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# Locally Correct Fréchet Matchings

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## Abstract

The Fréchet distance is a metric to compare two curves, which is based on monotonous matchings between these curves. We call a matching that results in the Fréchet distance a Fréchet matching. There are often many different Fréchet matchings and not all of these capture the similarity between the curves well. We propose to restrict the set of Fréchet matchings to “natural” matchings and to this end introduce *locally correct* Fréchet matchings. We prove that at least one such a matching exists for two polygonal curves and give an algorithm to compute it.

## 1 Introduction

Many problems ask for the comparison of two curves. Consequently, several distance measures have been proposed for the similarity of two curves  $P$  and  $Q$ , for example, the Hausdorff and the Fréchet distance. Such a distance measure simply returns a number indicating the (dis)similarity. However, the Hausdorff and the Fréchet distance are both based on matchings of the points on the curves. The distance returned is the maximum distance between any two matched points. The Fréchet distance uses monotonous matchings (and limits hereof): if point  $p$  on  $P$  and  $q$  on  $Q$  are matched, then any point on  $P$  after  $p$  must be matched to  $q$  or a point on  $Q$  after  $q$ . The Fréchet distance is the maximal distance between two matched points minimized over all monotonous matchings of the curves. We call a matching resulting in the Fréchet distance a Fréchet matching.

There are often many different Fréchet matchings for two curves. However, as the Fréchet distance is determined only by the maximal distance, not all of these matchings capture the similarity between the curves well (see Fig. 1). There are applications that directly use a matching, for example, to compute the average distance [5] or to morph between the curves [3]. In such situations it is particularly important to be able to compute a “good” matching.

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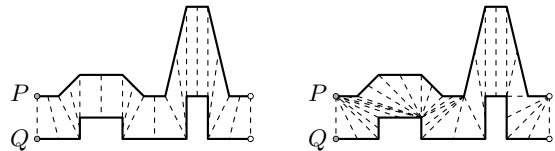


Figure 1: Two Fréchet matchings for curves  $P$  and  $Q$ .

**Results.** We propose to restrict the set of Fréchet matchings to “natural” matchings and to this end introduce *locally correct* Fréchet matchings: matchings that are Fréchet matchings for any two matched sub-curves. We prove the following theorem.

**Theorem 1** *For any two polygonal curves  $P$  and  $Q$ , there exists a locally correct Fréchet matching.*

This theorem follows directly from Lemma 3 in Section 3. The proof of this lemma results in a recursive algorithm to compute a locally correct matching.

**Related work.** The first algorithm to compute the Fréchet distance was given by Alt and Godau [1]. Since then, the Fréchet distance has received significant attention. Here we focus on approaches that restrict the allowed matchings. Efrat *et al.* [3] introduced Fréchet-like metrics, the geodesic width and link width, to restrict to matchings suitable for curve morphing. Their method is suitable only for non-intersecting polylines. Moreover, geodesic width and link width do not resolve the problem illustrated in Fig. 1: both matchings also have minimal geodesic width and minimal link width. Maheshwari *et al.* [4] studied a restriction by “speed limits”, which may exclude all Fréchet matchings and may cause undesirable effects near “outliers” (see Fig. 2). Buchin *et al.* [2] describe a framework for restricting Fréchet matchings, which they illustrate by restricting slope and path length. The former corresponds to speed limits. We briefly discuss the latter in Section 3.

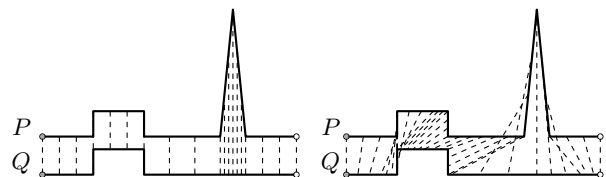


Figure 2: Two Fréchet matchings. The right one results from speed limits and is not locally correct.

