H2, H-infinity and \( l_1 \) optimal control for active suspension system

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$H_2$, $H_{\infty}$ and $l_1$ optimal control for active suspension system

Arnaud Valenti

Internal report 96/07

Final Project.
Supervisor: Dr. S. Weiland.

Arnaud Valenti
Eindhoven University of Technology
September 1996
Presentation of the project

I have carried out my final project at the University of Technology of Eindhoven. This university is very well known for its size, its modern equipment and the quality the researchers who use it. Nearly all the scientific and technological areas are represented, be it mathematical, physical, chemical and obviously electrical engineering.

It happens that I had already done my second year course in this university last year, a course which proved to be very beneficial. I then found myself among the electrotechnics and power electronics group; this year specialization in automatic control has led to any working in the measurement and control group, in a pleasant, friendly and above all, serious atmosphere.

As for myself, I worked under the supervision of Dr. Siep Weiland on a very interesting subject: the $H_\infty$, $H_2$ and $l_1$ control whose aim is to minimize the induced norm of a multivariable system. After having spent many hours studying this new type of theory which introduces a good number of mathematical notions, I was able to apply this type of control to a concrete example: a suspension system which forms a relevant practical support.

The aim of this project is in fact to compare the different type of control, $H_\infty$, $H_2$ and $l_1$ control, in order to determine in which case it is preferable to use the one or the other. The $H_2$ control being perfectly mastered, it no longer constitutes a real field of investigation. On the other hand, the $l_1$ control is a very new theory which needs implementations.

Organisation of the report:

The subject being both theoretical and technological, it was necessary to combine these two aspects. This is why I have begun by presenting the data of the technological problem: modelisation of the system (suspension + rest of the vehicle), simplifications with a view to the control design, goals and constraints,... Then before going on directly with the results of each type of control, I have evoked the main outline of the theory on the subject of the $H_\infty$, $H_2$ and $l_1$ control in order to better present its relevance as well as the way I have proceeded in order to obtain the final results.

My main working tool was Matlab and I especially used the functions described in the Matlab user’s guide used for the automatic control. I also needed to use Maple because I had to do a lot of mathematical calculations on matrices.
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Chapter I-Introduction.

I-1-\(H_\infty, H_2\) and \(l_1\) optimal controller.

The aim of this project is to compare three types of control: \(H_\infty, H_2\) and \(l_1\) optimal control which are all of them based on minimizing the induced norm, the \(H_\infty, H_2\) and \(l_1\) norm, respectively, of whatever multivariable system, that's why they are called "optimal". Moreover, the solution for such or such controller must be obviously a stabilizing solution, that means the closed loop system with the state feedback controller must be stable. (Otherwise it makes no sense). This notion of optimal control enables to reduce the effects of the inputs over the outputs, that's why this type of control is relevant if the exogenous inputs correspond to perturbations or disturbances because the optimal control has got the property to minimize their influence on the outputs.

The suspension system for transport vehicle is thus an interesting application for this type of control because the road profil can be seen as a perturbation by the vehicle.

I-2-The suspension systems for transport vehicles.

I-2-1-passive suspension and active suspension.

With almost all the suspension the required performances can be achieved if the speed of the vehicle is not too large. For a given speed the chosen design parameters are optimal for certain types of road, but not for each one. It is possible to improve this if an active suspension is used instead of a passive one. An actuator is added, which can generate a controller-specified force between the two relevant components of the system, axle and chassis. The different objectives can be translated in terms of norm bounds on certain transfert fonctions or impulse responses. This will be explained later in chapter II-3.

I-2-2-Historic and commercial aspect

The design of active suspensions using optimal control theory was proposed for the first time several decades ago. At that time, a realisation was not economically relevant because of the high cost of the required equipment. Nowadays, the situation has changed. Several passagers manufacturers already provide some types of active suspensions as option. Thoses suspensions are also developped for trucks by any manufactures. One of them is DAF Truck BV in The Nederlands and this rapport is about one of their researchs.
Chapter II-Problem formulation

II-1-Models specifications

It is not really known which part plays the largest role in determining the passenger and load comfort. Let us assume that the rear suspension is the critical one, which is therefore the prime target for an active suspension. To make all three parts of the suspension system active is not necessary and moreover it is too much expensive.

After a few simplifications, a linear model of the suspension system can be defined with ten second order linear differentials equations as follows:

\[ M \ddot{q} + B \dot{q} + K q = B_u u + B_w w \]

where M is the mass matrix, B is the damping matrix, and K is the stiffness matrix. The matrices \( B_u \) and \( B_w \) indicate how the control input u and the road disturbances w act on the system.

In my case, this equation can be simplified again to make the II optimal control problem feasible. I have done some new simplifications to reach a problem with only one degree of freedom which will be introduced in the next chapter.

II-2-Road models

It is necessary to find a model for the main road disturbances which correspond to an input of the suspension system. Two approaches are taken into account in order to describe as well as possible the road irregularities:

- The height of the road can be considered as a stochastic process. It is characterized by a Gaussian probability density function for its amplitude and a power spectral density.
as a function of spatial frequency for its spatial characteristics. For more informations see [1].

The second approach corresponds to the deterministic road irregularities which are caused for instance by pot holes, bumping or damaged roads. For this kind of disturbances a commonly used signal is a step input, but it is better, in my opinion, to use models which represent as accurately as possible the road profil. The most popular ones are rounded pulses, where the road height $q_r$ is defined as a function of the horizontal position:

$$q_r(r) = q_{\text{max}} \left( \frac{r}{l_d} \right)^2 e^{-\left(\frac{2\pi r}{l_d}\right)^2}$$

If the speed $v$ is constant ($r=vt$), then the road profil is a function of the time $t$ and is parameterized by $q_{\text{max}}$ and $l_d/v$.

The table II.2.1 presents the values of parameters for different type of rounded pulse and figure II.2.1 shows the five corresponding profiles as functions of time. See [1].

