On the relationship between k-planar and k-quasi planar graphs

On the Relationship between k-Planar and k-Quasi Planar Graphs*

Patrizio Angelini† Michael A. Bekos† Franz J. Brandenburg‡ Giordano Da Lozzo§
Giuseppe Di Battista¶ Walter Didimo‖ Giuseppe Liotta‖ Fabrizio Montecchiani‖
Ignaz Rutter**

Abstract

A graph is k-planar ($k \geq 1$) if it can be drawn (in the plane) such that no edge has more than k crossings. A graph is k-quasi planar ($k \geq 2$) if it can be drawn without k pairwise crossing edges. We prove that, for $k \geq 3$, every k-planar graph is $(k+1)$-quasi planar.

1 Introduction

An emerging research area, informally recognized as *beyond planarity* (see e.g. [6, 8]), concentrates on different models of graph planarity relaxation, which allow edge crossings but forbid specific configurations. Forbidden crossing configurations can be, for example, a single edge that is crossed too many times [9], a group of mutually crossing edges [5, 10], a group of adjacent edges crossed by another edge [4], or an edge that crosses two independent edges [2, 3, 7].

Different models give rise to different families of "beyond planar" graphs. Two of the most popular families introduced in this context are the *k-planar* graphs and the *k-quasi planar* graphs, which are usually defined in terms of *topological graphs*, i.e., graphs with a geometric representation in the plane with vertices as points and edges as Jordan arcs connecting their endpoints. A topological graph is k-planar ($k \geq 1$) if no edge is crossed more than k times, while it is k-quasi planar ($k \geq 2$) if it can be drawn in the plane without k pairwise crossing edges.

A graph is *k-planar* (*k-quasi planar*) if it is isomorphic to some k-planar (k-quasi planar) topological graph. Clearly, k-planar graphs are $(k+1)$-planar and k-quasi planar graphs are $(k+1)$-quasi planar. This naturally defines corresponding hierarchies.

*The research described in this paper started at the Dagstuhl Seminar 16452 "Beyond-Planar Graphs: Algorithmics and Combinatorics".

†Universität Tübingen, Germany, {angelini,bekos}@informatik.uni-tuebingen.de
‡University of Passau, Germany, brandenb@fim.uni-passau.de
§University of California, Irvine, CA USA, gdalozzo@uci.edu
¶Roma Tre University, Italy, gdb@dia.uniroma3.it
‖Universitá degli Studi di Perugia, Italy, {walter.didimo, giuseppe.liotta, fabrizio.montecchiani}@unipg.it
**TU Eindhoven, The Netherlands, i.rutter@tue.nl

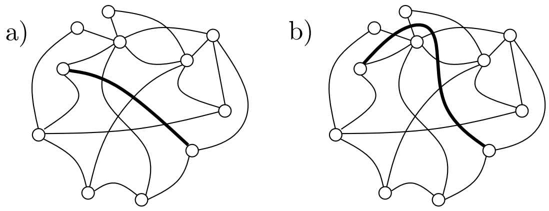

Figure 1: Rerouting the thick edge in the 3-planar graph (a) yields a 4-quasi planar graph (b).

The k-planarity and k-quasi planarity hierarchies have been widely explored in graph theory, graph drawing, and computational geometry, mostly in terms of edge density. While k-planar graphs are known to have at most a linear number of edges [9], the same is not known to be true for k-quasi planar graphs. While linear density upper bounds have been achieved for $k \leq 4$ [1], the best known upper bounds for $k \geq 5$ are super-linear [10]. Despite the fact that both graph hierarchies are well-researched little is known about their relationships.

Contribution. We focus on simple topological graphs and prove the first non-trivial inclusion relationship between the k-planarity and the k-quasi planarity hierarchies. Namely, we show that every k-planar graph is $(k+1)$-quasi planar, for every $k \geq 3$; see Fig. 1.

After some basic terminology in Section 2, Section 3 describes our proof strategy and introduces an edge rerouting technique for removing so-called untangled $(k+1)$-crossings (a $(k+1)$-*crossing* is a set of $(k+1)$ pairwise crossing edges). Section 4 shows that all $(k+1)$-crossings in a k-planar topological graph can be untangled and Section 5 then shows a global rerouting technique to remove all untangled $(k+1)$-crossings.

