolve the problem illustrated in Fig. 1: both matchings also have minimal geodesic width and minimal link width. Maheshwari *et al.* [4] studied a restriction by "speed limits", which may exclude all Fréchet matchings and may cause undesirable effects near "outliers" (see Fig. 2). Buchin *et al.* [2] describe a framework for restricting Fréchet matchings, which they illustrate by restricting slope and path length. The former corresponds to speed limits. We briefly discuss the latter in Section 3.

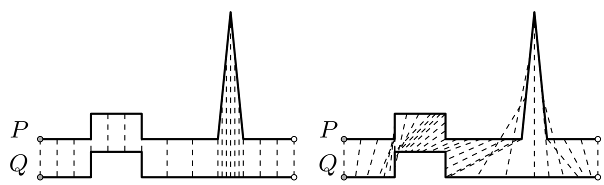

Figure 2: Two Fréchet matchings. The right one results from speed limits and is not locally correct.

2 Preliminaries

Curves. Let C be a polygonal curve with n edges, defined by vertices $c_0, \ldots, c_n$. We treat a curve as a continuous map $C : [0, n] \to \mathbb{R}^2$. In this map, $C(i)$ equals c_i for integer i. Furthermore, $C(i + \lambda)$ equals $(1-\lambda) \cdot c_i + \lambda \cdot c_{i+1}$, for integer i and $0 < \lambda < 1$. As a reparametrization $\sigma : [0, 1] \to [0, n]$ of a curve C, we allow any continuous, non-decreasing function such that $\sigma(0) = 0$ and $\sigma(1) = n$. We denote by $C_\sigma(t)$ the actual location according to reparametrization σ: $C_\sigma(t) = C(\sigma(t))$. By $C_\sigma[a, b]$ we denote the subcurve of C in between $C_\sigma(a)$ and $C_\sigma(b)$.

Fréchet matchings. We are given two polygonal curves P and Q with m and n edges. A (monotonous) *matching* μ between P and Q is a pair of reparametrizations (σ, θ), such that $P_\sigma(t)$ matches to $Q_\theta(t)$. The Euclidean distance between two matched points is denoted by $d_\mu(t) = |P_\sigma(t) - Q_\theta(t)|$. The maximum distance over a range of values is denoted by $d_\mu[a, b] = \max_{a \leq t \leq b} d_\mu(t)$. The *Fréchet distance* between two curves is defined as $\delta_{\mathrm{F}}(P, Q) = \inf_\mu d_\mu[0, 1]$. A *Fréchet matching* is a matching μ that realizes the Fréchet distance, that is, $d_\mu[0, 1] = \delta_{\mathrm{F}}(P, Q)$ holds.

Free space diagrams. Alt and Godau [1] describe an algorithm to compute the Fréchet distance based on the decision variant (that is, solving $\delta_{\mathrm{F}}(P, Q) \leq \varepsilon$ for some given ε). Their algorithm uses a *free space diagram*, a two-dimensional diagram on the range $[0, m] \times [0, n]$. Every point (x, y) in this diagram is either "free" (white) or not (indicating whether $|P(x) - Q(y)| \leq \varepsilon$). The diagram has m columns and n rows; every cell (c, r) ($1 \leq c \leq m$ and $1 \leq r \leq n$) corresponds to the edges $p_{c-1}p_c$ and $q_{r-1}q_r$. To compute the Fréchet distance, one finds the smallest ε such that there exists an x- and y-monotone path from point $(0, 0)$ to (m, n) in free space. For this, only certain *critical values* have to be checked. These values correspond to emergence of potentially new allowed paths, so-called *critical events*. The three event types are illustrated in Fig. 3. We scale every row and column in the diagram to correspond to the (relative) length of the actual edge of the curve instead of using unit squares for cells.