<table>
<thead>
<tr>
<th>rounded pulse</th>
<th>$q_{\text{max}}$ [m]</th>
<th>$l_d/v$ [s]</th>
</tr>
</thead>
<tbody>
<tr>
<td>tiny</td>
<td>0.0095</td>
<td>0.0116</td>
</tr>
<tr>
<td>small</td>
<td>0.0315</td>
<td>0.049</td>
</tr>
<tr>
<td>medium</td>
<td>0.0909</td>
<td>0.237</td>
</tr>
<tr>
<td>large</td>
<td>0.1216</td>
<td>0.500</td>
</tr>
<tr>
<td>huge</td>
<td>0.1886</td>
<td>1.000</td>
</tr>
</tbody>
</table>

Table II.2.1: values of the parameters.

![Fig II.2.1: Deterministic road profil](image)

II-3-Goals
- Position two system components with respect to each other or to the environment (connection goal).
- Prevent the transmission of vibrations between the components (isolation goal).

These goals are in conflict. Indeed, the best isolation occurs when the components are not connected and the best connection that means the more stiff one does not isolate the vibrations.

These position constraints can be translated in terms of limitations on the dynamic tire forces and on the suspension deflections. The isolation goal involves low levels of accelerations on driver and cargo (for comfort and limited packaging requirements). Moreover, low acceleration levels extend the lifetime of the mechanical parts of the system, especially the chassis and the axles.

Requirements:
- Comfort: It operates at the level of vertical acceleration. The study carried out (see [1]) showed which levels of acceleration were acceptable to human beings during a specific time. The output filter corresponding to the vertical acceleration derives from this study. We mean to find a vertical acceleration as small as possible with noise as low as possible.
- Spatial limits concerning the distance between the chassis and the cabin: From a certain point on, compression or extension is no longer possible. This would damage the device. We want a maximum compression of 0.15 m and an extension of 0.09 m.
- Road holding of the vehicle: The problem is not with a bend (bidimensional problem); on the other hand, the braking must be taken into consideration, it is only possible if the force exercised by the tires on the road is non-zero, and given the stiffness of the spring, the considered force can be expressed in terms of tire deflection, whose extension should not be more than 0.024 m and compression 0.08 m. See [1].
- Practical realization: The active system must not need too much power, if not, it will overheat (cooling problem). Consequently, the force exercised by the actuator must also be limited: 100 kN.

Thus we should have four signals as output to-be-controlled.
Chapter III-Simplified version

III-1-Schem and specifications

After a lot of simplifications it is quite relevant to use the following model to implement active suspensions.

The actuator is between the chassis and the tire axle.

Specifications:

Ms=1000 kg, chassis mass.

Mu=1500 kg, rear axle mass.

Kt=5.10^6 N/m, rear tire stiffness.

Ks=5.10^5 N/m, rear suspension stiffness.

bs=5.10^4 Ns/m, rear suspension damping.

mechanical equations:
III-2-Inputs and outputs definition.

We shall here elaborate a generalized plant of the system, which includes the model using for the suspension system as well as the different filter, (block G) and the regulator, (block K). The diagram appears under the following general form:

\[
M_s \ddot{q}_2 + b_s (\ddot{q}_2 - \dot{q}_1) + k_s (q_2 - q_1) = u
\]  

(III-1.1)

\[
M_u \ddot{q}_1 + b_s (\dot{q}_1 - \ddot{q}_2) + k_s (q_2 - q_1) + k_i (q_1 - q_0) = -u
\]  

(III.1.2)

- \( w \) corresponds to the so called input to the system. In our case, \( w \) is dimension 3: we shall have on one hand \( d_1 \) which gives the deterministic profil of the road \( q_0 \). And on the other hand, we shall have two signals called \( d_2 \) and \( d_3 \) which correspond to white noises (model for the vibrations) and which are added respectively to the suspension deflexion \( q_2 - q_1 \) and to the vertical acceleration.

- \( u \) corresponds to the control input, \( u \) represents the force of the actuator of the active suspensions.

- \( z \) corresponds to the output to-be-controlled:

\[
\ddot{q}_2 : \text{vertical acceleration}
\]

\[
q_2 - q_1 : \text{suspension deflection}
\]

\[
q_1 - q_0 : \text{tire deflection}
\]

\[
u : \text{actuator force}
\]

- \( y \) corresponds to the measurement output to calculate the values of the parameters of the controller K. Since the actuator is situated between the chassis and the axle of the wheels, fig III.1.1, its action will depend on \( q_2 - q_1 \) and on the vertical acceleration, both with their respective noise:
III-3-State-space equation.

We first want to translate the mechanical system of III-1 into a state-space representation for the open loop system.

\[ y = \begin{pmatrix} q_2 - q_1 + d_2' \\ q_2 + d_2' \end{pmatrix} \]

The chosen state-vector is the following:

\[
\begin{pmatrix} q_1 \\ q_2 \\ m_u \dot{q}_1 \\ m_s \dot{q}_2 \end{pmatrix}
\]

We want to obtain:

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u \\
y &= C_1x + D_{11}w + D_{12}u \\
z &= C_2x + D_{21}w + D_{22}u 
\end{align*}
\]

Assuming

\[
B = (B_1 \ B_2) \quad \text{and} \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}
\]

and
The state-space representation is the following:

\[
D = \begin{pmatrix} D_{11} & D_{21} \\ D_{12} & D_{22} \end{pmatrix}
\]

\begin{align*}
A &= \begin{pmatrix} 0 & 0 & \frac{1}{m_u} & 0 \\ 0 & 0 & 0 & \frac{1}{m_s} \\ -(k_s + k_t) & k_s & -\frac{b_s}{m_u} & \frac{b_s}{m_u} \\ k_s & -k_s & \frac{b_s}{m_s} & -\frac{b_s}{m_s} \end{pmatrix} \\
B &= \begin{pmatrix} 0 \\ 0 \\ k_t \\ 0 \end{pmatrix}
\end{align*}

and with:

\[
\ddot{q}_2 = \frac{1}{m_s} \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \dot{X} = \frac{1}{m_s} \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} (AX + BU)
\]

it holds:

\[
C = \begin{pmatrix} k_s & k_s & \frac{b_s}{m_s^2} & \frac{b_s}{m_s^2} \\ m_s & m_s & m_s^2 & m_s^2 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ k_s & k_s & \frac{b_s}{m_s^2} & \frac{b_s}{m_s^2} \\ m_s & m_s & m_s^2 & m_s^2 \\ -1 & 1 & 0 & 0 \end{pmatrix}
\]

\[
D = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{m_s} \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

**III-4-The input and output weighting filters.**

Those filters are only used for the computation of the controller (to act on the influence of the corresponding input or output in this computation). They are not used during
the real time simulation.