2 Preliminaries

We only consider graphs with neither parallel edges nor self-loops and without loss of generality we assume all graphs to be connected. We identify the vertices and edges of a topological graph with the points and arcs representing them, respectively. Two edges *cross* if they share one interior point and alternate around this point. Two edges *intersect* if they either cross or share a common endpoint. Graph G is *almost simple* if any two edges cross at most once, and it is *simple* if any two edges intersect at most once. Graph G divides the plane into topologically connected regions, called *faces*. The unbounded region is the *outer face*. Note that the boundary of a face can contain both vertices of the graph and crossing points between edges.

For a subgraph X of a graph G, the *arrangement* $\mathcal{A}_X$ *of* X, is the arrangement of the curves corresponding to the edges of X. We denote the vertices and edges of X by $V(X)$ and $E(X)$. A *node* of $\mathcal{A}_X$ is either a vertex or a crossing point of X. A *segment* of $\mathcal{A}_X$ is a part of an edge of X that connects two nodes.

A k-crossing X is *untangled* if in $\mathcal{A}_X$ all nodes corresponding to vertices in $V(X)$ are incident to a common face, otherwise it is *tangled*; see Sec. 4. A *fan* is a set of edges that share a common endpoint.

Observation 1 *Let $G = (V, E)$ be a k-planar simple topological graph and let X be a $(k+1)$-crossing in G. An edge in $E(X)$ cannot be crossed by any edge in $E \setminus E(X)$. In particular, for any two distinct $(k+1)$-crossings X and Y in G, $E(X) \cap E(Y) = \emptyset$ holds.*

3 Edge Rerouting Operations and Proof Strategy

We introduce an edge rerouting operation that will serve as a basic tool for our proof strategy. Let G be a k-planar simple topological graph and consider an untangled $(k+1)$-crossing X in G; see Fig. 2a.

Let $e = \{u, v\} \in E(X)$ and let $w \in V(X) \setminus \{u, v\}$. Denote by $\mathcal{A}'_X$ the arrangement obtained from $\mathcal{A}_X$ by removing all nodes corresponding to vertices in $V(X) \setminus \{u, v, w\}$, together with their incident segments, and by removing edge (u, v). The operation of *rerouting* e *around* w consists of redrawing e sufficiently close to the boundary of the outer face of $\mathcal{A}'_X$, choosing the routing that passes close to w, in such a way that e does not cross any edge in $E \setminus E(X)$ except for a fan incident to w; see Fig. 2b. More precisely, let D be a topological disk that encloses all crossing points of X and such that each edge in $E(X)$ crosses the boundary of D exactly twice. Then, the rerouted edge keeps unchanged the parts of e that go from u to the boundary of D and from v to the boundary of D. We call the unchanged parts of a rerouted edge its *tips* and the part that routes around w its *hook*.

Lemma 2 *Let $G' \simeq G$ be the topological graph obtained from G by rerouting an edge $e = \{u, v\} \in E(X)$ around a vertex $w \in V(X) \setminus \{u, v\}$. Let d be the edge of $E(X)$ incident to w. Graph G' has the following properties. (i) Edges e and d do not cross; (ii) The edges that are crossed by e in G' but not in G form a fan at w; (iii) G' is almost simple.*

Note that G' may be non-simple as e may possibly cross its adjacent edges (u, w) and (v, w). We fix this in Section 5 by redrawing (u, w) and (v, w) along e.

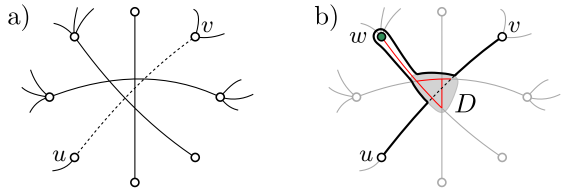

Figure 2: The rerouting operation for dissolving untangled k-crossings. (a) An untangled k-crossing X. (b) The rerouting of the dashed edge (u, v) around the marked vertex w. The arrangement $\mathcal{A}'_X$ is thin red, the removed nodes and segments are gray.

