Fréchet isotopies to monotone curves

Fréchet Isotopies to Monotone Curves*

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1 Introduction

We study the *isotopic Fréchet distance*, which is a distance measure between two curves f and g that captures one notion of an optimal morph between these two curves. The classic Fréchet distance between f and g, also called the "dog leash distance", measures the length of the shortest possible straight leash needed to connect a man and a dog which are walking forward along f and g. Any two feasible walks using such a shortest leash induce a *Fréchet matching* between f and g. One can now imagine to build a morph between f and g by sliding each point of f along the leash that connects it to its matched point on g. Such an approach will work well in unrestricted Euclidean space, however, it is not suitable for more general spaces that might contain obstacles. In the presence of obstacles the leashes of the classic Fréchet distance can jump discontinuously and hence the resulting morph would be discontinuous as well.

The *homotopic Fréchet distance* [3, 6] forces leashes to move continuously. More formally, for two curves f and $g\colon [0,1] \to \mathbb{R}^2$ in the plane a homotopy $h\colon [0,1]^2 \to \mathbb{R}^2$ is a continuous map between f and g. Such a homotopy essentially morphs one curve into the other: each point of f traces a path $h(p,\cdot)$ to a point on g. The length of a homotopy is the length of the longest such path, and a *Fréchet homotopy* is one that minimizes this length. The homotopic Fréchet distance between f and g is then the length of a Fréchet homotopy between f and g. The homotopic Fréchet distance and the classic Fréchet distance are equivalent in $\mathbb{R}^2$. The morph that results from a Fréchet homotopy is continuous, but it may change the structure of the input curves during the morph: intermediate curves can self-intersect or collapse to a point, even if f and g are simple curves.

A homotopy is an isotopy if all its intermediate curves $h(\cdot,t)$ are simple. The *isotopic Fréchet distance* measures the length of an optimal isotopy between f and g; we call an optimal isotopy a *Fréchet isotopy*. The study of Fréchet isotopies was initiated in [4]. The authors gave some simple observations and examples and showed that the isotopic Fréchet distance in the plane can be arbitrarily larger than the homotopic Fréchet distance.

Results. In this paper we revisit the isotopic Fréchet distance and refute a conjecture posed in [4]. We also give the first algorithms to compute short isotopies in some restricted cases. Specifically, we compute optimal isotopies if there is a direction in which both input curves are monotone. Furthermore, given a curve in $\mathbb{R} \times [0,\varepsilon]$ (for infinitesimally small ε), we construct an isotopy to a monotone curve using minimal length.

Related work. Closely related are morphs based on *geodesic width* [5]: the intermediate curves are not allowed to cross the input curves f and g, and they are restricted to the area between the leashes connecting the endpoints of f and g. This restriction naturally enforces intermediate curves without self-intersections since "geodesic leashes" do not cross each other. Morphs based on geodesic width minimize the maximum leash length. However, they are restricted to input curves that do not intersect each other; in contrast, Fréchet isotopies are also well-defined for input curves that intersect each other.

A variety of morphs have been considered in the graph drawing and computational geometry literature. For instance, it is well known that any two drawings of the same planar graphs can be morphed into one another. More recent work focused on bounding the number of steps in the optimal morph between any two input graphs [1, 2]. Here the intermediate curves are homeomorphic to the input and vary continuously. However, in contrast to Fréchet isotopies, the morphs do not minimize length.

2 Preliminaries

A *curve* in the plane is a continuous map $f\colon [0,1] \to \mathbb{R}^2$. We denote the x and y-coordinates of $f(p)$ by $f_x(p)$ and $f_y(p)$, respectively. A continuous nondecreasing surjection $\alpha\colon [0,1] \to [0,1]$ is called a *reparameterization* of a curve. A *homotopy* is a continuous map $h\colon [0,1] \times [0,1] \to \mathbb{R}^2$. We denote its level curves by $h_t\colon p \mapsto h(p,t)$, and say h goes from curve f to g if $h_0 = f$ and $h_1 = g$. A homotopy is an *isotopy* if each curve h_t is simple.

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A homotopy from f to g traces paths $\lambda_{h,p}\colon t \mapsto h(p,t)$ between the points $f(p)$ and $g(p)$, and such a path is traditionally referred to as a *leash*. Let the *length* of a homotopy h be the length $\mathit{length}(h) = \sup_p \mathit{length}(\lambda_{h,p})$ of its longest leash. We are interested in homotopies h minimizing this length and define the *homotopic Fréchet distance* between f and g as

$$d_{hom}(f,g) = \inf_{\substack{\alpha,\beta,h \\ h_0=f\circ\alpha,\ h_1=g\circ\beta}} \mathit{length}(h),$$

where h ranges over homotopies and α and β range over reparameterizations. The *isotopic Fréchet distance* d_{iso} is defined similarly, except that h ranges over isotopies.