III-4-1-The use of weighting filter. see [2].

The $l_1$, $H_2$ and $H_\infty$ optimal control theory is based on the computation of multivariable system norms. For instance the $H_\infty$ controller has to minimize more or less the highest gain of all the input-output transfer functions over all the frequencies. But because of the different sorts of input and output, we can find one or several transfer functions whose gain is pretty much larger than the other and hence the controller will always minimize this gain without taking the other part of the system into account. The idea is to balance these gains by using weighting filters. These filters decrease or increase the weight of the corresponding input/output transfer function for the computation of the controller.

Therefore we can, by varying the parameters of the filters, adapt the different transfer functions linking one or another input and output as we see fit, and choose the one which will determine the maximum value of the system norm between such and such a frequency.

For example, in my project the output signal related to the actuator force is about 1000 times larger than the others, that’s why an output weighting filter is required to reduce the influence of this signal in the controller design, otherwise the controller would only minimize the gain of the transfer functions related to this output signal.

III-4-2-generalized plant with filters

\[ \begin{align*}
  w & \quad \rightarrow \quad V \\
  u & \quad \rightarrow \quad P \\
  y & \quad \leftarrow \quad k \\
  z & \quad \rightarrow \quad W \\
\end{align*} \]

\[ \text{fig III.4.1 generalized plant with filters} \]

P: generalized plant without filters.
k: controller.
V: input filter.

\[
\begin{pmatrix}
  w_1 \\
  w_2 \\
  w_3
\end{pmatrix}
= 
\begin{pmatrix}
  V & 0 & 0 \\
  0 & V & 0 \\
  0 & 0 & V
\end{pmatrix}
\begin{pmatrix}
  d_0 \\
  d_2 \\
  d_3
\end{pmatrix}
\]

W: output filter.
\[
\begin{pmatrix}
\dot{q}_2 \\
q_2 - q_1 \\
q_1 - q_0 \\
u
\end{pmatrix} =
\begin{pmatrix}
W1 & 0 & 0 & 0 \\
0 & W2 & 0 & 0 \\
0 & 0 & W3 & 0 \\
0 & 0 & 0 & W4
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{pmatrix}
\]

III-4-3-Input filter specifications.

\( V_1(s) \) is a low-pass filter representing the road profil stochastic point of view.see[1].

\[
V_1(s) = \frac{\nu_0}{s + \omega_0}
\]

\( V_2(s) \) and \( V_3(s) \) are basic gain which modulate the influence of the white-noise (vibrations between components) in the computation of the controller.

\[
V_2(s) = 2.6 \times 10^{-3}, 0
\]

\[
V_3(s) = 3.9 \times 10^{-1}, 0
\]

III-4-4-Output filter specifications.

\( W_1(s) \) represents actually the confort for both human driver and cargo: it corresponds in fact to the fatigue decreased proficiency boundary.see[1].

\[
W_1(s) = \frac{s + \omega_0}{\omega_2} + \frac{s + \omega_1}{\omega^2 + 2\xi\omega_1 + \omega_0^2}
\]

\[
W_2(s) = \rho_3
\]

\[
W_3(s) = \rho_4
\]

\[
W_4(s) = \frac{s + 1}{\omega_4}
\]

This last filter is designed so that we consider the positionnement problems at low frequencies.
and the vibration problems at high frequencies.

### III-4-5-Parameter specifications.

\[
\begin{align*}
\rho_1 &= 10; & \omega_1 &= 20\pi \text{ rad/sec}; & \gamma_0 &= 0.008; \\
\rho_2 &= 100; & \omega_2 &= 10\pi \text{ rad/sec}; & \omega_{10} &= 0.4; \\
\rho_3 &= 100; & \omega_3 &= 10\pi \text{ rad/sec}; & \xi &= 1; \\
10^{-7} < \rho_4 < 10^{-5}; & \omega_4 &= 1000\pi \text{ rad/sec}; & \theta < 10^{-1}
\end{align*}
\]

Thoses have been provided by the firm DAF but some may change, for in any case they do not strictly conform to the reality. (The firm does not wish to reveal the work which was carried out).

Consequently, for my project I have looked for the most relevant values of each parameters. And after a lot of real time simulations and a lengthy work of analysis of the different transfer functions like in appendix C, the following results have been reached:

\[
\begin{align*}
\theta &= 10^{-1}; \\
\rho_5 &= 10^{-5}; \\
\omega_4 &= 0.25 \pi;
\end{align*}
\]

The other parameters remain the same.

### III-5-Conclusion.

In this chapter we have, as it were, stated the data of the problem in the language of automatic control. Now it is a question to answering the requirements of section II-3 which can be achieved if we manage to minimize whichever induced norm of the system.
Chapter IV-The $H_2$ optimal control

IV-1-The $H_2$ norm of a system

For more explanations about this chapter, see [2].

IV-1-1-The $H_2$ norm of a single input single output system

Let us consider the convolution system $A-1$ (see Appendix A). We are only interested in the impulse response of this system and in the $L_2$ norm of the corresponding output $y=h$. The induced norm of $H$ is then given by

$$
\|H\|_2 = \|h\|_2 = \sup_{u \in \delta} \frac{\|Hu\|_2}{\|u\|}
$$

Using the Parseval's identity we obtain:

$$
\|H\|_2 = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)H(-j\omega)d\omega \right\}^{1/2}
$$

And this defines the $H_2$ norm of $H$, denoted $\|H\|_2$ which is equal to the 2-norm of the impulse response of the system.

