Aligned drawings of planar graphs

Aligned Drawings of Planar Graphs

Tamara Mchedlidze* Marcel Radermacher* Ignaz Rutter†

Abstract

Let G be a planar embedded graph and $\mathcal{A}$ be a set of pseudolines passing through G. An *aligned* drawing of G and $\mathcal{A}$ is a planar polyline drawing Γ of G with an arrangement A of lines so that Γ and A have the same topological properties as G and $\mathcal{A}$. We study this problem restricted to two pseudolines. We show that if every edge of a graph is intersected by at most one pseudoline, then the instance has a straight-line aligned drawing. This implies that every configuration of a planar graph with two pseudolines has an aligned drawing with at most one bend. In order to prove this result, we strengthen the result of Da Lozzo et al. [3], and prove that a planar graph G and a pseudoline $\mathcal{C}$ have an aligned drawing with a prescribed convex drawing of the outer face.

1 Introduction

Two fundamental primitives for highlighting structural properties of a graph in a drawing are alignment of vertices such that they are collinear and geometrically separating unrelated graph parts, e.g., making them separable by a straight line. Not surprisingly, both these techniques have been previously considered from a theoretical point of view in the case of planar straight-line drawings.

Da Lozzo et al. [3] study the problem of producing a planar straight-line drawing of a given embedded graph $G = (V,E)$, i.e., G has a fixed combinatorial embedding and outer face, such that a given set $S \subseteq V$ of vertices is collinear. It is clear that if such a drawing exists, then the line containing the vertices in S is a curve starting and ending at infinity that for each edge e of G either fully contains e or intersects e in at most one point, which may be an endpoint. We call such a curve a *pseudoline* with respect to G. Further, the pseudoline contains all the vertices in S. Da Lozzo et al. [3] show that this is a full characterization of the alignment problem, i.e., a straight-line drawing where the vertices in S are collinear exists if and only if there exists a pseudoline $\mathcal{L}$ with respect to G that contains the vertices in S.

Likewise, for the problem of separation, Biedl et al. [1] considered so-called *HH*-drawings, where given an embedded graph $G = (V,E)$ and a partition $V = A \dot\cup B$, one seeks a planar straight-line drawing of G in which A and

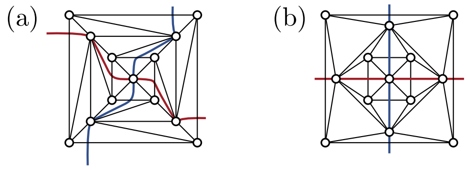

Figure 1: Aligned Drawing (b) of a 2-aligned planar graph (a). The pseudolines $\mathcal{R}$ and $\mathcal{B}$ and the corresponding lines in the drawing are drawn red and blue, respectively.

B can be separated by a line. Again, it turns out that such a drawing exists if and only if there exists a pseudoline $\mathcal{L}$ with respect to G such that the vertices in A and B are separated by $\mathcal{L}$ in the sense that they are in different *halfplanes* defined by $\mathcal{L}$.

In particular, the results of Da Lozzo et al. [3] show that given a pseudoline $\mathcal{L}$ with respect to G one can always find a planar straight-line drawing of G such that the vertices on S are collinear and the vertices contained in the halfplanes defined by $\mathcal{L}$ can be separated by a line L. In other words, a topological configuration consisting of a planar graph G and a pseudoline with respect to G can always be stretched. In this paper we initiate the study of this stretchability problem with more than one given pseudoline.

More formally, a tuple $(G, \mathcal{C}_1, \dots, \mathcal{C}_k)$ is a *k-aligned graph* if $G = (V,E)$ is a planar embedded graph and $\mathcal{C}_1, \dots, \mathcal{C}_k$ are pseudolines with respect to G. We further require that each pair $\mathcal{C}_i$ and $\mathcal{C}_j$ with $i \neq j$ intersects precisely once. If the number k of curves is clear from the context, we drop it from the notation and simply speak of *aligned graphs*. A tuple $(\Gamma, L_1, \dots, L_k)$, where each L_i is a line and Γ is a planar drawing of G, is an *aligned drawing of* $(G, \mathcal{C}_1, \dots, \mathcal{C}_k)$ if and only if the following properties hold; refer to Fig. 1.

