Computing representative networks for braided rivers

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1 Introduction

Geomorphology is the study of the shape of natural terrains and the processes that create them. One of these processes is erosion due to water flow. The combination of terrain shape and water flow gives rise to various computational problems that have been studied in geomorphology, geographic information science (GIS), and computational geometry.

One prominent problem is the computation of drainage networks (flow) [1]. Here computations are based on elevation data only and the shape of the terrain is used to determine where rivers will form (see [11] for an overview). A second problem concerns local minima. Due to erosion local minima are more rare in natural terrains than local maxima; minor local minima are often measurement errors. Such errors are clearly undesirable when studying flow on terrains and hence, minor local minima are removed by computational means [6]. A third commonly studied problem deals with watersheds and their boundaries [2, 11].

Braided rivers. The usual models for water flow in terrains allow rivers to merge, which is natural because side valleys join main valleys. And clearly, if water always follows the direction of steepest descent, a river cannot split (except due to degeneracies). Yet splitting happens in deltas and various river types, in particular braided river systems [4]. Such systems have islands called bars, separating different channels of the same river over their length after which the channels confluence. Modeling braided rivers, where channels can both split and merge, is considerably more complex than modeling standard drainage networks, where all rivers flow only downhill and do not bifurcate. In this paper we initiate the study of braided rivers from the perspective of computational geometry and topology.

To model a braided river we first need a representation of the basic geometry, independent of water level. We hence use the elevation of the river bed as a starting point. In meandering rivers the so-called *thalweg* is often used as a basic representation. The thalweg is defined as the deepest part of a continuous channel, which is a linear feature. We are striving for a similar representation for braided rivers, consisting of linear features along *lowest paths* in each channel. These linear features can merge and bifurcate, that is, they form a planar graph or network. The use of graphs to model and analyze braided rivers was recently pioneered in [7]. We define lowest paths intuitively as the paths that do not go higher than they need to go to.

A representative network for a braided river should not necessarily contain all possible channels. Topologically speaking a tiny local maximum in the river bed creates two channels. We can use persistence to simplify our input and avoid such situations. But still, too many channels may remain. We wish to select a set of channels which are sufficiently different from each other, which we model with a function (the *sand function*) that relates to the volume of sediment the river has to move before the two channels become one. More volume needs more time to be removed [5]. A bar of very small volume separating two channels requires insignificant time to be removed, but a large bar with a large volume may require multiple floods to be shaved off or cut through by a new channel, meaning that the two channels separated by this bar are significantly different.

Our objective now is to compute a representative network of channels that is optimal in some sense. We require that (*i*) each channel is *locally lowest*, (*ii*) any two channels are sufficiently *different* (specified by a parameter δ and the sand function), and (*iii*) the representative network is *maximum*. Unfortunately, solving this problem exactly is NP-hard. As an alternative we first compute a *striation*: a left-bank-to-right-bank sequence of non-crossing paths for the whole river bed. We then define a sand function to measure how different two channels are, which allows us to extract a representative network from the striation using a greedy algorithm. Due to space restrictions we present only one heuristic for the striation and one model for the sand function in this short abstract. The other heuristics, models, and omitted proofs can be found in the full version of the paper.

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2 Definitions and problem statement

Let $G = (V, E)$ be a triangulation of a topological disk M in the plane, and let $h: M \to \mathbb{R}$ be the *height* of the points in M, where the edges and triangles of G interpolate h linearly between its vertices. So G can be viewed as a simplicial 2-complex in $\mathbb{R}^3$ by adding h as a third dimension. Let $\sigma \in V$ be a *source* and $\tau \in V$ be a *sink*, both on the boundary ∂M. We refer to (G, h) as a *terrain*, and to the volume $\{(x, y, z) \mid (x, y) \in M, z \in \mathbb{R}, z \leq h(x, y)\}$ as *sand*.

Let $\pi^-_{\partial M}$ and $\pi^+_{\partial M}$ (respectively clockwise and counterclockwise) be the two paths from σ to τ along the boundary of M. We call a path π from σ to τ *semi-simple* if it has no self-intersections, but it may coincide with itself, see the red path in Figure 1. Let $\mathcal{P}$ be a set of semi-simple, pairwise non-crossing paths from σ to τ along edges of G. For two semi-simple paths π_0 and π_1 in $\mathcal{P}$, let $D(\pi_0, \pi_1)$ be the region bounded by and including π_0 and π_1. So two semi-simple paths π_0 and π_1 from σ to τ have no proper crossings if and only if $D(\pi^+_{\partial M}, \pi_0) \subseteq D(\pi^+_{\partial M}, \pi_1)$ or $D(\pi^+_{\partial M}, \pi_1) \subseteq D(\pi^+_{\partial M}, \pi_0)$.