**Definition IV-1**

$$
\|H\|_2 = \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega)H(-j\omega)d\omega \right\}^{1/2}
$$

This norm has a very interesting interpretation in terms of stochastic signals because it is equal to the covariance of the signal if its mean is 0 (white noise process).see [2]. Thus the $H_2$ norm of the transfert function $H(s)$ is equal to the norm $\|y\|$ of the output $y$, when taking white noise as input to the system.

We can also anticipate that this norm is very helpful if we want to minimize the effects of white noise on a system.

IV-1-2-The $H_2$ norm for multivariable system.

The $H_2$ norm of a $p \times m$ transfert function is defined as follows.

**Definition IV-2** Let $H(s)$ be a stable multivariable transfert function of dimension $p \times m$. The $H_2$ norm of $H(s)$ is defined as:
\[ \|H(s)\|_2 = \left\{ \frac{1}{2} \int_{-\infty}^{\infty} \text{trace}\{H^*(-j\omega)H(j\omega)\}d\omega \right\}^{1/2} \]

Indeed let be m inputs such that

\[ \mu(i)(t) = O(t) \quad i=1,\ldots,m \]

then the corresponding output is a p dimensional signal whose the square two-norm is

\[ \|y^{(i)}\|_2^2 = \int_{-\infty}^{\infty} \sum_{j=1}^{p} |y_j^{(i)}(t)|^2 dt \]

and the $H_2$ norm of the transfert function is the square root of the sum of the two norms of these outputs:

\[ \|H(s)\|_2^2 = \sum_{i=1}^{m} \|y^{(i)}\|_2^2 dt \]

Moreover, like in the case of a single input single output system, this norm has the interpretation of the $\mathcal{L}_2$ norm of the trace of the signal spectrum of the output $y$ when the input is a white noise signal with a variance equal to I (identity matrix). see [2].

**IV-1-3-The computation of the $H_2$ norm**

Let $H(s)$ be a stable transfert function of dimension $p \times m$ and suppose that

\[ H(s) = C(I-A)^{-1}B + D \]

where $A, B, C, D$ are real matrices defining the traditional state space equations

\[
\begin{align*}
\dot{X} &= AX + BU \\
Y &= CX + DU
\end{align*}
\]

The computation of this norm uses the observability gramian $M$ of the system which is a
solution of the Lyapunov equation (algebraic task) \( MA + A^T M + C^T C = 0 \).

**Theorem IV-1** Let \( H(s) \) be a stable transfer function of the system. Suppose that \((A,B,C,D)\) is a minimal representation of \( H(s) \). Then

1. \( \|H(s)\|_2 < \infty \) if and only if \( D = 0 \)
2. If \( M \) is the observability gramian then

\[ \|H(s)\|_2 = \text{trace} \left( B^T MB \right) \]

We obtain the same results with the controllability gramian \( W \) which is also a solution of the Lyapunov equation and which enables the computation of the \( H_2 \) norm (=trace (\( CWC^T \))).

**IV-2-The computation of the \( H_2 \) optimal controllers.**

It coincides in fact with the well known LQG controller.

Let us consider the general control configuration as seen in section III-4-2:

\[
\begin{array}{ccc}
 & G & \\
\downarrow & & \downarrow \\
 w & \rightarrow & z \\
 & u & \\
\downarrow & & \downarrow \\
 & K & \\
\end{array}
\]

**fig IV.2.1 general control configuration**

If \( T_{cl} \) denotes the closed loop transfer function \( T_{cl} : w \rightarrow z \), and if we consider the obvious partitioning of \( G \)

\[
\begin{pmatrix}
  z \\
y
\end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix}
  w \\
u
\end{pmatrix}
\]

then \( T_{cl} = G_{11} + G_{12} K (I - G_{22}K)^{-1} G_{21} \).

The \( H_2 \) optimal control problem is formalized as follows:

*Synthesize a stabilizing controller \( K \) for the generalized plan \( G \) such that \( \|T_{cl}\|_2 \) is minimum.*

The solution of this problem is split into two independent problems:

- First, obtain an "optimal estimate" \( x_{est} \) of the state variable \( x \), based on the measurements \( y \).
- Second, use this estimate \( x_{est} \) as if the controller would have been perfect knowledge of the full state \( x \) of the system.

The Kalman filter is the optimal solution to the first problem and the state feedback linear quadratic regulator is the solution to the second one. see[2].
The Matlab routine "h2syn" computes the $H_2$ optimal controller in this way.

**IV-3-Practical results for the suspension system**

From the open loop system which has been defined in section III-4-1, Matlab computes the $H_2$ optimal controller and we obtain the following continuous time simulations of closed loop system for each output. (with the input signals defined in section II-2, the chosen deterministic road profil is the huge one because it is apparently the most unfavorable one and for the input $d_2$ we use a white noise whose amplitude is at the most 10% of that of the vertical acceleration, idem for $d_3$: white noise whose amplitude is at the most 10% of that of the signal $q_2 - q_1$.

The parameters of the filters have been specified in section III-4-3 and in Appendix C.

The results correspond to those which would be obtained in the reality. I mean by this, that the filters seen before only sense for the computation of the controller but do not play a part to the practical experiments.

![Fig IV.3.1 $d^2q_2/dt^2$](#)

![Fig IV.3.2 $q_2(t) - q_1(t)$](#)
We obtain in my opinion an interesting compromise between the cancellation of the noise and the damping of the bump. Whatever the case may be, the goals are reached, cf section VII.
Chapter V-The $H_\infty$ optimal control

V-1-The $H_\infty$ norm of a system

V-1-1-The $H_\infty$ norm of a single input single output system.

If $h: \mathbb{R} \to \mathbb{R}$ defined in the convolution system A-1 (Appendix A) $\in \mathcal{L}_1$ ($\|h\|_1 < \infty$) then $H$ is stable in the sense that bounded inputs provide bounded outputs, and in particular a bounded energy signal as input produces a bounded energy output signal. Thus under this condition, if we consider $H: \mathcal{L}_2 \to \mathcal{L}_2$ we can define the $\mathcal{L}_2$-induced norm

$$
\|H\|^2 = \sup_{u \in \mathcal{L}_2} \frac{\|H(u)\|^2}{\|u\|^2}
$$

Theorem V-1 Let $H$ be defined by (A-I-1) and suppose that $h \in \mathcal{L}_1$. Then the $\mathcal{L}_2$-induced norm of $H$ is given by

$$
\|H\|^2 = \max_{\omega \in \mathbb{R}} |\hat{h}(j\omega)|
$$

where $h(j\omega) = \int_{-\infty}^{\infty} h(t) e^{j\omega t} dt$ is the Fourier transform of $h$.