1. the arrangement of $L_1, \dots, L_k$ is isomorphic to the arrangement of $\mathcal{C}_1, \dots, \mathcal{C}_k$,
2. Γ is homeomorphic to the planar embedding of G,
3. each line L_i intersects in Γ the same vertices and edges as $\mathcal{C}_i$ in G, and it does so in the same order.

We focus on straight-line aligned drawings. For brevity, unless stated otherwise, the term aligned drawing refers to a straight-line drawing throughout this paper.

This convention generalizes the problems studied by Da Lozzo et al. and Biedl et al. who concentrated on the case of a single line. We study a natural extension of their setting and ask for an alignment on two lines. We note that Da Lozzo et al. and Biedl and et al. focus on alignment and separation of given vertices, respectively. Their characteri-

*Department of Computer Science, Karlsruhe Institute of Technology, Germany, radermacher@kit.edu, mched@iti.uka.de
†Department of Mathematics and Computer Science, TU Eindhoven, The Netherlands, i.rutter@tue.nl

zations in terms of existence of pseudolines allows them to abstract from geometry and to construct the pseudolines in a purely combinatorial setting. In contrast to that, we are given multiple pseudolines as part of the input.

In Section 3, we strengthen the result of Da Lozzo et al. and Biedl et al. and show that there exists an aligned drawing of G with a fixed convex drawing of the outer face. In Section 4, we show that every aligned graph $(G, \mathcal{R}, \mathcal{B})$ with each edge intersected by at most one pseudoline has an aligned drawing. This immediately implies that every aligned graph $(G, \mathcal{R}, \mathcal{B})$ has an aligned drawing with at most one bend.

In addition to the strongly related work mentioned above, there are several other works that are related to the alignment of vertices in drawings. Dujmović [4] shows that every n-vertex planar graph $G = (V, E)$ has a planar straight-line drawing such that $\Omega(\sqrt{n})$ vertices are aligned, and Da Lozzo et al. [3] show that in planar treewidth-3 graphs, one can align $\Theta(n)$ vertices and that in treewidth-k graphs one can align $\Omega(k^2)$ vertices. Chaplik et al. [2] study the problem of covering all edges of a planar graph with a small set of lines. They show that it is $\mathcal{NP}$-hard to decide whether a graph has such a cover of size at most k. The complexity of covering all vertices by at most k lines is open. Dujmović et al. [5] show that there is no set of n lines intersecting in one point that supports all n-vertex planar graphs.

2 Preliminaries and Proof Strategy

Let $G = (V, E)$ be a planar embedded graph with a vertex set V and an edge set E. We call $v \in V$ *interior*, if v does not lie on the boundary of the outer face of G. An edge $e \in E$ is *interior*, if e does not lie entirely on the boundary of the outer face of G. An interior edge is a *chord* if it connects two vertices on the outer face. A point p of an edge e is an *interior* point of e, if p is not an endpoint of e. A *pseudosegment* is a connected bounded subset of a pseudoline. A *triangulation* is a planar graph whose inner faces are all triangles. A *triangulation* of a graph G is a triangulation that contains G as a subgraph. For a graph G and an edge e of G, the graph G/e is obtained from G by contracting e and merging the resulting multiple edges. A *k-wheel* is a wheel graph W_k with k vertices on the outer face and one interior vertex.