We measure the similarity between two paths using a *sand function* $d: \mathcal{P} \times \mathcal{P} \to \mathbb{R}$, and we say a path π_0 is δ*-dissimilar* to π_1 if and only if $d(\pi_0, \pi_1) \geq \delta$. Since it is generally not a metric, we do not call d a distance.

Our goal is to compute a non-crossing set of paths $\Pi \subseteq \mathcal{P}$ that represent channels in a river. For that we use lowest paths, paths that minimize the distance spent at high elevations. Formally, for a path π, let $\rho(\pi, z)$ be the length of π for which the height is at least z. We say a path π_0 is *lower* than π_1 if and only if there exists a $z^* \in \mathbb{R}$, such that for all $z \geq z^*$, $\rho(\pi_0, z) = \rho(\pi_1, z)$ and for all $\varepsilon > 0$, there is some $z' \in (z^* - \varepsilon, z)$ with $\rho(\pi_0, z') < \rho(\pi_1, z')$ [8]. We call Π a δ*-network* if no pair of paths $\pi_0, \pi_1 \in \Pi$ has proper crossings, and $d(\pi_0, \pi_1) \geq \delta$ if π_1 is lower than π_0. A delta-network is representative unless replacing a subset of paths by a lower path yields a delta-network.

We assume that all vertices in our terrain have different height, and all edges have different slope. We attach a vertex v_∞ to the boundary of the terrain that is higher than all other vertices. Given such a modified terrain, a source σ, a sink τ, a parameter δ, and a sand function d, we study the problem of computing a representative δ-network over the edges of the terrain.

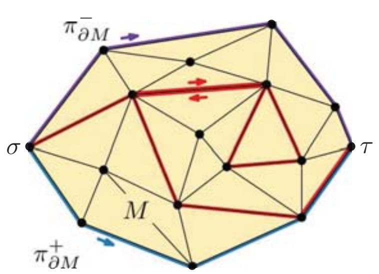

Figure 1: The disk M and three paths of $\mathcal{P}$ without proper crossings, including the two paths $\pi^-_{\partial M}$ and $\pi^+_{\partial M}$ and a backtracking path.

3 Morse-Smale complex and lowest paths

Let $\mathbb{M}$ be a smooth, compact 2-dimensional manifold without boundary, and let $h: \mathbb{M} \to \mathbb{R}$ be a height function on $\mathbb{M}$. A point p on $\mathbb{M}$ is *critical* with respect to h if all partial derivatives vanish at p. Otherwise, p is called *regular*. There are three types of critical points: (local) minima, (local) maxima, and saddle points. For each regular point p we can define the path of *steepest ascent* (or *steepest descent* in the opposite direction) as the path that follows the gradient of h at p. These paths are also known as *integral lines* and are open at both ends, with at each end a critical point. Using these integral lines, we can subdivide $\mathbb{M}$ as follows: two regular points p and q belong to the same cell if the integral lines through p and q end at the same critical points on both sides. The resulting complex is known as the *Morse-Smale complex*, or MS-complex in short. Note that if one of the endpoints of an integral line is a saddle point, then the corresponding cell is 1-dimensional. We refer to 2-dimensional cells of the MS-complex as *MS-cells*, and 1-dimensional cells as *MS-edges*. It can be shown [3] that every MS-cell is a quadrilateral with a minimum, a saddle, a maximum, and again a saddle along the boundary of the cell.

Note that our paths need to follow the edges of the terrain. Therefore, instead of the standard MS-complex, we use (a subset of) a quasi MS-complex as defined in [3], which does follow the edges of the terrain. Let v be a vertex, and let $S(v)$ be the *edge star* of v consisting of the set of edges incident to v. The *lower edge star* $S^\downarrow(v)$ consists of the subset of edges whose endpoints are lower than v with respect to h. Symmetrically, we can define the upper edge star $S^\uparrow(v) = S(v) \setminus S^\downarrow(v)$. The lower edge star can naturally be subdivided into *wedges* of consecutive edges in $S^\downarrow(v)$ separated by edges in $S^\uparrow(v)$. Given these definitions, we construct a *descending* [10] quasi MS-complex as follows. From every saddle point v we construct a steepest descent path in every wedge of $S^\downarrow(v)$ until it reaches a minimum. Note that steepest descent paths may partly overlap, but cannot cross. The cells of this complex are bounded by alternating minima and saddle points, and every cell contains exactly one maximum. In the remainder of this paper we refer to the descending quasi MS-complex as constructed above simply as the MS-complex, unless stated otherwise. The same rule applies to the components of the complex, namely the MS-cells and MS-edges.