The *Fréchet distance* $d_F(f,g) = \inf_{\alpha,\beta} \sup_p \|f \circ \alpha(p) - g \circ \beta(p)\|$ is a related measure that does not require leashes to trace out a homotopy, so each leash can be assumed to be a shortest path. The pair (α,β) is called a *matching*. We define the *cost* of a matching (α,β) between f and g as $\mathit{cost}_{f,g}(\alpha,\beta) = \sup_p \|f\circ\alpha(p) - g\circ\beta(p)\|$. A *Fréchet matching* between curves f and g in the plane is one with cost $d_F(f,g)$.

In the plane, the map $\mathit{Aff}^{f,g}(p,t) = (1-t)\cdot f(p) + t\cdot g(p)$ using line segments (shortest paths) as leashes is a homotopy since it is an affine interpolation between continuous maps. We call $\mathit{Aff}^{f,g}$ the *affine homotopy* from f to g, and its length is $\mathit{length}(\mathit{Aff}^{f,g}) = \sup_p \|f(p) - g(p)\|$. It follows that the homotopic Fréchet distance and the Fréchet distance are equivalent in $\mathbb{R}^2$. On the other hand, the isotopic Fréchet distance in the plane can be arbitrarily larger than the homotopic Fréchet distance [4].

We call a homotopy h from $f\circ\alpha$ to $g\circ\beta$ a *Fréchet homotopy* if $\mathit{length}(h) = d_{hom}(f,g)$, and call h a *Fréchet isotopy* if h is an isotopy with $\mathit{length}(h) = d_{iso}(f,g)$. Since every isotopy is a homotopy, we have $d_{hom}(f,g) \le d_{iso}(f,g)$ and any isotopy that is a Fréchet homotopy is also a Fréchet isotopy. However, Fréchet isotopies need not be Fréchet homotopies since there might exist a homotopy shorter than any isotopy.

For a curve f and a unit vector $(x,y) \in S^1$, we define the *directional length* of f in the direction (x,y) to be the total length that f moves *forward* in the direction of the vector, given by $\mathit{length}_{(x,y)}(f) = \int_0^1 \max(0, \langle \frac{\mathrm{d}f(p)}{\mathrm{d}p}, (x,y)\rangle)\mathrm{d}p$, where $\langle\cdot,\cdot\rangle$ is the inner product. We define the horizontal length of a curve as $\mathit{length}_{\mathrm{hor}}(f) = \mathit{length}_{(-1,0)}(f) + \mathit{length}_{(1,0)}(f)$ and define the *horizontal* homotopic and isotopic Fréchet distances using the horizontal length function. As usual, a horizontal Fréchet homotopy (respectively isotopy) is one minimizing the horizontal homotopic (respectively isotopic) Fréchet distance.

Throughout the paper, we assume all input curves to be simple.

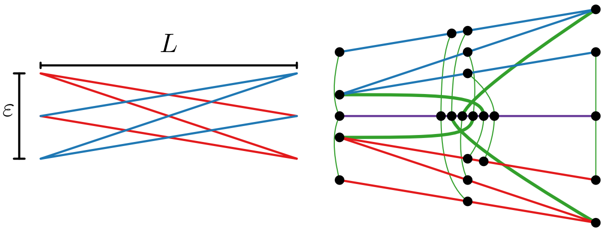

Figure 1: An isotopy of length $L/2$ (as ε approaches 0) between two 'opposite' zig-zags. The fat arcs have horizontal length roughly $L/2$, whereas the others have negligible horizontal length.

3 Disproving a conjecture

In Figure 1 we show an example of two zig-zag curves, originally presented in [4]. The Fréchet distance between these curves is at most ε, as there is a matching whose leashes are all vertical. However, this Fréchet mapping yields a homotopy that collapses the zig-zag to a flat line before re-expanding to the other zig-zag, which does not result in an isotopy, as the three segments coincide halfway along the isotopy.

In [4], the authors conjectured that the isotopic Fréchet distance between the zig-zags is $\sqrt{L^2+\varepsilon^2}$. However, the isotopy demonstrated by the green leashes on the right side of Figure 1 has length arbitrarily close to $\sqrt{L^2+\varepsilon^2}/2 + \varepsilon/2$.

