Self-approaching paths in simple polygons

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1 Introduction

The problem of finding an optimal obstacle-avoiding path in a polygonal domain is one of the fundamental problems of computational geometry. Often a desired path has to conform to certain constraints. For example, a path may be required to be monotone [3], curvature-constrained [9], have no more than k links [16], etc. A natural requirement to consider is that a point moving along a desired path must always be getting closer to its destination. Such *radially monotone paths* appear, for example, in greedy geographic routing in network setting [10] and beacon routing in geometric setting [5]. A strengthening of a radially monotone path is a *self-approaching path* [12, 13, 1]: a point moving along a self-approaching path is always getting closer not only to its destination but also to all the points on the path ahead of it. There are several reasons to prefer self-approaching paths over radially monotone paths. First, unlike for a radially monotone path, any subpath of a self-approaching path is self-approaching. Therefore, if the destination is not known in advance and the desired path is required to be radially monotone, one will have to resort to using self-approaching paths. Second, the length of a radially monotone path can be arbitrarily large in comparison with the Euclidean distance between the source and the destination points, whereas self-approaching paths have a bounded detour.

In this paper we study self-approaching paths that are contained in a simple polygon. We consider the following questions:

- Given two points s and t inside a simple polygon P, does there exist a self-approaching s-t path inside P?
- Find the shortest self-approaching s-t path.
- Given a polygon P, test if it self-approaching, *i.e.*, that there exists a self-approaching path between any two points in P.

Related work. Self-approaching curves were first introduced in the context of online searching for a kernel of a polygon [12], and further studied in [13]. An equivalent definition of a self-approaching path is that for every point on the path there has to be a 90° angle containing the rest of the path. Aichholzer et al. developed a generalization of self-approaching paths for an arbitrarily fixed angle α instead of 90°. A relevant type of paths are increasing chords paths [18], which are self-approaching in both directions. The nice properties of self-approaching and increasing chords paths and their potential to be applied in network routing were recognized by the graph drawing community. As a result, a number of papers appeared in the recent years on self-approaching and increasing chords graphs [2, 8, 17].

This paper is organized in the following way. We introduce a few definitions and concepts in Section 2. In Section 3, we characterize a shortest self-approaching path between two points in a simple polygon. In Section 4 we present an algorithm to construct the shortest self-approaching path between two points if it exists, or to report that it does not exist, by assuming a model of computation in which we can solve certain transcendental equations. Finally, in Section 5 we present a linear-time algorithm to decide if a polygon is self-approaching, that is, if there is a self-approaching path between any two point of the polygon. Refer to the full version of this paper for the omitted proofs.

2 Preliminaries

A *self-approaching path* π in a continuous domain is a piece-wise smooth[1] oriented curve such that for any three points a, b, and c that appear on the curve in this order: $|ac| \geq |bc|$, where $|ac|$ and $|bc|$ are Euclidean distances.

Icking et al. showed the following *normal property* of a self-approaching path, that we will be using extensively in this paper,

Lemma 1 (the normal property [13]) *An s-t path π is self-approaching if and only if any normal to π at any point $a \in \pi$ does not cross $\pi(a, t)$.*

A normal h to a directed curve π at some point $a \in \pi$ defines two half-planes. Let the *positive half-plane* h^+ be the open half-plane which is congruent with the direction of π at point a. We can rephrase the normal property in the following way.

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[1] Some previous works do not require the curve to be smooth. However in this paper we will be mostly considering shortest self-approaching paths, and thus the requirement on smoothness is justified.

Lemma 2 (the half-plane property) *An s-t path π is self-approaching if and only if, for any normal h to π at any point $a \in \pi$, the subpath $\pi(a, t)$ lies completely in the positive half-plane h^+.*

A *bend* of a self-approaching path π is a point of discontinuity of the first derivative of π.