Proof: see [2]. (It comes from the Parseval's theorem).

This $\mathcal{L}_2$-induced norm is therefore the highest gain in a Bode diagram and we define the right side of the equality (V.1.1.1) as the $H_\infty$ norm of the system.

Definition V-1 Let $H(s)$ be the transfer function of a stable SISO system. The $H_\infty$ norm of $H$, denoted $\|H\|_\infty$ is the number

$$
\|H\|_\infty = \|H(s)\|_\infty = \max_{\omega \in \mathbb{R}} |\hat{h}(j\omega)|
$$

The $\mathcal{L}_2$-induced norm of an operator $H$ is equal to the $H_\infty$ norm of the transfer function $H(s)$ (subject to the previous theorem).

V-1-2-The $H_\infty$ norm for multivariable systems.

Let us consider the singular values decomposition (Appendix A-II) of a p×m stable transfer function $H(s)$. For each $\omega \in \mathbb{R}$, the singular values are ordered according to:

$$
\sigma_1(\omega) \geq \sigma_2(\omega) \geq \ldots \geq \sigma_r(\omega) > 0.
$$
From the section Appendix A-I- we can write, for each \( \omega \in \mathbb{R} \):

\[
\|H(j\omega)\tilde{u}(j\omega)\| \leq \sigma(j\omega)\|\tilde{u}(j\omega)\| = \sigma(j\omega)\|\tilde{u}(j\omega)\|
\]

We can now generalize the results of the previous section in the multivariable case.

**Definition V.2**- Let \( H(s) \) be a stable multivariable transfer function. The \( H_{\infty} \) norm of \( H(s) \) is defined as

\[
\|H(s)\|_{\infty} := \sup_{\omega \in \mathbb{R}} \sigma(H(j\omega))
\]

And moreover, we keep the analog theorem:

**Theorem V.2**- For a multivariable transfer function \( H(s) \), the \( \mathcal{L}_2 \)-induced norm of its associated mapping is equal to the \( H_{\infty} \) norm of \( H(s) \).

\[
\|H\|_{(2,2)} = \|H(s)\|_{\infty} \quad (V.1.2.1)
\]

**V-1-3-The computation of the \( H_{\infty} \) norm.**

Let \((A,B,C,D)\) be the state space representation seen in section IV-1-3. Instead of searching for an exact expression of \( \|H\|_{\infty} \), we will find an algebraic condition whether or not \( \|H\|_{\infty} < \gamma \) for some real number \( \gamma \geq 0 \). By performing a test checking if the condition is reached for some values of \( \gamma \) as small as possible, we may get arbitrarily close to the norm \( \|H\|_{\infty} \).

We have proved that

\[
\|H(s)\|_{\infty} = \|H\|_{(2,2)} = \sup_{w \in \mathbb{L}_2} \frac{\|H(w)\|_2}{\|w\|_2}
\]

that means

\[
\|H(s)\|_{\infty} \leq \gamma \iff \|H(w)\|_2^2 - \gamma^2 \|w\|_2^2 = \|z\|_2^2 - \gamma^2 \|w\|_2^2 \leq 0
\]

for all \( w \in \mathbb{L}_2 \).

This last inequality is reached if there exists a matrix \( K \) which reaches the following Riccati equation (see [2])

\[
A^TK + KA + (B^TK - D^TC)[\gamma^2I-D^TD]^{-1}(B^TK - D^TC) + C^TC = 0 \quad (V.1.2.2)
\]

Moreover, the system has to be stable. It is the case if \( K \) is non negative definite and such a solution is a stabilizing solution.

**Theorem V.3**- Let \( H(s) \) be represented by the minimal state model (), then
1. \( \|H\|_\infty < \infty \) if and only if the eigenvalues \( \sigma(A) \in \mathbb{C} \).
2. \( \|H\|_\infty < \gamma \) if and only if there exist a stabilizing solution of the Ricati equation.

Proof: see ref[2] or ref [4].

V-2-The computation of the \( H_\infty \) optimal controller.

The aim is to establish the optimal control which corresponds to a particular solution to the algebraic Ricati equation.

Let us consider again the general control configuration seen in section IV-2 with the obvious partitioning of \( G \)

\[
\begin{pmatrix}
\zeta \\
y
\end{pmatrix} =
\begin{pmatrix}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{pmatrix}
\begin{pmatrix}
w \\
u
\end{pmatrix}
\]

Let \( M \) be the closed loop transfert function \( T_{ct:w} \rightarrow z \), \( T_{ct} = G_{11} + G_{12} K (I - G_{22}K)^{-1} G_{21} \).

The \( H_\infty \) control problem is formulized as follows:

\text{Find a stabilizing controller } K \text{ such that } \|T_{ct}\|_\infty < \gamma \text{ with } \gamma > 0 \text{ as small as possible.}

The synthesis of \( H_\infty \) suboptimal controllers is based on the following two Ricati equations

\begin{align*}
A^TX + XA - X[B_2 B_2^T - \gamma^{-2} B_1 B_1^T]X + C_1^TC_1 &= 0 \\
AY + YA^T - Y[C_2^TC_2 - \gamma^{-2} C_1^TC_1]Y + B_1 B_1^T &= 0
\end{align*}

(V-2-1) \hspace{1cm} (V-2-2)

Where the matrices \( C_1, C_2, B_1, B_2 \) are defined like in section III-3.

Notes:X and Y are symmetric and if they exist, they are unique.

We are particulary interested in the stabilizing solutions of these equations .