We will show that the isotopy of Figure 1 is arbitrarily close to optimal. Consider a convex region D and an isotopy h between curves f and g in the plane, where the endpoints of all intermediate curves lie in D; that is, $\mathrm{Im}(\lambda_{h,0}) \subseteq D$ and $\mathrm{Im}(\lambda_{h,1}) \subseteq D$. Fix some $p \in (0,1)$ and denote by poly_t the polyline with an edge from $h(0,t)$ to $h(p,t)$, and an edge from $h(p,t)$ to $h(1,t)$. Let θ_t be the (counterclockwise) angle at $h_t(p)$ between the two edges of poly_t (plus a multiple of 360 degrees), such that θ_t varies continuously with t. We show in Lemma 1 that (in any isotopy from poly_0 to poly_1) the leash $\lambda_{h,p}$ must intersect D if θ_0 and θ_1 differ by at least 180 degrees, see Figure 2.

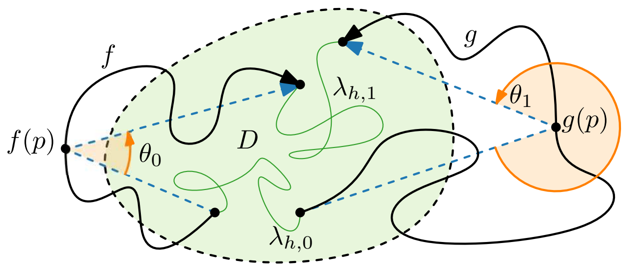

Figure 2: Curves $f = h_0$, $g = h_1$ and polylines poly_0 and poly_1 with endpoints in convex region D.

Lemma 1 *If f is isotopic to $poly_0$ relative to its vertices*[1] *and g is isotopic to $poly_1$ relative to its vertices, and $|\theta_1 - \theta_0| \geq 180$, then $h(p,t) \in D$ for some t.*

Proof. Because f and g are isotopic to $poly_0$ and $poly_1$ respectively (relative to their vertices), we may assume without loss of generality that $0 \leq \theta_0 < 360$ and $0 \leq \theta_1 < 360$. Because θ_t varies continuously, we have by the intermediate value theorem that $\theta_t = 180$ for some $t \in [0,1]$. Hence, $h(p,t)$ lies on the line segment between $h(0,t) \in D$ and $h(1,t) \in D$. By convexity, this segment lies completely in D, so $h(p,t) \in D$. ◻

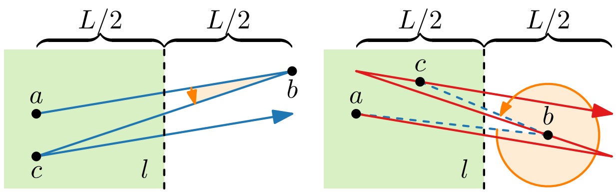

Figure 3: The vertices and region D used to obtain a lower bound for the curves of Figure 1.

Using Lemma 1, we can show that our isotopy for the zig-zags of Figure 1 is optimal as ε approaches 0. For this, we show that any Fréchet isotopy has length at least $L/2$. Assume that the zig-zags f and g are parameterized such that an isotopy h of length less than $L/2$ between them exists. Let l be the vertical line centered between the vertices, such that each vertex has distance $L/2$ to l, and let D be the half-plane to the left of l, see Figure 3. Let $f(a)$, $f(b)$ and $f(c)$ be the first three vertices of f. If $length(h) < L/2$, the leashes $\lambda_{h,a}$ and $\lambda_{h,c}$ lie completely inside D, and $\lambda_{h,b}$ lies completely outside D. Since $a < b < c$ and $g(b) \notin D$, a and c lie in different components of $g^{-1}(D)$. Isotopy h induces a restricted isotopy between the subcurves of f and g from a to c, and these subcurves satisfy the conditions required by Lemma 1. Therefore, $\mathrm{Im}(\lambda_{h,b})$ intersects D, so $length(h) \geq L/2$.

4 Isotopies between monotone curves

A curve f is *strictly x-monotone* if $f_x(p) < f_x(p')$ for all $p < p'$. Lemma 2 implies that for such curves, the isotopic and homotopic Fréchet distances are equal.