A *reachable region* $\mathcal{R}(s) \subseteq P$, for a given point s in a polygon P, is a set of all points $t \in P$ for which there exists a self-approaching s-t path $\pi \in P$.

A *reverse-reachable region* $\mathcal{R}^{-1}(t) \subseteq P$, for a given point t in a polygon P, is a set of all points $s \in P$ for which there exists a self-approaching s-t path $\pi \in P$.

2.1 Involutes

In the following sections we will show that shortest self-approaching paths consist of straight-line segments, circular arcs, and involutes of circular arcs of some order. In the full version of this paper we introduce involute curves, and show the derivation of the following formula for an involute of a circle of order k:

$$I_k(\theta) = \sum_{0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i a_{2i}(\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - \sum_{1}^{\lceil \frac{k}{2} \rceil} (-1)^{i-1} a_{2i-1}(\theta) \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix},$$

where $a_i(\theta) = r_0 \frac{\theta^i}{i!} + c_1 \frac{\theta^{i-1}}{(i-1)!} + \cdots + c_i$.

Given a point $p_i(r_i, \varphi_i)$ for each involute I_i of order i (for all $1 \le i \le k$), the constants c_i can be found from the following equations:

$$\begin{aligned} r_i \cos(\theta_i - \varphi_i) &= a_0(\theta_i) - a_2(\theta_i) + \dots, \\ r_i \sin(\theta_i - \varphi_i) &= a_1(\theta_i) - a_3(\theta_i) + \dots, \end{aligned} \quad (1)$$

where θ_i is the parameter at which involute I_i passes through p_i. The length of the tangent segment from the point p_k to the involute I_{k-1} is $|a_k(\theta_k)|$.

3 Properties of a shortest self-approaching path

Next, we prove that a shortest self-approaching path is unique, and that the shortest self-approaching path consists of straight segments, circular arcs and involutes to the later pieces of the path.

We begin with proving several lemmas:

Lemma 3 *For any two points p_1 and p_2 (in this order) on a self-approaching s-t path π in $\mathbb{R}^2$, the perpendicular bisector of straight-line segment $\overline{p_1 p_2}$ does not intersect sub-path $\pi(p_2, t)$.*

Lemma 4 *Bends of a shortest self-approaching path in a simple polygon P form a subset of vertices of P.*

Lemma 5 *A shortest self-approaching s-t path in a simple polygon P cannot have an inflection point (or an inflection segment) that is interior to P.*

Define the *inflection* points of a directed geodesic path γ from s to t as the first points of the inflection segments of γ, *i.e.*, the set of last points in the maximal subchains of γ with the same direction of turn.

Lemma 6 *A shortest self-approaching path from s to t in a simple polygon P contains all the inflection points of the geodesic path from s to t.*

Consider two self-approaching paths π_1 and π_2 from s to t in a simple polygon P that do not have other points in common. Let γ be a geodesic path from s to t inside the area bounded by π_1 and π_2.

Lemma 7 *Geodesic path γ between two self-approaching paths π_1 and π_2 is also self-approaching.*

As a corollary to this lemma, for two self-approaching paths from s to t, a path, composed of geodesics in the areas bounded by subpaths of the two paths between each pair of consecutive intersection points, is also self-approaching. In other words, let $s = p_0, p_1, \dots, p_k, p_{k+1} = t$ be all the intersection points of π_1 and π_2 in the order they appear on π_1 and π_2. Observe, that the intersection points must appear in the same order along the both paths, otherwise there would exist three points on one of these paths for which the inequality in the definition of a self-approaching path would not be satisfied. Let γ_i be the geodesic from p_i to p_{i+1} in the area between two subpaths $\pi_1(p_i, p_{i+1})$ and $\pi_2(p_i, p_{i+1})$. Then,

Lemma 8 *The concatenation of the geodesics $\gamma = \gamma_0 \oplus \gamma_1 \oplus \cdots \oplus \gamma_k$ is self-approaching.*

Lemma 9 *For a given polygon P and points s and t in it, the convex hull of the shortest self-approaching path π^* is contained in the convex hull of any self-approaching s-t path π. That is $CH(\pi^*) \subseteq CH(\pi)$.*

The next theorem is a direct corollary of Lemma 8.