The symetric matrix \( X \) is a stabilizing solution of (V-2-1) if the eigenvalues

\[ \sigma(A - B_2 B_2^TX + \gamma^{-2} B_1 B_1^TX) \subset \mathbb{C}^- \]

Simiraly, the symetric matrix \( Y \) is a stabilizing solution of (V-2-2) if

\[ \sigma(A - YC_2 C_2^TY + \gamma^{-2} YC_1 C_1^TY) \subset \mathbb{C}^- \]

It is time now to emphasize the main result of this section which has been considered as one of the main contributions in \( H_\infty \) optimal control theory during the last 10 years.

**Theorem V-4**- Under a few conditions (like controllability, observability, see [2]), there exist an internally stabilizing controller \( K \) that achieves \( \|T_{ct}\|_\infty < \gamma \) if and only if

1. Equation (V-2-1) has a stabilizing solution \( X = X^T \geq 0 \).
2. Equation (V-2-2) has a stabilizing solution \( Y = Y^T \geq 0 \).
3. \( \sigma(XY) < \gamma' \).
Moreover in that case one such controller is given by

\[
\begin{align*}
\dot{\xi} &= (A + \gamma^{-2}B_1B_1^TX)\xi + B_2u + ZL(C_2\xi - y) \\
u &= F\xi
\end{align*}
\]  

(V-2.3)

where \( F := -B_2^TX \)

\( L := YC_2^T \)

\( Z := (I - \gamma^2 YX)^{-1} \)

In short, the \( H_\infty \) control design looks as follows:

**Algorithm**

**INPUT:** generalized plant \( G \) in state space form III-3

- tolerance level \( \varepsilon > 0 \).

**step 1.** Find \( \gamma_1, \gamma_h \) such that \( M:w\rightarrow z \) satisfies \( \gamma_1 < \|T_{el}(s)\|_\infty < \gamma_h \).

**step 2.** Let \( \gamma = (\gamma_1 + \gamma_h)/2 \) and verify if there exist matrices \( X = X^T \) and \( Y = Y^T \) satisfying the conditions of theorem V-4.

**step 3.** If so, then set \( \gamma_h = \gamma \). If not then set \( \gamma = \gamma_1 \).

**step 4.** If \( \gamma_h - \gamma > \varepsilon \) then go to step 2.

**step 5.** Put \( \gamma = \gamma_h \) and let

\[
\begin{align*}
\dot{\xi} &= (A + \gamma^{-2}B_1B_1^TX)\xi + B_2u + ZL(C_2\xi - y) \\
u &= F\xi
\end{align*}
\]

define the state space equations of a controller \( K(s) \).

**OUTPUT:** \( K(s) \) is a stabilizing controller which achieves \( \|T_{el}\|_\infty < \gamma \).

The program mhc.mat from Matlab ref[5] enables these computations and is able to show the corresponding real time simulations.

**V-3-The implementation of \( H_\infty \) optimal control on the suspension system.**

Similarly to the previous section, Matlab and especially the program \( mhc \), computes the \( H_\infty \) optimal controller and we obtain the following continuous time simulations of closed loop system for each output. (with the input signals defined in section II-2, the chosen deterministic road profil is the huge one because it is apparently the most disturbing one). The parameters of the filters have been specified in section III-4.
Therefore under the same assumptions concerning the inputs as in the previous section related to $H_2$ control, we obtain about the same results.
Chapter VI-The $l_1$ optimal control

VI-1-The $l_1$ norm of a system.

Let us consider again the convolution system A.1 of Appendix A and the general configuration of section III-4-2 (with the same partitioning).

We are now interested in minimizing the effects of $w$ to $z$ by considering amplitudes of signals and this amounts to minimizing the induced norm:

$$\|H\|_{\infty} = \sup_{w \in \mathbb{L}_1} \frac{\|z\|_{\infty}}{\|w\|_{\infty}}$$  \hspace{1cm} (VI-1-1)

We want to find a stabilizing controller $C$ such that this norm is minimal subject to the system constraints. Since the minimum may not exist, it is better to search for the infimum $\mu$ of (VI-1-1), i.e., we define:

$$\mu = \inf_{C_{\text{stab.}}} \sup_{w \in \mathbb{L}_1} \frac{\|z\|_{\infty}}{\|w\|_{\infty}}$$  \hspace{1cm} (V-1-2)

The $l_1$ control theory is based on the following theorem: see [3].

**Theorem VI-1** - Let us consider discrete time signals. If $T_{cl}$ maps the closed loop plant of the above general control configuration. $T_{cl} : w \rightarrow z$ and $z(t)=[t_{cl} * d](t)$. Then

$$\|t_{cl}\|_{l_1} = \sup_{w \in \mathbb{L}_1} \frac{\|z\|_{l_1}}{\|w\|_{l_1}}$$  \hspace{1cm} (VI-1-3)

where $t_{cl}$ is the impulse response of the closed loop system and where the $l_1$-norm is the $l_1$-norm of the corresponding transfer function.
\[ \| t_{cl} \|_{l_1} = \max_i \sum_j \sum_{k=0}^{\infty} |[t_{cl}]_{ij}(k)| \quad (VI-1-4) \]

The \( l_1 \) norm of a system is actually equal to the \( \infty \)-induced norm.

**Proof:** for SISO case.

\[
\| z \|_{\infty} = \| t_{cl} \ast w \|_{\infty} = \sup_n \biggl| \sum_{l=0}^{n} t_{cl}(n-l) w(l) \biggr| 
\]

\[
\| z \|_{\infty} \leq \sup_n \biggl| \sum_{l=0}^{n} t_{cl}(n-l) \biggr| \cdot \| w \|_{\infty} = \| w \|_{\infty} \cdot (|t_{cl}(0)| + \ldots + |t_{cl}(n)|) 
\]

\[
\| z \|_{\infty} \leq \| w \|_{\infty} \cdot \| t_{cl} \|_{l_1} 
\]

Next, we need to show that there exist \( w' \in l_{\infty}, \| w' \|_{\infty} = 1 \), which reaches \( \| z \|_{\infty} = \| t_{cl} \|_{\infty} \).