Lemma 2 *For strictly x-monotone curves f and g, each curve h_t of $h = \mathit{Aff}^{f,g}$ is strictly x-monotone.*

Proof. Recall that $h_t(p) = (1-t)\cdot f(p) + t \cdot g(p)$. Consider the x-coordinates $x_t(p)$ of $h_t(p)$ and $x_t(p')$ of $h_t(p')$ for $p < p'$. Let $s_t = x_t(p') - x_t(p)$. Because f and g are strictly x-monotone, we have $x_0(p) < x_0(p')$ and $x_1(p) < x_1(p')$, so $s_0 > 0$ and $s_1 > 0$. Since s is affine, we have $s(t) > 0$ for $t \in [0,1]$, so $x_t(p) < x_t(p')$. Hence, each level curve h_t is strictly x-monotone. ◻

[1] That is, there exists an isotopy from f to $poly_0$ that does not move $f(0)$, $f(p)$ or $f(1)$.

Theorem 3 *For strictly x-monotone curves, the homotopic and isotopic Fréchet distances are equivalent.*

5 Isotopies to monotone curves

In this section, we consider the problem of monotonizing curves using minimal horizontal movement. Specifically, we show how to construct a short isotopy from an input curve to some x-monotone curve. In Section 5.1 we argue that this problem is interesting already if we measure only horizontal length by showing that an optimal isotopy may have non-monotone leashes, even though we can choose any x-monotone target curve. We construct an isotopy of minimal horizontal length to an x-monotone curve in Section 5.2. We first give a lower bound for homotopies to x-monotone curves.

Lemma 4 *For any homotopy h from f to any x-monotone curve g, $length_{\mathrm{hor}}(h) \geq \frac{1}{2}\sup_{p \leq p'} f_x(p) - f_x(p')$.*

Proof. We have $length_{\mathrm{hor}}(\lambda_{h,p}) \geq |f_x(p) - g_x(p)|$ and because g is x-monotone, $g_x(p) \leq g_x(p')$. Because $length_{\mathrm{hor}}(h) \geq length_{\mathrm{hor}}(\lambda_{h,p}) \geq f_x(p) - g_x(p)$ and $length_{\mathrm{hor}}(h) \geq length_{\mathrm{hor}}(\lambda_{h,p'}) \geq g_x(p') - f_x(p')$, we have $2 \cdot length_{\mathrm{hor}}(h) \geq f_x(p) - g_x(p) + g_x(p') - f_x(p') \geq f_x(p) - f_x(p')$. ◻

5.1 Non-monotone isotopies

Consider the curve $f = h_0$ of Figure 4 with $f(0) = p_0$ and $f(1) = p_5$, morphing into an x-monotone curve $g = h_1$ as depicted. No matter how small we pick $\varepsilon > 0$, the depicted isotopy has length at most $r+\varepsilon$ if $w < r$. By Lemma 4, there exists no homotopy to an x-monotone curve of length less than r, even if we pick a different x-monotone curve g. In this context, the lemma states that because $f_x(p_1) \geq f_x(p_2) + 2r$ and $g_x(p_1) \leq g_x(p_2)$, one of the leashes λ_{h,p_1} or λ_{h,p_2} has length at least r in any homotopy h.

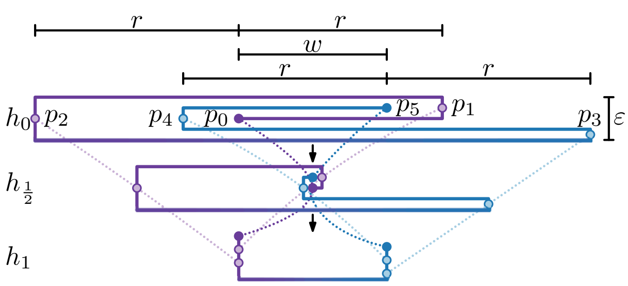

Figure 4: A curve for which any Fréchet isotopy to an x-monotone curve moves some point for distance $w/2-\varepsilon$ in both the positive and the negative x-direction.

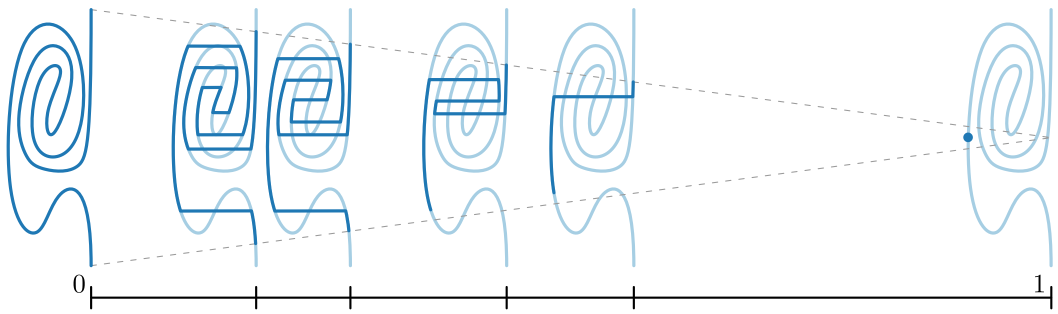

Figure 5: The major critical events of our isotopy (rotated 90 degrees) on a spiral.