We now describe our strategy for transforming a k-planar simple topological graph G into a simple topological graph $G' \simeq G$ that is $(k+1)$-quasi planar. Note that G is trivially $(k+2)$-quasiplanar, but it may contains $(k+1)$-crossings. The idea is to pick from each $(k+1)$-crossing X in G an edge e_X and a vertex w_X and then to apply the above rerouting operation *simultaneously* for all pairs (e_X, w_X). We call this operation *global rerouting*. Note that this is well-defined due to Observation 1.

There are several constraints that have to be satisfied for such a global rerouting to have the desired effect. First, the rerouting operation works only for untangled $(k+1)$-crossings; we deal with this problem in Section 4. Second, even if all $(k+1)$-crossings are untangled, the graph G' resulting from the global rerouting may be non-simple and/or contain new $(k+1)$-crossings. We overcome these issues in Section 5.

4 Untangling $(k+1)$-Crossings

We show how to untangle $(k+1)$-crossings in k-planar topological graphs. The main idea is as follows. Consider a $(k+1)$-crossing X in a k-planar topological graph G. Since the edges in $E(X)$ already have k crossings, the faces of the arrangement $\mathcal{A}_X$ partition $G - E(X)$ into disjoint subgraphs G_f, one for each face f of $\mathcal{A}_X$; see Fig. 3a. Further, G_f is drawn inside a topological disk D_f whose boundary contains only the vertices in $V(G_f) \cap X$. By suitably deforming and rearranging these subgraphs, e.g., on the outside of a circle, the edges in $E(X)$ can be reinserted so that X either has fewer crossings or is untangled; see Fig. 3b. Note that this does not tangle other $(k+1)$-crossings.

Lemma 3 *Let G be a k-planar simple topological graph. There exists a k-planar simple topological graph $G' \simeq G$ without tangled $(k+1)$-crossings.*

5 Removing Untangled $(k+1)$-Crossings

Let G be a k-planar simple topological graph without tangled $(k+1)$-crossings. First, we show how to remove all $(k+1)$-crossings in G when $k \geq 3$ to obtain a

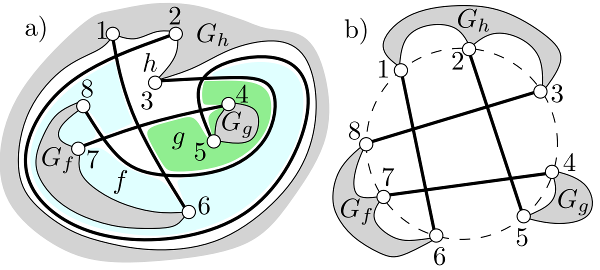

Figure 3: (a) A tangled 4-crossing X in a 3-planar graph G partitions $G - E(X)$ into disjoint subgraphs. (b) Transformation that untangles X.

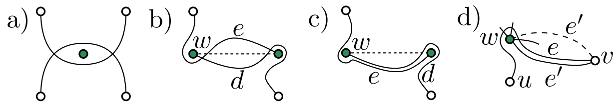

Figure 4: (a–b) Topological graphs that are not almost simple, arising from a global rerouting. (c) Avoiding the non-simplicity in (b) by redrawing one of the two rerouted edges. The vertices used for rerouting are filled green. (d) Illustration for the proof of Lemma 8.

$(k+1)$-quasi planar almost-simple topological graph $G' \simeq G$. Then, we describe how to make this graph simple, without introducing $(k+1)$-crossings.

Let G' be the topological graph obtained from G by performing a global rerouting that picks an edge vertex pair (e_X, w_X) from each $(k+1)$-crossing X in G.

Conditions on the Global Rerouting. We establish conditions on the global rerouting that guarantee that G' is almost simple and $(k+1)$-quasi planar. We start by describing which edge pairs cross newly in G'.

Lemma 4 *If e and d are two edges that cross in G' but not in G, then either both are rerouted around the same vertex, or one of them is rerouted around an endpoint of the other.*

Lemma 4 gives rise to the following criterion for determining whether a global rerouting results in an almost-simple topological graph G'; see Fig. 4a,b.