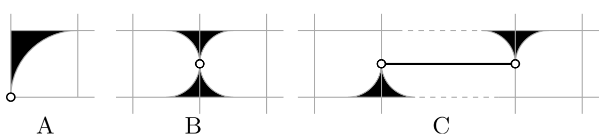

Figure 3: Three event types. (A) Endpoints come within range of each other. (B) Passage opens on cell boundary. (C) Passage opens in row (or column).

3 Locally correct matchings

We introduce *locally correct* matchings, which are a Fréchet matching for any two matched subcurves.

Definition 1 (Local correctness) *Given two polygonal curves P and Q, a matching $\mu = (\sigma, \theta)$ is* locally correct *if for all a, b with $0 \leq a \leq b \leq 1$*

$$d_\mu[a, b] = \delta_{\mathrm{F}}(P_\sigma[a, b], Q_\theta[a, b]).$$

Note that not every Fréchet matching is locally correct. The question arises whether a locally correct matching always exists and if so, how to compute it.

Existence. We prove that there always exists a locally correct matching for any two curves by induction on the number of edges in the curves. First, we present two simple observations for the two base cases.

Observation 1 (P is a point) *For two polygonal curves P and Q with $m = 0$, a locally correct matching is (σ, θ), where $\sigma(t) = 0$ and $\theta(t) = t \cdot n$.*

Observation 2 (Line segments) *For two polygonal curves P and Q with $m = n = 1$, a locally correct matching is (σ, θ), where $\sigma(t) = \theta(t) = t$.*

For induction, we split the two curves based on events (see Fig. 4). Since each split must reduce the problem size, we ignore any events on the left or bottom boundary of cell $(1, 1)$ or on the right or top boundary of cell (m, n). This excludes both events of type A. A free space diagram is *connected* at value ε, if a monotonous path exists from the boundary of cell $(1, 1)$ to the boundary of cell (m, n). A *realizing event* is a critical event at the minimal value ε such that the corresponding free space diagram is connected.

Let $\mathcal{E}$ denote the set of concurrent realizing events for two curves. A *realizing set* E_{r} is a subset of $\mathcal{E}$ such that the free space admits a monotonous path from cell $(1, 1)$ to cell (m, n) without using an event

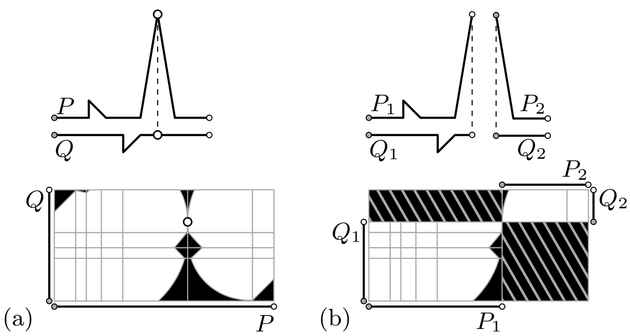

Figure 4: (a) Curves with the free space diagram for $\varepsilon = \delta_{\mathrm{F}}(P, Q)$ and the realizing event. (b) The event splits each curve into two subcurves. The hatched areas indicate parts that disappear after the split.

in $\mathcal{E}\backslash E_\mathrm{r}$. Note that a realizing set cannot be empty. When $\mathcal{E}$ contains more than one realizing event, some may be "insignificant": they are never required to actually make a path in the free space diagram. A realizing set is *minimal* if it does not contain a strict subset that is a realizing set. Such a minimal realizing set contains only "significant" events. We omit the (straightforward) proof of the following lemma.

Lemma 2 *For two polygonal curves P and Q with $m > 1$ and $n \geq 1$, there exists a minimal realizing set.*

In the remainder, we use realizing set to indicate a minimal realizing set, unless indicated otherwise. To prove the existence of a locally correct Fréchet matching, we prove the following, stronger lemma. Informally, it states that curves have a locally correct matching that is "closer" (except in cell $(1, 1)$ or (m, n)) than the distance of their realizing set. In addition, this matching is linear inside every cell.