The relation between lowest paths and the MS-complex is summarized in the following lemma.

Lemma 1 *The lowest path between two vertices u and v in G follows MS-edges, except for the head and tail of the path, which follow steepest descent paths.*

Due to this relation, we refer to paths on the MS-edges as *locally lowest* paths. As a result, all paths in a representative δ-network must follow MS-edges.

4 Striation

Consider an input terrain consisting of a sequence of pyramids with different heights as shown in Figure 2, where all non-peak vertices are at height 0. Let π_0 and π_1 be two paths from source to sink at height 0 and let P be the set of pyramids in $D(\pi_0, \pi_1)$. Then, for any "reasonable" sand function, $d(\pi_0, \pi_1) = \sum_{p \in P} vol(p)$. It is now easy to see that computing a representative δ-network for a terrain of this type is NP-hard by reduction from PARTITION.

To make the problem tractable, we put a restriction on the paths that can be used in a representative δ-network. We use a *striation*: a left-bank-to-right-bank sequence of non-crossing paths for the whole terrain. Formally, a striation $\mathcal{S}$ is an ordered set of non-crossing paths $\mathcal{S} = \{\pi_0, \ldots, \pi_r\}$ from σ to τ with $\pi_0 = \pi^-_{\partial M}$ and $\pi_r = \pi^+_{\partial M}$. Every path in a striation must be composed of MS-edges and between every two consecutive paths π_i and π_{i+1} there can be at most one MS-cell and possibly several one-dimensional features. The one-dimensional features arise from overlapping MS-edges or from the way the striation is computed. We then restrict a representative δ-network to choose paths only from the striation.

Computing a striation. The hardness result of the previous section implies that computing a striation that contains a representative δ-network (with the most lowest paths) is NP-hard. We therefore use a heuristic to compute a striation. Our heuristic uses the persistence of local maxima, which can easily be computed from the standard MS-complex [3].

The first path π is obtained by computing the lowest path from source to sink that passes through the maximum q with the highest persistence (excluding v_∞). Since π actually consists of two lowest paths (from source to q, and from q to sink), π has the form of a path π' with a special vertex v from which there is a path to q and back to v (Lemma 1). We now subdivide G as follows. Let c be the MS-cell containing q, and let

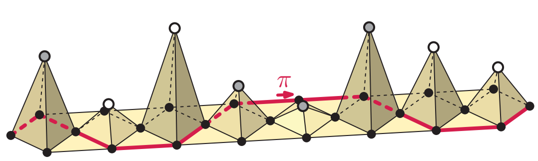

Figure 2: The path π forms a partition of the total sand volume.

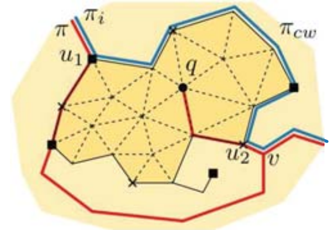

Figure 3: The lowest path π through q and π_0.

u_1 and u_2 be the first and last vertices of π that are on the boundary of c, respectively (see Fig. 3). Furthermore, let π_{cw} and π_{ccw} be the paths between u_1 and u_2 along the boundary of c in clockwise and counterclockwise direction, respectively. We can now obtain the path π_i as the concatenation of the subpath of π from σ to u_1, the path π_{cw}, and the subpath of π from u_2 to τ. Similarly, we can obtain π_{i+1} by replacing π_{cw} by π_{ccw} in π_i. If $u_1 = u_2$, then either π_i or π_{i+1} may backtrack from u_1. In that case we can replace the respective path with π'. The paths π_i and π_{i+1} subdivide G into three parts (see Fig. 4): $D_1 = D(\pi^-_{\partial M}, \pi_i)$, $D_2 = D(\pi_i, \pi_{i+1})$, and $D_3 = D(\pi_{i+1}, \pi^+_{\partial M})$. Since D_2 contains only one MS-cell, we recurse only in D_1 and D_3 to obtain $\mathcal{S}_1$ and $\mathcal{S}_3$. The final striation then consists of $\mathcal{S} = \{\pi^-_{\partial M}, \mathcal{S}_1, \pi_i, \pi_{i+1}, \mathcal{S}_3, \pi^+_{\partial M}\}$.