What makes this instance interesting is that any optimal isotopy moves some points in both the forward (positive) and backward (negative x-direction) for a considerable distance. Formally, for any isotopy h from f to g, we have both $length_{(1,0)}(\lambda_{h,p}) \geq w/2 - \varepsilon$ and $length_{(-1,0)}(\lambda_{h,p}) \geq w/2 - \varepsilon$ for some p. In particular, consider the two endpoints $f(p_0)$ and $f(p_5)$ in Figure 4. Based on Lemma 1, these points must 'untangle' with respect to each other somewhere in the isotopy h. For this, the x-coordinates of $h_t(p_0)$ and $h_t(p_5)$ must be equal for some value of t, say for $t = t^*$. Because g is x-monotone, and an optimal isotopy has length at most $r + \varepsilon$, we have $g_x(p_0) \leq g_x(p_2) \leq f_x(p_2) + r + \varepsilon = f_x(p_0) + \varepsilon$ and symmetrically $f_x(p_5) - \varepsilon = f_x(p_3) - r - \varepsilon \leq g_x(p_3) \leq g_x(p_5)$.

Let $\gamma = h_{t^*}$ and $x^* = \gamma_x(p_0) = \gamma_x(p_5)$. Since $f_x(p_5) - f_x(p_0) = w$, we have $x^* - f_x(p_0) \geq w/2$ or $f_x(p_5) - x^* \geq w/2$. If $x^* - f_x(p_0) \geq w/2$, we also have $g_x(p_0) - x^* \geq w/2 - \varepsilon$, so p_0 moves forward for a distance of at least $w/2$ and backwards for a distance of at least $w/2 - \varepsilon$. Otherwise, $f_x(p_5) - x^* \geq w/2$, and p_5 moves backwards for $w/2$ and forwards for $w/2 - \varepsilon$. The total distance such points move approaches $d_{iso}(f, g)$ as ε approaches 0.

5.2 Shrinking based isotopies

For a curve f, we define an isotopy $Shr^f(p,t) = \mathcal{P}_{|\mathrm{Im}(f_x)|t/2}(f)(p)$, where $\mathcal{P}_l(f)$ is a curve intuitively obtained from f by moving all local maxima of f_x in the negative x-direction, and all local minima of f_x in the positive x-direction for distance l. Define $N_l^f(p)$ to be the component of $f^{-1}((f_x(p) - l, f_x(p) + l))$ containing p; that is, the subpath of f reachable from $f(p)$ using only x-coordinates at distance less than l from $f_x(p)$. Let l^* be the horizontal length of the longest monotone[2] subpath of f. We can define $\mathcal{P}_l(f)$ more formally by recursively defining $\mathcal{P}_l(f) = \mathcal{P}_{l-l^*}(\mathcal{P}_{l^*}(f))$ if $l > l^*$; and if $l \leq l^*$, replacing, for each minimum or maximum p of f_x, the arc $N_l^f(p)$ of f by a vertical segment between its endpoints with f_x at distance l from $f_x(p)$ (or by the endpoint if only one such endpoint exists).

[2] Monotone in either the positive or the negative x-direction.

In the full paper, we show that after infinitesimal perturbation, Shr^f is an isotopy. Figure 5 illustrates its behavior for a curve based on an example from [4].

5.3 Optimality of the isotopy to a monotone curve

Lemma 4 gave us a lower bound on $length_{\mathrm{hor}}(h)$ for a homotopy h turning f into an x-monotone curve. Theorem 5 (proof in the full paper) tells us that the horizontal length of the isotopy we construct matches this lower bound, meaning that the isotopy yields an x-monotone curve using minimal horizontal movement.

Theorem 5 *$Shr^f(t)$ yields an x-monotone curve at $t = |\mathrm{Im}(f_x)|^{-1}\frac{1}{2}\sup_{p \leq p'} f_x(p) - f_x(p')$, using only $\frac{1}{2}\sup_{p \leq p'} f_x(p) - f_x(p')$ horizontal movement.*

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