Our general strategy for proving the existence of aligned drawings is as follows. First, we show that we can triangulate our instance by adding vertices and edges without invalidating its properties. We can thus assume that our graph G is a triangulation. Second we show that, unless G is very small, e.g., a k-wheel or a triangle, it contains a specific type of edge, namely an edge that is contained in a pseudoline, or an edge that is not intersected by any of the pseudolines. Third, we exploit the existence of such an edge to inductively prove the existence of an aligned drawing of G. Depending on whether the edge is contained in a separating triangle or not, we either decompose along that triangle or contract the edge. In both cases the problem reduces to smaller instances that are almost independent. In order to combine solutions it is, however, crucially important to use the same line R for both of them.

3 Aligned Drawings with One Pseudoline

We show that every aligned graph $(G, \mathcal{R})$ has an aligned drawing (Γ, R). For this extended abstract, we assume that G is 2-connected.

Depending on their relationship to $\mathcal{R}$ the edges of $(G, \mathcal{R})$ are characterized as follows. An edge e of G is *gray* if it does not intersect the curve $\mathcal{R}$. The edge e of G is *red* if it lies entirely on the red curve. Otherwise an edge intersects with $\mathcal{R}$ once and we refer to this edge as *monochromatic*. A vertex on $\mathcal{R}$ is *red* and otherwise *gray*. An *aligned triangulation* of $(G, \mathcal{R})$ is an aligned graph $(G_T, \mathcal{R})$ with G_T a triangulation of G and the same outer face as G.

The following Lemma 1, Lemma 2 and Theorem 3 , in this order, implement the three steps of our proof strategy as outlined above. The proofs are omitted since proofs with similar arguments are given in Section 4.

Lemma 1 *Every aligned graph $(G, \mathcal{R})$ admits an aligned triangulation whose size is linear in G.*

Lemma 2 *Let $(G, \mathcal{R})$ be an aligned triangulation with k vertices on the outer face without a chord. If G is neither a triangle nor a k-wheel, then $(G, \mathcal{R})$ contains a red or a gray edge.*

Proof sketch. Observe the following two properties of aligned triangulation without chords: (i) two red vertices that are consecutive on $\mathcal{R}$ are connected by a red edge, (ii) if G has no red edge, every interior gray vertex is incident to an interior gray edge. With these observations we can show that, if $(G, \mathcal{R})$ does not contain a red or gray edge, there is at most one interior vertex which is red. ☐

Theorem 3 *Let $(G, \mathcal{R})$ be an aligned triangulation and let (Γ_O, R) be an aligned convex drawing of the aligned outer face $(O, \mathcal{R})$ of G. There exists an aligned drawing (Γ, R) of $(G, \mathcal{R})$ with the same line R and the outer face drawn as Γ_O.*

Given an arbitrary aligned graph $(G, \mathcal{R})$, we can first triangulate it using Lemma 1 and then draw it with Theorem 3. Then the drawing of $(G, \mathcal{R})$ is obtained from this drawing by removing additional vertices and edges.

Corollary 4 *Every aligned graph $(G, \mathcal{R})$ admits an aligned drawing (Γ, R) with a fixed aligned convex drawing (Γ_O, R) of the aligned outer face $(O, \mathcal{R})$.*

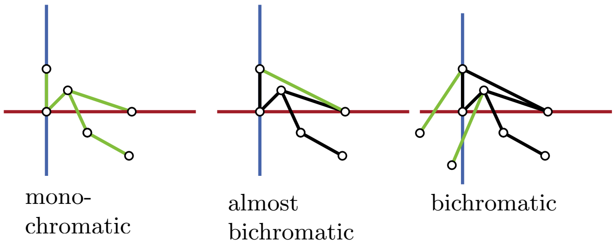

Figure 2: Examples of a monochromatic and a (almost-) bichromatic graph.

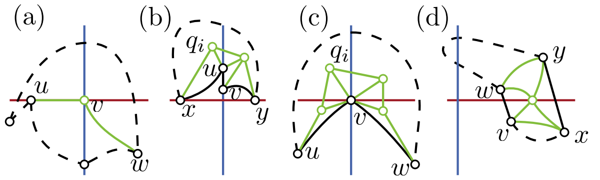

Figure 3: Steps for triangulating almost-bichromatic graphs (black) with monochromatic edges (green).