## 2 Preliminaries

**Curves.** Let  $C$  be a polygonal curve with  $n$  edges, defined by vertices  $c_0, \dots, c_n$ . We treat a curve as a continuous map  $C : [0, n] \rightarrow \mathbb{R}^2$ . In this map,  $C(i)$  equals  $c_i$  for integer  $i$ . Furthermore,  $C(i + \lambda)$  equals  $(1 - \lambda) \cdot c_i + \lambda \cdot c_{i+1}$ , for integer  $i$  and  $0 < \lambda < 1$ . As a reparametrization  $\sigma : [0, 1] \rightarrow [0, n]$  of a curve  $C$ , we allow any continuous, non-decreasing function such that  $\sigma(0) = 0$  and  $\sigma(1) = n$ . We denote by  $C_\sigma(t)$  the actual location according to reparametrization  $\sigma$ :  $C_\sigma(t) = C(\sigma(t))$ . By  $C_\sigma[a, b]$  we denote the subcurve of  $C$  in between  $C_\sigma(a)$  and  $C_\sigma(b)$ .

**Fréchet matchings.** We are given two polygonal curves  $P$  and  $Q$  with  $m$  and  $n$  edges. A (monotonous) *matching*  $\mu$  between  $P$  and  $Q$  is a pair of reparametrizations  $(\sigma, \theta)$ , such that  $P_\sigma(t)$  matches to  $Q_\theta(t)$ . The Euclidean distance between two matched points is denoted by  $d_\mu(t) = |P_\sigma(t) - Q_\theta(t)|$ . The maximum distance over a range of values is denoted by  $d_\mu[a, b] = \max_{a \leq t \leq b} d_\mu(t)$ . The *Fréchet distance* between two curves is defined as  $\delta_F(P, Q) = \inf_\mu d_\mu[0, 1]$ . A *Fréchet matching* is a matching  $\mu$  that realizes the Fréchet distance, that is,  $d_\mu[0, 1] = \delta_F(P, Q)$  holds.

**Free space diagrams.** Alt and Godau [1] describe an algorithm to compute the Fréchet distance based on the decision variant (that is, solving  $\delta_F(P, Q) \leq \varepsilon$  for some given  $\varepsilon$ ). Their algorithm uses a *free space diagram*, a two-dimensional diagram on the range  $[0, m] \times [0, n]$ . Every point  $(x, y)$  in this diagram is either “free” (white) or not (indicating whether  $|P(x) - Q(y)| \leq \varepsilon$ ). The diagram has  $m$  columns and  $n$  rows; every cell  $(c, r)$  ( $1 \leq c \leq m$  and  $1 \leq r \leq n$ ) corresponds to the edges  $p_{c-1}p_c$  and  $q_{r-1}q_r$ . To compute the Fréchet distance, one finds the smallest  $\varepsilon$  such that there exists an x- and y-monotone path from point  $(0, 0)$  to  $(m, n)$  in free space. For this, only certain *critical values* have to be checked. These values correspond to emergence of potentially new allowed paths, so-called *critical events*. The three event types are illustrated in Fig. 3. We scale every row and column in the diagram to correspond to the (relative) length of the actual edge of the curve instead of using unit squares for cells.

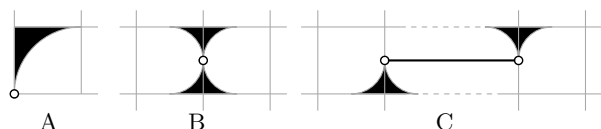


Figure 3: Three event types. (A) Endpoints come within range of each other. (B) Passage opens on cell boundary. (C) Passage opens in row (or column).

## 3 Locally correct matchings

We introduce *locally correct* matchings, which are a Fréchet matching for any two matched subcurves.

**Definition 1 (Local correctness)** Given two polygonal curves  $P$  and  $Q$ , a matching  $\mu = (\sigma, \theta)$  is locally correct if for all  $a, b$  with  $0 \leq a \leq b \leq 1$

$$d_\mu[a, b] = \delta_F(P_\sigma[a, b], Q_\theta[a, b]).$$

Note that not every Fréchet matching is locally correct. The question arises whether a locally correct matching always exists and if so, how to compute it.

**Existence.** We prove that there always exists a locally correct matching for any two curves by induction on the number of edges in the curves. First, we present two simple observations for the two base cases.