Theorem 10 *A shortest self-approaching s-t path is unique.*

Fig. 1 shows an example of a shortest self-approaching path inside a polygon. In the next two theorems and lemma we give its characterization.

Theorem 11 *The shortest self-approaching s-t path in a simple polygon consists of straight segments, circular arcs and circle involutes of some order.*

A path π is called *geodesically convex* when the shortest path connecting any two points of π lies completely on one (and the same side) of π.

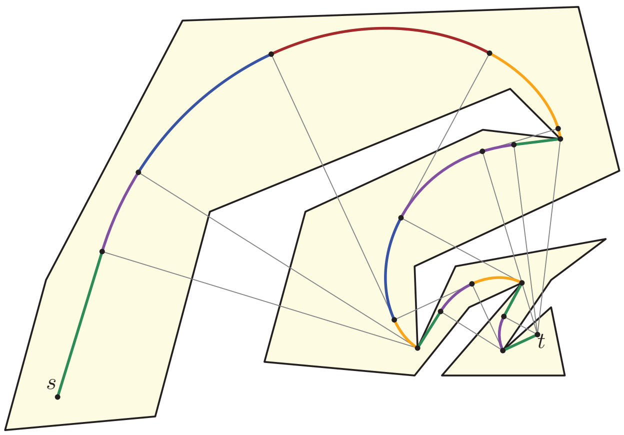

Figure 1: The shortest self-approaching path from s to t consists of line segments (green), circular arcs (purple), and involutes of a circle of some order (1st order in orange, 2nd—in blue, and 3rd—in brown).

Lemma 12 *The shortest self-approaching s-t path in a simple polygon P consists of geodesically convex paths between inflection points of the s-t geodesic.*

Theorem 13 *The shortest self-approaching s-t path in a simple polygon P consists of $O(n^2)$ segments. A shortest self-approaching s-t path may require $\Omega(n^2)$ segments.*

4 Existence of a self-approaching path

In this section we consider the question of testing whether, for given points s and t in a polygon P, they can be connected with a self-approaching path. In Theorem 11 we proved that a shortest self approaching path can consist of involutes of a circle of a high order, and in Section 2 we showed that such an involute is defined by a system of transcendental equations. In [15] Laczkovich proved a strengthening of Richardson's theorem, that states that the statement $\exists x : f(x) = 0$ is undecidable, where $f(x)$ is an expression generated by the rational numbers, the variable x, the operations of addition, multiplication, and composition, and the sine function. The Equations (1) that we need to solve to obtain formulas for the involutes are a special case of the class of expressions in Laczkovich's theorem. Nevertheless, it strongly suggests that an involute of a circle of order higher than one cannot be computed.

Next, we show an algorithm to test whether there exists a self-approaching path connecting two points s and t, and if so, to compute the shortest path, under the assumption that we can solve Equations (1). Subsequently, it may be possible to release this assumption, and modify the algorithm to build an approximation to the shortest path.

4.1 Shortest path algorithm

The proof of Theorem 11 is constructive. Let us assume that we can solve equations of the form as Equations (1) for an involute of order k in time $O(f(k))$, and evaluate the formula of the involute of order k for a given parameter θ in time $O(g(k))$. Then, we can decide if two points s and t can be connected by a self-approaching path, and we can construct the shortest path between the points. The outline of the algorithm:

Starting at t, move backwards along a geodesic s-t path γ. Maintain the convex hull CH of the final part of the shortest self-approaching path π^* to the destination t built so far. At every bend point p_ℓ:

(a) Calculate the appropriate branch of an involute I_{CH} of CH. If I_{CH} intersects the opposite boundary of the polygon, thus, cutting off s from t, report that a self-approaching path from s to t does not exist and terminate the algorithm.