\[
\sup_{w \in l_{\infty}, \| w \|_{\infty} = 1} \frac{\| z \|_{\infty}}{\| w \|_{\infty}} = \sup_{w \in l_{\infty}, \| w \|_{\infty} = 1} \| z \|_{\infty}
\]

with

\[
z(n) = \sum_{l=0}^{n} t_{cl}(n-l) w(l) 
\]

If we fix \( T > 0 \) and if we define \( w(k) \) as follows:

\[
w_{T}(k) = \begin{cases} 
0 & \text{for } k > T \\
\text{sign}(t_{cl}(T-k)) & \text{for } k \leq T 
\end{cases}
\]

Then
And if $T \to \infty$, then it holds

\[ \|z_T\|_\infty = \frac{\|z_T\|_\infty}{\|w_T\|_\infty} \cdot \|t_c\|_1 \]

Indeed, because it is a sum of positive numbers,

\[ \lim_{T \to \infty} (z_T(T)) = \|z_T\|_\infty \]

Therefore, as $T_{cl} = G_{11} + G_{12} K (I - G_{22} K)^{-1} G_{21}$, we actually want to find

\[ \mu = \inf_{q \in l_1} \| g_{11} + g_{12} * q * g_{21} \| \]  \hspace{1cm} \text{(OPT)}

When this minimum is found for a certain $q$, we find the optimum controller $C_{opt}$ using the Youla parametrization.

**VI-2-Solution to the discrete time $l_1$ problem.**

**VI-2-1-Problem formulation.**

The starting point is the previous equation (OPT). The problem can be formulated as follows: $g_{11}$ is seen as an element of a vector space and we want to find the minimum 1-norm distance from this vector to the span of the vector $g_{12} q g_{21}$ where $q \in l_1$ is a free design parameter. Then if we find such a $q$ and hence the corresponding $Q$ (Laplace transform) the Youla
parametrization (see [3] p277 th 6.3) ensures that $C$ corresponds to an internally stable closed loop system if we choose the compensator transfer matrix as follows:

$$C = (Y - M Q) (X - N Q)^{-1}$$

$$C = (X - Q \bar{Y}^{-1}) (Y - Q \bar{M})$$

This is equivalent to the following minimum distance problem:

$$\mu_{opt} = \inf_{M \in \mathcal{S}} \| G_{11} - M \|$$

(OPT1)

where $\mathcal{S} = \{M$ rational stable / $\exists Q$ rational stable satisfying $M = G_{12} Q G_{21}\}$.

VI-2-2-Special cases.

The dimensions of the inputs and outputs of the standard system of $(G_{12})$ determine what kind of rank (full row rank or full column rank or both) $G_{12}$ and $G_{21}$ have, and we can classify problems (OPT) by rank as follows. In the case $n_u \geq n_z$, or at least as many control inputs as regulated outputs, $G_{12}$ will be a "fat" matrix with full row rank. If we have $n_u \geq n_w$ or at least as many measured outputs as exogeneous inputs, $G_{21}$ will be a "skinny" matrix with full column rank. We call this the good rank case.
If $n_u < n_z$, $G_{12}$ will be a skinny matrix with full column rank and we will say this $G_{12}$ has bad rank. Similarly, $n_z < n_w$ will result in a fat $G_{21}$ with full row rank and we will say that $G_{21}$ has bad rank.
In our case either both $G_{12}$ and $G_{21}$ have bad rank (four-block problem).

VI-2-3-Existence of a minimizer.

The solution is found by dealing with the dual spaces and by making use of the following theorem.

Theorem VI-2 Let $M$ be a subspace in a real norm linear space $X$. Let $x^* \in X^*$ and let $d$ denote its distance from $M^\perp$. Then:

$$d = \min_{m^* \in M^\perp} \| m^* - m^* \| = \sup_{x \in BM} < x, x^* >$$

VI-2-1

where the minimum on the left is achieved for some $m^*_0 \in M^\perp$. If the supremum on the right is achieved for some $x_0 \in M$, then $x^* \cdot m^*_0$ is aligned with $x_0$.

This rapport does not get into the details of this theory. See [7].

In short, in the bad rank case, we consider the related minimum distance problem.
\[
\mu_{opt} = \inf_{M \in S_1} \| G_{11} - M \|
\]

where \( S_1 = \{ M \text{ stable} \mid \exists Q \text{ stable satisfying } M = G_{12}QG_{21} \} \), and the theorem VI-2 establishes the existence of a minimizer for this problem precluding zeros on the unit circle.

**VI-2-4-Our case.**

For our system (section III-3), the computations (using MAPLE) give in discrete time after the Tustin transformation (trapezoidal rule):

\[
s = \frac{2}{T} \frac{(z-1)}{(z+1)}
\]

with \( T = 0.001s \) which is acceptable in comparison to the lowest time constant of the mechanical system.

\[
G_{12} = \begin{pmatrix}
3.786 (609z^4 - 2417z^3 + 3599z^2 - 2383z + 5892)(z+1)(1.0015z - 0.98932) \\
\Delta(z)(0.26594z^2 - 0.49951z + 0.2345) \\
-0.002(z+1)^2(1183z^2 - 2400z + 1217) \\
\Delta(z)10^9 \\
-0.0002(z+1)^2(13657z^2 - 2797z + 13946) \\
\Delta(z)10^9 \\
0.25488z - 0.25668 \\
0.81847z + 0.18153
\end{pmatrix}
\]

with \( \Delta(z) = 0.14689 z^4 - 0.58152 z^3 + 0.86373 z^2 - 0.57048 z + 0.14137 \) whose roots are inside the unit disk (stable poles).

\[
G_{21} = \begin{pmatrix}
0.96(z+1)^2(31z^3 - 91z^2 + 89z - 29) \\
\Delta(z)10^8 (0.25498z - 0.25458) \\
-2400(z+1)^3(8017z^2 - 16000z + 7983) \\
\Delta(z)10^9 (0.25498z - 0.25458) \\
0.00026 & 0 \\
0.039 & 0
\end{pmatrix}
\]

The complexity of the \( L_1 \) optimum problem depends on the number of unstable zeros of \( G_{12} \) and \( G_{21} \) with their corresponding multiplicity. Appendix B shows what we call zeros of a
multivariable system. In short, we need to determine the Smith and Mac Millan form of the the matrices $G_{12}$ and $G_{21}$.