**Observation 1 (P is a point)** For two polygonal curves  $P$  and  $Q$  with  $m = 0$ , a locally correct matching is  $(\sigma, \theta)$ , where  $\sigma(t) = 0$  and  $\theta(t) = t \cdot n$ .

**Observation 2 (Line segments)** For two polygonal curves  $P$  and  $Q$  with  $m = n = 1$ , a locally correct matching is  $(\sigma, \theta)$ , where  $\sigma(t) = \theta(t) = t$ .

For induction, we split the two curves based on events (see Fig. 4). Since each split must reduce the problem size, we ignore any events on the left or bottom boundary of cell  $(1, 1)$  or on the right or top boundary of cell  $(m, n)$ . This excludes both events of type A. A free space diagram is *connected* at value  $\varepsilon$ , if a monotonous path exists from the boundary of cell  $(1, 1)$  to the boundary of cell  $(m, n)$ . A *realizing event* is a critical event at the minimal value  $\varepsilon$  such that the corresponding free space diagram is connected.

Let  $\mathcal{E}$  denote the set of concurrent realizing events for two curves. A *realizing set*  $E_r$  is a subset of  $\mathcal{E}$  such that the free space admits a monotonous path from cell  $(1, 1)$  to cell  $(m, n)$  without using an event

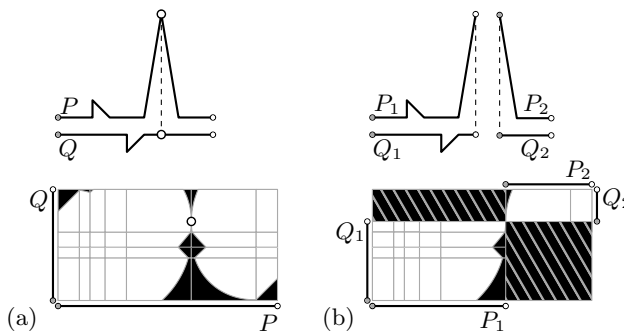


Figure 4: (a) Curves with the free space diagram for  $\varepsilon = \delta_F(P, Q)$  and the realizing event. (b) The event splits each curve into two subcurves. The hatched areas indicate parts that disappear after the split.

in  $\mathcal{E} \setminus E_r$ . Note that a realizing set cannot be empty. When  $\mathcal{E}$  contains more than one realizing event, some may be “insignificant”: they are never required to actually make a path in the free space diagram. A realizing set is *minimal* if it does not contain a strict subset that is a realizing set. Such a minimal realizing set contains only “significant” events. We omit the (straightforward) proof of the following lemma.

**Lemma 2** *For two polygonal curves  $P$  and  $Q$  with  $m > 1$  and  $n \geq 1$ , there exists a minimal realizing set.*

In the remainder, we use realizing set to indicate a minimal realizing set, unless indicated otherwise. To prove the existence of a locally correct Fréchet matching, we prove the following, stronger lemma. Informally, it states that curves have a locally correct matching that is “closer” (except in cell  $(1, 1)$  or  $(m, n)$ ) than the distance of their realizing set. In addition, this matching is linear inside every cell.

**Lemma 3** *If the free space diagram of two polygonal curves  $P$  and  $Q$  is connected at value  $\varepsilon$ , then there exists a locally correct Fréchet matching  $\mu = (\sigma, \theta)$  such that  $d_\mu(t) \leq \varepsilon$  for all  $t$  with  $\sigma(t) \geq 1$  or  $\theta(t) \geq 1$ , and  $\sigma(t) \leq m - 1$  or  $\theta(t) \leq n - 1$ . Furthermore,  $\mu$  is linear in every cell.*

**Proof.** We prove this by induction on  $m + n$ . The base cases ( $m = 0$ ,  $n = 0$ , and  $m = n = 1$ ) follow from Observation 1 and Observation 2.