Lemma 5 *Graph G' is an almost-simple topological graph if and only if the following conditions hold.*
C1. No two edges are rerouted around the same vertex.
C2. There is no pair of edges e, d such that e is rerouted around an endpoint of d and d is rerouted around an endpoint of e.

Moreover, G' is $(k+1)$-quasi planar (though not necessarily almost-simple) if no two edges are rerouted around the same vertex.

Lemma 6 *Let G be a k-planar simple topological graph without tangled $(k+1)$-crossings, and let G' be the graph obtained by a global rerouting operation on G. If G' does not contain two edges rerouted around the same vertex and $k \geq 3$, then G' does not contain any $(k+1)$-crossing.*

Note that Lemma 6 does not hold for $k = 2$.

Obtaining simplicity. Lemmas 3 and 6 imply that, for $k \geq 3$, a given k-planar simple topological graph G can be redrawn such that the resulting topological graph $G' \simeq G$ contains no $(k+1)$-crossings, assuming that no two edges are rerouted around the same vertex. The graph G' may however be not simple, and even not almost simple. We first show how to remove from G' pairs of edges crossing more than once, without introducing any $(k+1)$-crossings. Afterwards we show how to remove crossings between adjacent edges still without introducing any $(k+1)$-crossings.

Lemma 7 *There exists a $(k+1)$-quasi planar almost-simple topological graph G^* such that $G^* \simeq G'$.*

Proof. If G' is not almost simple, then by Lemma 5 there exist pairs of edges such that each of them is rerouted around an endpoint of the other one. We now show how to resolve all such pairs. Let e, d be any of these pairs; see also Fig. 4b. We redraw one of the two edges, say e, sufficiently close along d between the two crossings. More precisely, the tip of e crossed by the hook of d is redrawn following the tip of d crossed by the hook of e, without crossing it (see Fig. 4c). Hence e and d do not cross. We apply this transformation to all such pairs of tips. We claim that the resulting graph G^* does not contain new $(k+1)$-crossings.

Observe first that all such pairs of tips are pairwise disjoint, since no two edges are rerouted around the same vertex. This implies that no tip of an edge is transformed twice in G^*, and that no two transformed edges cross each other. Hence, if a $(k+1)$-crossing exists in G^* then it contains exactly one transformed edge. We prove that this is not the case.

Consider again a pair of edges e, d that cross twice in G' and such that e is transformed in G^*. For an edge l, let X_l denote the $(k+1)$-crossing of G containing l. The edges that cross e are: (i) a set X'_d of edges that cross the tip of d crossed by the hook of e and thus that are part of X_d, (ii) a set X'_e of edges in X_e that cross the tip of e not crossed by d, (iii) a set E_w of edges incident to the vertex w around which e is rerouted (and thus cross the hook of e).

Note that X'_d contains the edges that cross e in G^* and not in G'. These are at most $k-1$ edges. The edges in X'_e do not cross the edges in X'_d, because they are non-rerouted edges that belong to distinct $(k+1)$-crossings of G, and any edge in E_w crosses at most one edge in X'_e. Thus G^* is $(k+1)$-quasi planar.

Finally, we claim that the edges in X'_d are crossed only once by e. Recall that none of these edges crosses

e in G'. Since the tip of e crossed by the hook of d is transformed by following the tip of d crossed by the hook of e, an edge h of X'_d can cross e only once on this tip. On the other hand, it could be that also the other tip of e has been transformed along the tip of an edge l such that it crosses h. But then h crosses tips of two rerouted edges d, l in G', and by Lemma 4 also in G, contradicting the disjointness of X_d and X_l. ☐

Lemma 8 *There exists a $(k+1)$-quasi planar simple topological graph $\overline{G}$ such that $\overline{G} \simeq G^*$.*

Proof. Since G is simple, if a pair of crossing edges e and e' share an endpoint u, then at least one of them, say e, has been redrawn. Suppose first that only one of them has been redrawn. Under this assumption, we distinguish between two subcases: either e has also been transformed when going from G' to G^* or not.