4 Aligned Drawings with Two Pseudolines

In this section, we show that every aligned graph $(G, \mathcal{R}, \mathcal{B})$, where each edge is intersected by at most one of two pseudolines, has an aligned drawing.

The intersection of the four halfplanes defined by $\mathcal{R}$ and $\mathcal{B}$ define four *quadrants*. A vertex or an edge is *red* or *blue* if it lies entirely on $\mathcal{R}$ or $\mathcal{B}$, respectively. A *gray* vertex or edge lies entirely in the interior of a quadrant. An edge is *monochromatic* if it either lies on a pseudoline or shares exactly one point with only one pseudoline, or one of its endpoints lies on the intersection of $\mathcal{R}$ and $\mathcal{B}$. Thus, a monochromatic edge is not necessarily red or blue. *Almost-bichromatic* edges have both endpoints on different pseudolines. Accordingly, a *bichromatic* edge e is always intersected by both pseudolines with at least one intersection point in the interior of e. Every 2-aligned graph is a *bichromatic graph*. An *almost-bichromatic* graph is a bichromatic graph without bichromatic edges. A *monochromatic graph* is an almost-bichromatic graph without almost-bichromatic edges; see Fig. 2.

A *2-aligned triangulation* of $(G, \mathcal{R}, \mathcal{B})$ is a 2-aligned graph $(G_T, \mathcal{R}, \mathcal{B})$ with G_T a triangulation of G whose outer face is a 4-cycle with each vertex in a different quadrant. We start with the triangulation step of our proof strategy.

Lemma 5 *Let $(G, \mathcal{R}, \mathcal{B})$ be a monochromatic aligned graph. There exists a monochromatic aligned triangulation $G_T = (V_T, E_T)$ of G whose size is linear in the size G.*

Proof sketch. Insert a 4-cycle in the unbounded region of G with each vertex in a different quadrant. Since $(G, \mathcal{R}, \mathcal{B})$ is monochromatic we can connect each new vertex with a monochromatic or gray edge to the outer face of G.

If f is a non-triangular face whose interior contains the intersection of $\mathcal{R}$ and $\mathcal{B}$, we insert edges uv, vw as shown in Fig. 3(a).

If f is a non-triangular face with a red (blue) edge $e = uv$ we can split f into two faces f' and f'' as shown in Fig. 3(b) such that f' contains e on its boundary. Then we can triangulate f' with monochromatic edges. A similar approach works for red (blue) vertices; see Fig. 3(c).

If f is a non-triangular face whose interior contains a pseudosegment s, then we find two edges vw, xy as shown in Fig. 3(d) and we can triangulate by inserting a vertex on s and monochromatic edges.

If none of the cases above apply, then no non-triangular face contains a part of $\mathcal{R}$ or $\mathcal{B}$. Thus all remaining non-triangular faces can be triangulated with gray edges. ☐

Next we deal with step two of our proof strategy and show the existence of a specific type of edge unless the instance is very small.

Lemma 6 *Let $(G, \mathcal{R}, \mathcal{B})$ be a monochromatic aligned triangulation that does not contain an interior red, blue, or gray edge. Then G is isomorphic to the 4-wheel.*

Proof. Our first goal is to argue that both $\mathcal{R}$ and $\mathcal{B}$ alternately intersect vertices and the interiors of edges of G.