Lemma 3 *If the free space diagram of two polygonal curves P and Q is connected at value ε, then there exists a locally correct Fréchet matching $\mu = (\sigma, \theta)$ such that $d_\mu(t) \leq \varepsilon$ for all t with $\sigma(t) \geq 1$ or $\theta(t) \geq 1$, and $\sigma(t) \leq m - 1$ or $\theta(t) \leq n - 1$. Furthermore, μ is linear in every cell.*

Proof. We prove this by induction on $m + n$. The base cases ($m = 0$, $n = 0$, and $m = n = 1$) follow from Observation 1 and Observation 2.

For the induction, we assume that $m \geq 1$, $n \geq 1$, and $m + n > 2$. By Lemma 2, there exists a realizing set E_r for P and Q, say at value ε_r. The set contains realizing events $e_1, \ldots, e_k$ ($k \geq 1$), numbered in lexicographic order. By definition, $\varepsilon_\mathrm{r} \leq \varepsilon$ must hold. Suppose that E_r splits curve P into $P_1, \ldots, P_{k+1}$ and curve Q into $Q_1, \ldots, Q_{k+1}$. By definition of a realizing event, none of the events in E_r occur on the right or top boundary of cell (m, n). Therefore, the following holds for any i ($1 \leq i \leq k + 1$): $m_i \leq m$, $n_i \leq n$, and $m_i < m$ or $n_i < n$. Since there is a path in the free space diagram at ε_r through all events in E_r, the induction hypothesis implies that, for any i ($1 \leq i \leq k + 1$), there exists a locally correct matching $\mu_i = (\sigma_i, \theta_i)$ for P_i and Q_i such that μ_i is linear in every cell and $d_{\mu_i}(t) \leq \varepsilon_\mathrm{r}$ for all t with $\sigma_i(t) \geq 1$ or $\theta_i(t) \geq 1$, and $\sigma_i(t) \leq m_i - 1$ or $\theta_i(t) \leq n_i - 1$. Combining these matchings with the events in E_r yields a matching $\mu = (\sigma, \theta)$ for (P, Q). As we argue below, this matching is locally correct and satisfies the additional properties. The matching of an event corresponds to a single point (type B) or a horizontal or vertical line (type C). By induction, μ_i is linear in every cell. Since all events occur on cell boundaries, the cells of the matchings and events are disjoint. Therefore, the matching μ is also linear inside every cell.

For $i < k + 1$, d_{μ_i} is at most ε_r at the point where μ_i enters cell (m_i, n_i) in the free space diagram of P_i and Q_i. We also know that d_{μ_i} equals ε_r at the top right corner of cell (m_i, n_i). Since μ_i is linear inside the cell, $d_{\mu_i}(t) \leq \varepsilon_\mathrm{r}$ also holds for t with $\sigma_i(t) > m_i - 1$ and $\theta_i(t) > n_i - 1$. Analogously, for $i > 0$, $d_{\mu_i}(t)$ is at most ε_r for t with $\sigma_i(t) < 1$ and $\theta_i(t) < 1$. Hence, $d_\mu(t) \leq \varepsilon_\mathrm{r} \leq \varepsilon$ holds for t with $\sigma(t) \geq 1$ or $\theta(t) \geq 1$, and $\sigma(t) \leq m - 1$ or $\theta(t) \leq n - 1$.

To show that μ is locally correct, suppose for contradiction that we have values for a, b such that $\delta_\mathrm{F}(P_\sigma[a, b], Q_\theta[a, b]) < d_\mu[a, b]$. If a, b are in between two consecutive events, then we know that the submatching corresponds to one of the matchings μ_i. However, we know that these are locally correct and thus $\delta_\mathrm{F}(P_\sigma[a, b], Q_\theta[a, b]) = d_\mu[a, b]$ holds.