(b) Otherwise, find a geodesic path γ_ℓ from the preceding inflection point of γ to p_ℓ in $P \backslash I_{CH}$, and add its last segment qp_ℓ as a prefix to π^*.

(c) Update CH. Repeat for the new bend point q, until s is reached. Report the found path π^*.

To obtain an algorithm with an optimal running time, there are a few considerations to take into account when constructing the shortest path. First, instead of unnecessarily calculating the whole involute I_{CH} until the intersection point with the boundary of P, and then discarding the part of it under the tangent line from q, its segments can be calculated one by one as needed until the tangent point. Second, to optimally test if I_{CH} intersects the opposite boundary of the polygon, we can maintain a shortest path tree that will allow us to build funnels from the opposite sides of the polygon boundary. Third, it is not necessary to construct the whole geodesic γ_ℓ to be able to compute its last segment qp_ℓ. Instead, we can move backwards along γ, vertex by vertex, until we reach a point from which a tangent to I_{CH} can be computed (possibly with adding new points along it).

Theorem 14 *The shortest self-approaching s-t path, if it exists, can be constructed in $O(k + \frac{n \log k}{\sqrt{k}}(g(\sqrt{k}) + f(\sqrt{k})))$ time, where k is the size of the output.*

5 Self-approaching polygon

A polygon is self-approaching, if for any two points there exists a self-approaching path connecting them.

Theorem 15 *Polygon P is self-approaching if and only if for any disk D centered at any point $p \in P$, the intersection $D \cap P$ has one connected component.*

Corollary 16 *Any self-approaching polygon is also increasing-chord.*

Next, we present an algorithm to test whether a given simple polygon P is self-approaching. From the proof of Theorem 15 it follows that the polygon P is self-approaching iff, for all edges e on the boundary of P directed in the counter-clockwise order, an area bounded between the two normals to e at its two end points in the right half-plane of e is free of ∂P. We call this area the *half-strip* of e. We will use this property to test efficiently if the polygon is self-approaching.

Let P be given as a set of points $p_0, p_1, \ldots, p_{n-1}$ in the counter-clockwise order around the boundary. We will start at p_0, move along the boundary in the counter-clockwise order and maintain the union of all the half-strips of the edges visited so far. More precisely, we will maintain the left and the right sides, ρ_l and ρ_r, of the hour-glass shape that is the union of the half-strips; ρ_l and ρ_r are convex polygonal chains. Store the segments of ρ_l and ρ_r as two lists, the last segments in the lists are infinite rays.

At every iteration of the algorithm, perform the following steps. Let p_i be the current point of the polygon P. The chain ρ_r contains the right side of the union of all the half-strips up to point p_{i-1}. Consider the next boundary segment $\overline{p_{i-1}p_i}$, and a perpendicular ray h_i at the point p_i. To update the chain ρ_r, do the following: For each segment $\overline{c_j c_{j+1}}$ on ρ_r,

- if $\overline{p_{i-1}p_i}$ intersects $\overline{c_j c_{j+1}}$, then report that P is not self-approaching and terminate;
- if h_i intersects $\overline{c_j c_{j+1}}$, calculate the intersection point c', and replace the first elements of the list ρ_r up to $\overline{c_j c_{j+1}}$ with two segments, $\overline{p_i c'}$ and $\overline{c' c_{j+1}}$; repeat for the next point p_{i+1}.

Traverse the boundary of polygon P twice in the counter-clockwise order, and then repeat the same algorithm traversing the boundary of P twice in the clockwise order. If none of the segments $\overline{p_{i-1}p_i}$ intersected a segment of ρ_r, report that P is self-approaching.

Theorem 17 *Given a simple polygon P with n vertices, the presented algorithm tests in $O(n)$ time if it is self-approaching.*

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