Since $(G, \mathcal{R}, \mathcal{B})$ is a monochromatic triangulation, a vertex lies on the intersection of $\mathcal{R}$ and $\mathcal{B}$. As in the proof of Lemma 2 one can argue that if two vertices occur consecutively along $\mathcal{R}$ or $\mathcal{B}$, then we find a red or blue edge, respectively. Now assume that $\mathcal{R}$ intersects two edges e_1, e_2 consecutively. Since G is a triangulation, it follows that e_1 and e_2 share an endpoint x. Moreover, all endpoints of e_1 and e_2 must be gray. Further e_1 and e_2 are consecutive in the circular order of edges around x as otherwise we would either find an intersection with $\mathcal{R}$ between e_1 and e_2 or a gray edge. Thus, e_1 and e_2 bound a face, and hence their endpoints distinct from x are in the same quadrant and connected by an edge e, which is thus gray. Moreover, e cannot be an outer edge as the outer face is a monochromatic 4-cycle and thus does not contain gray edges. It thus follows, that both $\mathcal{R}$ and $\mathcal{B}$ alternate between vertices and edges of G. Moreover, if v is a vertex that is followed by an edge $e = uw$, then u, v, w form a triangle.

Let v be the vertex on the intersection of $\mathcal{R}$ and $\mathcal{B}$. Then there are four triangles $T_1, \dots, T_4$ around v whose edges opposite of v alternately intersect $\mathcal{R}$ and $\mathcal{B}$ in clockwise order. Let u, w be two vertices of these triangles that lie in the same quadrant, without loss of generality, $u \in T_1$ and $w \in T_2$. We show that $u = w$. If not, then let N denote the set of neighbors of v that lie clockwise between u and w together with u and w. Since G is monochromatic, all vertices in N are gray and lie in the same quadrant. Moreover, since G is a triangulation, they form a fan of triangles, and we hence find a gray edge. Since this argument applies to any two consecutive triangles, it follows that the four edges of $T_1, \dots, T_4$ opposite of v form a 4-cycle.

Consider a red (blue) vertex v not lying on the intersection of $\mathcal{R}$ and $\mathcal{B}$. The vertex is incident to two triangles $T_1 = (v, x, u), T_2 = (v, w, y)$ intersecting the pseudoline $\mathcal{R}$ ($\mathcal{B}$) with x and y in the same quadrant and u and w in the

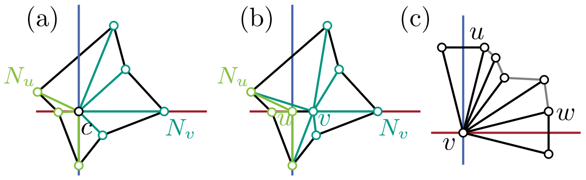

Figure 4: Unpacking an edge in a drawing Γ' of G/e (a) to obtain a drawing Γ of G (b). (c) Fan around vertex v on the intersection of $\mathcal{R}$ and $\mathcal{B}$.

same quadrant. The arguments from above immediately apply to T_1 and T_2, showing that the endpoints u,w and x,y of the triangles T_1, T_2 within the same quadrant are connected by a gray edge, if $u \neq w$ or $x \neq y$ respectively. Since G is a simple graph either $x \neq y$ or $u \neq w$. This concludes the proof. □

The next two lemmas implement step three of our proof strategy and show that the edges that exists by Lemma 6 can be used to reduce the size of the instance. The correctness of Lemma 7 follows from Corollary 4.

Lemma 7 *Let $(G,\mathcal{R},\mathcal{B})$ be a monochromatic aligned triangulation and let T be a separating triangle splitting G into subgraphs $G_{\text{in}}, G_{\text{out}}$ such that $G_{\text{in}} \cap G_{\text{out}} = T$ and G_{out} contains the outer face of G. Then $(G_{\text{out}},\mathcal{R},\mathcal{B})$ is a monochromatic aligned triangulation and G has an aligned drawing if and only if G_{out} has an aligned drawing.*

Lemma 8 *Let $(G,\mathcal{R},\mathcal{B})$ be a monochromatic triangulation and let e be a red, blue or gray edge of G that is not contained in a separating triangle. Then the graph $(G/e,\mathcal{R},\mathcal{B})$ is a monochromatic triangulation. Further, $(G,\mathcal{R},\mathcal{B})$ has an aligned drawing if $(G/e,\mathcal{R},\mathcal{B})$ has an aligned drawing.*

Proof. Observe that G/e is simple and triangulated since G does not have separating triangles. Further, if no endpoint of e lies on the intersection of $\mathcal{R}$ and $\mathcal{B}$, G/e remains aligned since e is red, blue or gray, in particular the vertex c resulting from contracting e has the same color as e. If an endpoint v of $e = uv$ lies on the intersection of $\mathcal{R}$ and $\mathcal{B}$ we place the vertex c obtained by contracting e on the intersection as well. Since every neighbor of u keeps its color, G/e remains a monochromatic triangulation.