Hence, assume that there is at least one event of E_r in between a and b. There are two possibilities: either $d_\mu[a, b] = \varepsilon_\mathrm{r}$ or $d_\mu[a, b] > \varepsilon_\mathrm{r}$. $d_\mu[a, b] < \varepsilon_\mathrm{r}$ cannot hold, since $d_\mu[a, b]$ includes a realizing event. First, assume $d_\mu[a, b] = \varepsilon_\mathrm{r}$ holds. If $\delta_\mathrm{F}(P_\sigma[a, b], Q_\theta[a, b]) < \varepsilon_\mathrm{r}$ holds, then there is a matching that does not use the events in between a and b and has a lower maximum. Therefore, the free space connects point $(\sigma(a), \theta(a))$ with point $(\sigma(b), \theta(b))$ at a lower value than ε_r. This implies that all events between a and b can be omitted, contradicting that E_r is a minimal realizing set.

Now, assume $d_\mu[a, b] > \varepsilon_\mathrm{r}$. Let t' denote the highest t for which $\sigma(t) \leq 1$ and $\theta(t) \leq 1$ holds, that is, the point at which the matching exits cell $(1, 1)$. Similarly, let t'' denote the lowest t for which $\sigma(t) \geq m - 1$ and $\theta(t) \geq n - 1$ holds. We know that $d_\mu(t) \leq \varepsilon_\mathrm{r}$ holds for any $t' \leq t \leq t''$. Hence, $d_\mu(t) > \varepsilon_\mathrm{r}$ can only hold for $t < t'$ or $t > t''$. Suppose that $d_\mu(a) > \varepsilon_\mathrm{r}$ holds. Thus, we have that $a < t'$ and μ is linear between a and t'. Therefore, $d_\mu(a) > d_\mu(t)$ holds for any t with $a < t < t'$. Analogously, if $d_\mu(b) > \varepsilon_\mathrm{r}$ holds, then $d_\mu(b) > d_\mu(t)$ holds for any t with $t'' < t < b$. Hence, we conclude that $d_\mu[a, b] = \max\{d_\mu(a), d_\mu(b)\}$. Since this gives a lower bound on the Fréchet distance, we conclude that the matching μ is locally correct. □

Further restrictions. We considered restricting the matchings to the "shortest" locally correct matching, where "shortest" refers to the length of the path in the free space diagram. However, Fig. 5 shows that such a restriction does not necessarily improve the

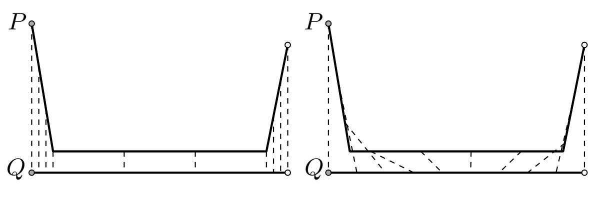

Figure 5: Two locally correct Fréchet matchings for curves P and Q. The right matching is the shortest.

quality of the matching. Another potential constraint we are currently investigating is "local optimality". Intuitively, a matching is locally optimal if no small change decreases the matched distance locally.

4 Algorithm

The existence proof results in a recursive algorithm with execution time $O((m+n) \cdot m \cdot n \cdot \log(m \cdot n))$. Fig. 1 (left), Fig. 2 (left), Fig. 5 (left), and Fig. 6 illustrate matchings computed with our algorithm.

Algorithm 1 `FindLocallyCorrectMatching`(P, Q)

Require: P and Q are curves with m and n edges
Ensure: A locally correct matching for P and Q

1: **if** $m = 0$ **or** $n = 0$ **then**
2:     **return** (σ, θ) where $\sigma(t) = t \cdot m$, $\theta(t) = t \cdot n$
3: **else if** $m = n = 1$ **then**
4:     **return** (σ, θ) where $\sigma(t) = \theta(t) = t$
5: **else**
6:     Find event e_{r} of a minimal realizing set
7:     Split P into P_1 and P_2 according to e_{r}
8:     Split Q into Q_1 and Q_2 according to e_{r}
9:     $\mu_1 \rightarrow$ `FindMatching`(P_1, Q_1)
10:     $\mu_2 \rightarrow$ `FindMatching`(P_2, Q_2)
11:     **return** concatenation of μ_1, e_{r}, and μ_2

Using the notation of Alt and Godau [1], $L^F_{i,j}$ denotes the interval of free space on the left boundary of cell (i,j); $L^R_{i,j}$ denotes the subset of $L^F_{i,j}$ that is reachable from point $(0,0)$ of the free space diagram with a monotonous path in the free space. Analogously, $B^F_{i,j}$ and $B^R_{i,j}$ are defined for the bottom boundary.