For the induction, we assume that  $m \geq 1$ ,  $n \geq 1$ , and  $m + n > 2$ . By Lemma 2, there exists a realizing set  $E_r$  for  $P$  and  $Q$ , say at value  $\varepsilon_r$ . The set contains realizing events  $e_1, \dots, e_k$  ( $k \geq 1$ ), numbered in lexicographic order. By definition,  $\varepsilon_r \leq \varepsilon$  must hold. Suppose that  $E_r$  splits curve  $P$  into  $P_1, \dots, P_{k+1}$  and curve  $Q$  into  $Q_1, \dots, Q_{k+1}$ . By definition of a realizing event, none of the events in  $E_r$  occur on the right or top boundary of cell  $(m, n)$ . Therefore, the following holds for any  $i$  ( $1 \leq i \leq k + 1$ ):  $m_i \leq m$ ,  $n_i \leq n$ , and  $m_i < m$  or  $n_i < n$ . Since there is a path in the free space diagram at  $\varepsilon_r$  through all events in  $E_r$ , the induction hypothesis implies that, for any  $i$  ( $1 \leq i \leq k + 1$ ), there exists a locally correct matching  $\mu_i = (\sigma_i, \theta_i)$  for  $P_i$  and  $Q_i$  such that  $\mu_i$  is linear in every cell and  $d_{\mu_i}(t) \leq \varepsilon_r$  for all  $t$  with  $\sigma_i(t) \geq 1$  or  $\theta_i(t) \geq 1$ , and  $\sigma_i(t) \leq m_i - 1$  or  $\theta_i(t) \leq n_i - 1$ . Combining these matchings with the events in  $E_r$  yields a matching  $\mu = (\sigma, \theta)$  for  $(P, Q)$ . As we argue below, this matching is locally correct and satisfies the additional properties. The matching of an event corresponds to a single point (type B) or a horizontal or vertical line (type C). By induction,  $\mu_i$  is linear in every cell. Since all events occur on cell boundaries, the cells of the matchings and events are disjoint. Therefore, the matching  $\mu$  is also linear inside every cell.

For  $i < k + 1$ ,  $d_{\mu_i}$  is at most  $\varepsilon_r$  at the point where  $\mu_i$  enters cell  $(m_i, n_i)$  in the free space diagram of  $P_i$  and  $Q_i$ . We also know that  $d_{\mu_i}$  equals  $\varepsilon_r$  at the top right corner of cell  $(m_i, n_i)$ . Since  $\mu_i$  is linear inside the cell,  $d_{\mu_i}(t) \leq \varepsilon_r$  also holds for  $t$  with  $\sigma_i(t) > m_i - 1$  and  $\theta_i(t) > n_i - 1$ . Analogously, for  $i > 0$ ,  $d_{\mu_i}(t)$  is at most  $\varepsilon_r$  for  $t$  with  $\sigma_i(t) < 1$  and  $\theta_i(t) < 1$ . Hence,  $d_\mu(t) \leq \varepsilon_r \leq \varepsilon$  holds for  $t$  with  $\sigma(t) \geq 1$  or  $\theta(t) \geq 1$ , and  $\sigma(t) \leq m - 1$  or  $\theta(t) \leq n - 1$ .

To show that  $\mu$  is locally correct, suppose for contradiction that we have values for  $a, b$  such that  $\delta_F(P_\sigma[a, b], Q_\theta[a, b]) < d_\mu[a, b]$ . If  $a, b$  are in between two consecutive events, then we know that the submatching corresponds to one of the matchings  $\mu_i$ . However, we know that these are locally correct and thus  $\delta_F(P_\sigma[a, b], Q_\theta[a, b]) = d_\mu[a, b]$  holds.

Hence, assume that there is at least one event of  $E_r$  in between  $a$  and  $b$ . There are two possibilities: either  $d_\mu[a, b] = \varepsilon_r$  or  $d_\mu[a, b] > \varepsilon_r$ .  $d_\mu[a, b] < \varepsilon_r$  cannot hold, since  $d_\mu[a, b]$  includes a realizing event. First, assume  $d_\mu[a, b] = \varepsilon_r$  holds. If  $\delta_F(P_\sigma[a, b], Q_\theta[a, b]) < \varepsilon_r$  holds, then there is a matching that does not use the events in between  $a$  and  $b$  and has a lower maximum. Therefore, the free space connects point  $(\sigma(a), \theta(a))$  with point  $(\sigma(b), \theta(b))$  at a lower value than  $\varepsilon_r$ . This implies that all events between  $a$  and  $b$  can be omitted, contradicting that  $E_r$  is a minimal realizing set.