We first argue about the subcase in which e has not been transformed. Note that, e crosses e' with its hook. Then we redraw e' by following e until reaching u. Now e' crosses only edges that cross the tip of e incident to u. This guarantees that no $(k+1)$-crossing is introduced and that no edge is crossed twice (because G^* is almost simple). See also Fig. 4d.

Consider now the subcase where e has been transformed. Then, there exists an edge e'' that crosses e twice in G', and that has not been transformed. Note that the endpoint around which e'' has been rerouted is not u, as otherwise e' would cross twice e'', which is not possible because G^* is almost simple. Then edge e' is part of the $(k+1)$-crossing of e'' in G. Then we redraw the first part of e from u to the crossing with e' by following e' and leave the rest of e' unchanged. Notice that all the new crossings are due to edges that cross e', and it can be argued that these cannot produce a $(k+1)$-crossing. However, we also need to show that none of these edges is crossed twice. Such an edge would have an endpoint in-between e and e'', which is impossible by the definition of the edge rerouting operation (given that such edge belongs to the same $(k+1)$-crossing of e'' in G).

If both e and e' have been redrawn, then they cannot be redrawn around the same vertex, and the case in which one is redrawn around the non-shared endvertex of the other can be handled similarly as above. ☐

Existence of global rerouting. It remains to show the existence of a global rerouting where no two edges are rerouted around the same vertex. Consider the bipartite graph H whose nodes correspond to the $(k+1)$-crossings of G and the vertices of G such that $(X, v) \in E(H)$ if and only if $v \in V(X)$. It can be shown that H is bipartite, planar, and each node corresponding to a $(k+1)$-crossing has degree $2k$. It is then not hard to see that Hall's criterion is satisfied and there exists a matching that assigns a vertex to each $(k+1)$-crossing.

Lemma 9 *Let G be a k-planar simple topological graph. There exists a global rerouting on G such that no two edges are rerouted around the same vertex.*

The lemmas from this section imply our main result.

Theorem 10 *Let G be a k-planar simple topological graph for $k \geq 3$, then there exists a $(k+1)$-quasi planar simple topological graph $\overline{G}$ such that $\overline{G} \simeq G$.*

6 Conclusion

We proved that, for any $k \geq 3$, k-planar graphs are $(k+1)$-quasiplanar. Our main open question is whether the same inclusion relation also holds for $k = 2$.

References

[1] E. Ackerman. On the maximum number of edges in topological graphs with no four pairwise crossing edges. *Discrete Comput. Geom.*, 41(3):365–375, 2009.

[2] M.A. Bekos, S. Cornelsen, L. Grilli, S.-H. Hong, and M. Kaufmann. On the recognition of fan-planar and maximal outer-fan-planar graphs. In *GD 2014*, volume 8871 of *LNCS*, pages 198–209. Springer, 2014.

[3] C. Binucci, E. Di Giacomo, W. Didimo, F. Montecchiani, M. Patrignani, A. Symvonis, and I.G. Tollis. Fan-planarity: Properties and complexity. *Theor. Comput. Sci.*, 589:76–86, 2015.

[4] O. Cheong, S. Har-Peled, H. Kim, and H.-S. Kim. On the number of edges of fan-crossing free graphs. *Algorithmica*, 73(4):673–695, 2015.

[5] J. Fox, J. Pach, and A. Suk. The number of edges in k-quasi-planar graphs. *SIAM J. Discrete Math.*, 27(1):550–561, 2013.

[6] M. Kaufmann, S. Kobourov, J. Pach, and S.-H. Hong. Beyond planar graphs: Algorithmics and combinatorics. Dagstuhl Seminar 16452, 2016.

[7] M. Kaufmann and T. Ueckerdt. The density of fan-planar graphs. *CoRR*, abs/1403.6184, 2014.

[8] G. Liotta. Graph drawing beyond planarity: some results and open problems. In *Proc. 15th Italian Conf. Theor. Comput. Sci.*, volume 1231 of *CEUR Workshop Proceedings*, pages 3–8, 2014.

[9] J. Pach and G. Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997.

[10] A. Suk and B. Walczak. New bounds on the maximum number of edges in k-quasi-planar graphs. *Comput. Geom.*, 50:24–33, 2015.