With a slight modification to the decision algorithm, we can compute the minimal value of ε such that a path is available from cell $(1,1)$ to cell (m,n). This requires only two changes: $B^R_{1,2}$ should be initialized with $B^F_{1,2}$ and $L^R_{2,1}$ with $L^F_{2,1}$; the answer should be "yes" if and only if $B^R_{m,n}$ or $L^R_{m,n}$ is non-empty.

Realizing set. By computing the Fréchet distance using the modified Alt and Godau algorithm, we obtain an ordered, potentially non-minimal realizing set $\mathcal{E} = \{e_1, \ldots, e_l\}$. Let E_k denote the first k events of $\mathcal{E}$. The algorithm must find an event that is contained in a realizing set. We use a binary search on $\mathcal{E}$ to find the r such that E_r contains a realizing set, but E_{r-1} does not. This implies that event e_r must be contained in a realizing set. We use e_r to split the curves. Note that r is unique due to monotonicity.

For correctness, the order of events in $\mathcal{E}$ must be consistent in different iterations, for example, by using a lexicographic order. Set E_r contains only realizing sets that use e_r. Therefore, E_{r-1} contains a realizing set to connect cell $(1,1)$ to e_r and e_r to cell (m,n). Thus any event found in subsequent iterations is part of E_{r-1} and is part of a realizing set with e_r.

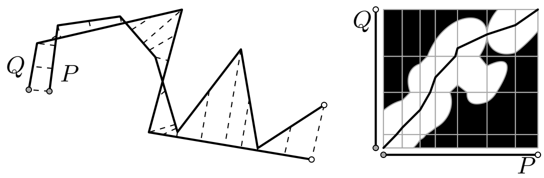

Figure 6: Locally correct matching produced by the algorithm. Free space diagram drawn at $\varepsilon = \delta_{\mathrm{F}}(P, Q)$.

To determine whether some E_k contains a realizing set, we check whether cells $(1,1)$ and (m,n) are connected without "using" the events of $\mathcal{E} \backslash E_k$. To do this efficiently, we further modify the Alt and Godau algorithm. We require only a method to prevent events in $\mathcal{E} \backslash E_k$ from being used. After $L^R_{i,j}$ is computed, we check whether the event e (if any) that ends at the left boundary of cell (i,j) is part of $\mathcal{E} \backslash E_k$ and necessary to obtain $L^R_{i,j}$. If this is the case, we replace $L^R_{i,j}$ with an empty interval. Event e is necessary if and only if $L^R_{i,j}$ is a singleton. To obtain an algorithm that is numerically more stable, we introduce entry points. The *entry point* of the left boundary of cell (i,j) is the maximal $i' < i$ such that $B^R_{i',j}$ is non-empty. These values are easily computed during the decision algorithm. Assume e starts on the left boundary of cell (i_{s}, j). Event e is necessary if and only if $i' < i_{\mathrm{s}}$. Therefore, we use the entry point instead of checking whether $L^R_{i,j}$ is a singleton. This process is analogous for horizontal boundaries of cells.

We silently assumed that each event in $\mathcal{E}$ ends at a different cell boundary. If multiple events end at the same cell boundary, then these events occur in the same row (or column) and it suffices to consider only the event that starts at the rightmost column (or highest row). This justifies the assumption and ensures that $\mathcal{E}$ contains $O(m \cdot n)$ events. Hence, computing e_{r} (line 6) takes $O(m \cdot n \cdot \log(m \cdot n))$ time.

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