Now, assume  $d_\mu[a, b] > \varepsilon_r$ . Let  $t'$  denote the highest  $t$  for which  $\sigma(t) \leq 1$  and  $\theta(t) \leq 1$  holds, that is, the point at which the matching exits cell  $(1, 1)$ . Similarly, let  $t''$  denote the lowest  $t$  for which  $\sigma(t) \geq m - 1$  and  $\theta(t) \geq n - 1$  holds. We know that  $d_\mu(t) \leq \varepsilon_r$  holds for any  $t' \leq t \leq t''$ . Hence,  $d_\mu(t) > \varepsilon_r$  can only hold for  $t < t'$  or  $t > t''$ . Suppose that  $d_\mu(a) > \varepsilon_r$  holds. Thus, we have that  $a < t'$  and  $\mu$  is linear between  $a$  and  $t'$ . Therefore,  $d_\mu(a) > d_\mu(t)$  holds for any  $t$  with  $a < t < t'$ . Analogously, if  $d_\mu(b) > \varepsilon_r$  holds, then  $d_\mu(b) > d_\mu(t)$  holds for any  $t$  with  $t'' < t < b$ . Hence, we conclude that  $d_\mu[a, b] = \max\{d_\mu(a), d_\mu(b)\}$ . Since this gives a lower bound on the Fréchet distance, we conclude that the matching  $\mu$  is locally correct.  $\square$

**Further restrictions.** We considered restricting the matchings to the “shortest” locally correct matching, where “shortest” refers to the length of the path in the free space diagram. However, Fig. 5 shows that such a restriction does not necessarily improve the

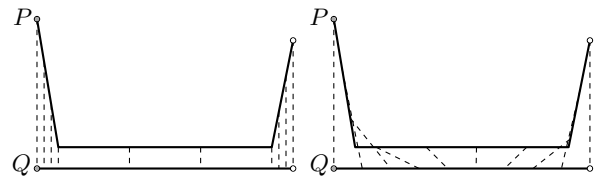


Figure 5: Two locally correct Fréchet matchings for curves  $P$  and  $Q$ . The right matching is the shortest.

quality of the matching. Another potential constraint we are currently investigating is “local optimality”. Intuitively, a matching is locally optimal if no small change decreases the matched distance locally.

#### 4 Algorithm

The existence proof results in a recursive algorithm with execution time  $O((m+n) \cdot m \cdot n \cdot \log(m \cdot n))$ . Fig. 1 (left), Fig. 2 (left), Fig. 5 (left), and Fig. 6 illustrate matchings computed with our algorithm.

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##### Algorithm 1 FindLocallyCorrectMatching( $P, Q$ )

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**Require:**  $P$  and  $Q$  are curves with  $m$  and  $n$  edges

**Ensure:** A locally correct matching for  $P$  and  $Q$

- 1: **if**  $m = 0$  **or**  $n = 0$  **then**
  - 2:     **return**  $(\sigma, \theta)$  where  $\sigma(t) = t \cdot m$ ,  $\theta(t) = t \cdot n$
  - 3: **else if**  $m = n = 1$  **then**
  - 4:     **return**  $(\sigma, \theta)$  where  $\sigma(t) = \theta(t) = t$
  - 5: **else**
  - 6:     Find event  $e_r$  of a minimal realizing set
  - 7:     Split  $P$  into  $P_1$  and  $P_2$  according to  $e_r$
  - 8:     Split  $Q$  into  $Q_1$  and  $Q_2$  according to  $e_r$
  - 9:      $\mu_1 \rightarrow \text{FindMatching}(P_1, Q_1)$
  - 10:     $\mu_2 \rightarrow \text{FindMatching}(P_2, Q_2)$
  - 11:    **return** concatenation of  $\mu_1$ ,  $e_r$ , and  $\mu_2$
- 

Using the notation of Alt and Godau [1],  $L_{i,j}^F$  denotes the interval of free space on the left boundary of cell  $(i, j)$ ;  $L_{i,j}^R$  denotes the subset of  $L_{i,j}^F$  that is reachable from point  $(0, 0)$  of the free space diagram with a monotonous path in the free space. Analogously,  $B_{i,j}^F$  and  $B_{i,j}^R$  are defined for the bottom boundary.