Let (Γ', R, B) be an aligned drawing of $(G/e,\mathcal{R},\mathcal{B})$. Let Γ'' denote the drawing obtained from Γ' by removing c together with its incident edges and let f denote the face of Γ'' where c used to lie. Since, G/e is triangulated, f is star-shaped and c lies inside the kernel of f; see Fig. 4. We construct a drawing Γ of G as follows. First, we place u at the position of c and insert all edges incident to u; if one of the two vertices u, v lies on the outer face or on the intersection of $\mathcal{R}$ and $\mathcal{B}$, we assume, without loss of generality, that vertex to be u. Note that since G is an aligned triangulation, there is no red or blue edge incident to the intersection of $\mathcal{R}$ and $\mathcal{B}$ or to the outer face. This leaves a uniquely defined inner face f' in which we have to place v. Since G is triangulated, the interior of f' is intersected by at most one pseudoline. Since, we only removed a fan of triangles from f to obtain f', the face f' remains star-shaped and furthermore, all points inside f' sufficiently close to c lie in the kernel of f'. Here it is important that, if the edge is red or blue, the line R or B intersects the kernel of f', respectively. □

Altogether the above lemmas and the observation that an aligned 4-wheel always has an aligned drawing prove our main result.

Theorem 9 *Every monochromatic 2-aligned graph has an aligned drawing.*

By splitting every bichromatic edge with a vertex in the interior of the intermediate quadrant, thus making the graph monochromatic, we get the following corollary.

Corollary 10 *Every bichromatic 2-aligned graph has a polyline aligned drawing with at most one bend per edge.*

5 Conclusion

In this paper we showed that every monochromatic 2-aligned graph $(G,\mathcal{R},\mathcal{B})$ has a straight-line aligned drawing. This immediately implies one-bend aligned drawings of bichromatic aligned graphs. As a tool we showed that an aligned graph $(G,\mathcal{R})$ has an aligned drawing with a fixed convex drawing of the outer face. We conjecture that every almost-bichromatic and possibly also every bichromatic graph $(G,\mathcal{R},\mathcal{B})$ has a straight-line aligned drawing.

References

[1] Biedl, T.C., Kaufmann, M., Mutzel, P.: Drawing planar partitions II: HH-drawings. In: Hromkovič, J., Sýkora, O. (eds.) WG'98. pp. 124–136. Springer (1998)

[2] Chaplick, S., Fleszar, K., Lipp, F., Ravsky, A., Verbitsky, O., Wolff, A.: Drawing graphs on few lines and few planes. In: Hu, Y., Nöllenburg, M. (eds.) GD'16. pp. 166–180. Springer (2016)

[3] Da Lozzo, G., Dujmović, V., Frati, F., Mchedlidze, T., Roselli, V.: Drawing planar graphs with many collinear vertices. In: Hu, Y., Nöllenburg, M. (eds.) GD'16. pp. 152–165. Springer (2016)

[4] Dujmović, V.: The utility of untangling. In: Di Giacomo, E., Lubiw, A. (eds.) GD'15. pp. 321–332. Springer (2015)

[5] Dujmović, V., Evans, W., Kobourov, S., Liotta, G., Weibel, C., Wismath, S.: On graphs supported by line sets. In: Brandes, U., Cornelsen, S. (eds.) GD'10. pp. 177–182. Springer (2011)