5 Representative network

To compute a representative δ-network conforming to a striation, we first need to define when two paths in the striation are δ-dissimilar.

Sand function. To define the dissimilarity for two paths π_i and π_{i+1} of the striation we define a sand function $d(\pi_i, \pi_{i+1})$. Intuitively, we define $d(\pi_i, \pi_{i+1})$ in such a way that it measures the volume of sand that lies between π_i and π_{i+1}. To that end we compute a homotopy (continuous morph) $\eta\colon [0,1]^2 \to M$ between π_i and π_{i+1}. We attach a height function $\zeta\colon [0,1]^2 \to \mathbb{R}$ to this homotopy, thus defining a surface in $\mathbb{R}^3$

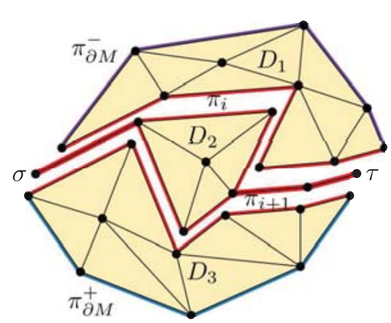

Figure 4: Splitting the triangulation by the striation paths around an MS-cell.

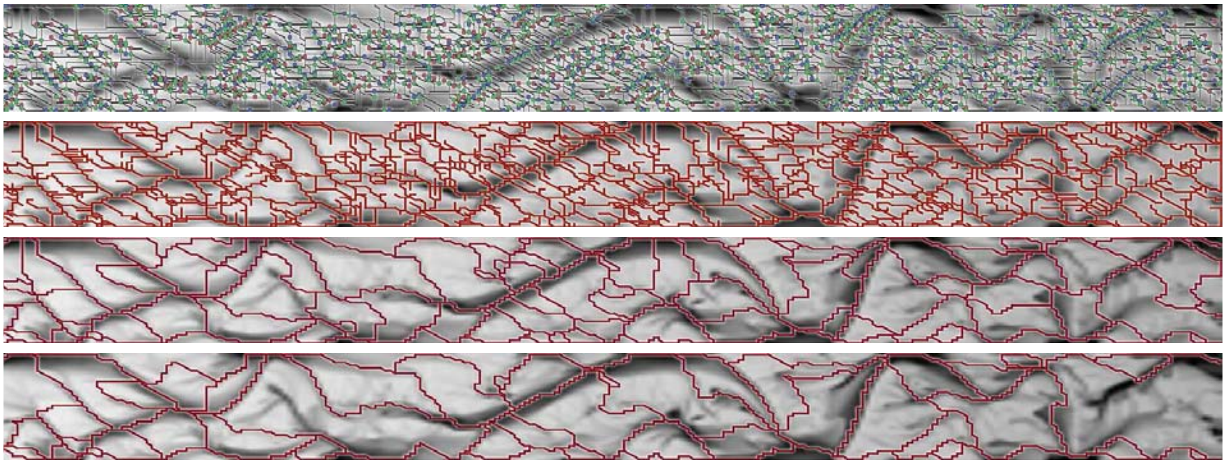

Figure 5: A modeled river with the MS-complex, the striation, and two representative δ-networks for different δ.

between π_i and π_{i+1}. We define $d(\pi_i, \pi_{i+1})$ as the volume of sand above this surface. To ensure that only sand between π_i and π_{i+1} is measured, and without multiplicity, we restrict η to be a monotone isotopy[1]. Choosing a suitable isotopy is non-trivial; details can be found in the full paper. We can extend the choices for π_i and π_{i+1} to apply to all paths π_i and π_j of the striation.

Representative network. Finally, we can easily compute a representative δ-network Π using a simple greedy algorithm. We consider all paths in the striation starting with the lowest path. We add a path π to Π if $d(\pi, \pi_i) \geq \delta$ for all $\pi_i \in \Pi$. It is easy to see that the resulting δ-network is representative.

6 Experimental results

We performed experiments on a numerically modeled river, created by a state-of-the-art model suite that is used in the civil engineering and fluvial and coastal morphology disciplines worldwide, indicating its usefulness and quality [9]. Figure 5 shows our results. First of all we observe that the representative networks capture the channels of the river very well. Second, we see that the channels which are removed in the sparser network cross big bars. Furthermore, the sparser network also avoids deep short channels which are no longer connected or did not form as a channel at all. Hence the sparser network is indeed more representative for the river than the more complex one.

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