With a slight modification to the decision algorithm, we can compute the minimal value of  $\varepsilon$  such that a path is available from cell  $(1, 1)$  to cell  $(m, n)$ . This requires only two changes:  $B_{1,2}^R$  should be initialized with  $B_{1,2}^F$  and  $L_{2,1}^R$  with  $L_{2,1}^F$ ; the answer should be “yes” if and only if  $B_{m,n}^R$  or  $L_{m,n}^R$  is non-empty.

**Realizing set.** By computing the Fréchet distance using the modified Alt and Godau algorithm, we obtain an ordered, potentially non-minimal realizing set  $\mathcal{E} = \{e_1, \dots, e_l\}$ . Let  $E_k$  denote the first  $k$  events of  $\mathcal{E}$ . The algorithm must find an event that is contained in a realizing set. We use a binary search on  $\mathcal{E}$  to find the  $r$  such that  $E_r$  contains a realizing set, but  $E_{r-1}$  does not. This implies that event  $e_r$  must be contained in a realizing set. We use  $e_r$  to split the curves. Note that  $r$  is unique due to monotonicity.

For correctness, the order of events in  $\mathcal{E}$  must be consistent in different iterations, for example, by using a lexicographic order. Set  $E_r$  contains only realizing sets that use  $e_r$ . Therefore,  $E_{r-1}$  contains a realizing set to connect cell  $(1, 1)$  to  $e_r$  and  $e_r$  to cell  $(m, n)$ . Thus any event found in subsequent iterations is part of  $E_{r-1}$  and is part of a realizing set with  $e_r$ .

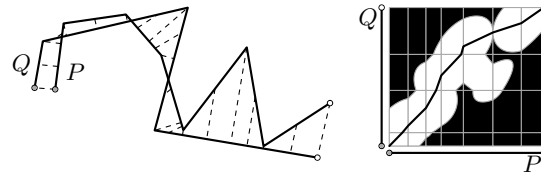


Figure 6: Locally correct matching produced by the algorithm. Free space diagram drawn at  $\varepsilon = \delta_F(P, Q)$ .

To determine whether some  $E_k$  contains a realizing set, we check whether cells  $(1, 1)$  and  $(m, n)$  are connected without “using” the events of  $\mathcal{E} \setminus E_k$ . To do this efficiently, we further modify the Alt and Godau algorithm. We require only a method to prevent events in  $\mathcal{E} \setminus E_k$  from being used. After  $L_{i,j}^R$  is computed, we check whether the event  $e$  (if any) that ends at the left boundary of cell  $(i, j)$  is part of  $\mathcal{E} \setminus E_k$  and necessary to obtain  $L_{i,j}^R$ . If this is the case, we replace  $L_{i,j}^R$  with an empty interval. Event  $e$  is necessary if and only if  $L_{i,j}^R$  is a singleton. To obtain an algorithm that is numerically more stable, we introduce entry points. The *entry point* of the left boundary of cell  $(i, j)$  is the maximal  $i' < i$  such that  $B_{i',j}^R$  is non-empty. These values are easily computed during the decision algorithm. Assume  $e$  starts on the left boundary of cell  $(i_s, j)$ . Event  $e$  is necessary if and only if  $i' < i_s$ . Therefore, we use the entry point instead of checking whether  $L_{i,j}^R$  is a singleton. This process is analogous for horizontal boundaries of cells.

We silently assumed that each event in  $\mathcal{E}$  ends at a different cell boundary. If multiple events end at the same cell boundary, then these events occur in the same row (or column) and it suffices to consider only the event that starts at the rightmost column (or highest row). This justifies the assumption and ensures that  $\mathcal{E}$  contains  $O(m \cdot n)$  events. Hence, computing  $e_r$  (line 6) takes  $O(m \cdot n \cdot \log(m \cdot n))$  time.

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