The computation of the invariant factors is achieved using the greatest commun divisor of the different minors of the matrix.

We obtain the following Smith and MacMillan form for $G_{12}$ and $G_{21}$, $S_{12}$ and $S_{21}$ respectively:

$$S_{12} = \begin{pmatrix} c_{12} \\ \Delta(z)(0.26594z^2-0.49951z+0.2345)(0.81847z+0.18153) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$S_{21} = \begin{pmatrix} c_{21} \\ \Delta(z)(0.25498z-0.25458) \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Where $c_{12}$ and $c_{21}$ are real constant numbers.

There are no zeros. Therefore there are no unstable zeros and no zeros on the unit circle neither.

But on the other hand, and this is very bad for us, $G_{12}$ and $G_{21}$ have both transmission zeros on the unit circle and now we need the following assumption:

**Assumption 1.** There exist one row of $G_{12}$ and two columns of $G_{21}$ which are linearly independant for all $z$ on the unit circle.

In our case there is only one possibility: $G_{12}$ and $G_{21}$ can be written in the following form:

$$G_{12} = \begin{pmatrix} Gh_{12} \\ Gb_{12} \end{pmatrix} \quad \text{with} \quad Gb_{12} = \begin{pmatrix} 0.25488z-0.25668 \\ 0.81847z+0.18153 \end{pmatrix}$$

$$G_{21} = \begin{pmatrix} Gh_{21} & Gb_{21} \end{pmatrix} \quad \text{with} \quad Gb_{21} = \begin{pmatrix} 0.00026 & 0 \\ 0 & 0.039 \end{pmatrix}$$

where $Gb_{12}$ and $Gb_{21}$ are invertible and have no zeros on the unit circle.

Moreover, $M$ can be written:

$$M = \begin{pmatrix} Gh_{12} \\ Gb_{12} \end{pmatrix} Q \begin{pmatrix} Gh_{21} & Gb_{21} \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & Mb \end{pmatrix}$$

and $Gb_{12}$ and $Gb_{21}$ define a good rank sub problem. Also we can define polynomial
factorization as follows:

\[ G_{h12} G_{b12}^{-1} = D_{12}^{-1} N_{12} \]

\[ G_{b21}^{-1} G_{h21} = N_{21} D_{21}^{-1} \]

Using these definitions we state the following result characterizing the feasible set \( S_1 \) of (OPT2) for this case.

**Theorem VI-3.** Given \( G_{12} \) and \( G_{21} \) as above, Assumption 1, and \( M \) stable, there exist \( Q \) stable satisfying \( M = G_{12} Q G_{21} \) if and only if:

\[ (i) \quad (D_{12} - N_{12}) \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{b} \end{pmatrix} = 0 \]

\[ (ii) \quad (K_{21} K_{b}) \begin{pmatrix} D_{21} \\ -N_{21} \end{pmatrix} = 0 \]

\[ (iii) \quad K_{b} \text{ interpolates } G_{b12} \text{ and } G_{b21} \]

Proof: see [7].

As \( G_{b12} \) and \( G_{b21} \) are have good rank case, the condition (iii) holds (see[7]).

Anyway in our case, as the Smith and McMillan form \( S_{12} \) and \( S_{21} \) have no zeros, the condition (iii) is always reached. For the definition of this interpolation condition, see [7].

Subject to the definition VI-2-4, \( M_b = G_{b12} Q G_{b21} \). \( Q \) has to be a rational and stable matrix to ensure the stability of the closed loop system (Youla parametrization), thus as \( G_{b12} \) and \( G_{b21} \) are rational and stable, \( M_b \) has to be a matrix of real rational functions with stable poles.

Then, as \( G_{b12} \) and \( G_{b21} \) are invertible:

\[ Q = G_{12}^{-1} M_b G_{21}^{-1} \]

\( M_b \) is actually a free parameter (matrix 1×2 in our case, whose elements are real rational functions with stable poles), which has to minimize the \( l_1 \) norm of the closed loop system.

The condition (ii) gives, according to the parametrization VI-2-3:

\[ M_{21} = M_b N_{21} D_{21}^{-1} = M_b G_{b12}^{-1} G_{h21} = f_{11}(M_b) \]

The condition (i) gives:

\[ M_{12} = G_{h12} G_{b12}^{-1} M_b = f_{12}(M_b) \]

\[ M_{11} = G_{h12} G_{b12}^{-1} M_b G_{b21}^{-1} G_{h21} = f_{11}(M_b) \]

Finally, our problem can be resumed in those terms: Find a real rational and stable \( M_b \) such that the \( l_1 \) norm of the following multivariable function \( ||T_c||_1 \) is minimum.
The computation of the $l_1$ norm is based on the computation of impulse responses which corresponds to an infinite sum of numbers and hence we can see that we need to use truncated problems.

Recalling that the closed loop transfert function $T_{cl} = G_{11} - M$ and using the definition of the $l_1$ norm, (OPT) can be written:
\[ \inf \max \sum_{i=1}^{n_i} \sum_{j=1}^{n_w} |t_{ij}(k)| \]

where \( \sum_{k=0}^{\infty} |t_{ij}(k)| \) is the impulse response of \( T_{ci}[i,j](z) \)

subject to \( G_{11} - T_{ci} = M \in S. \)

In the bad rank case, the condition (iii) of theorem VI-3 defines a certain set of constraints but in our case this set is empty. (Because there are no zeros which appear in the Smith and MacMillan form of \( G_{12} \) and \( G_{21} \).)

On the other hand condition (i) is clearly satisfied if and only if:

\[ [[d_{12} - n_{12}] * T_{ci}(k)] = [[d_{12} - n_{12}] * g_{11}(k) \]

where \( k = 0, 1, 2, ... \)

which defines a set of linear equality constraints, but in this case an infinite number. Condition (ii) defines a similar set of constraints.

We see now that (OPT) is equivalent to following linear program:

\[
\begin{align*}
\text{(LP):} & \quad \inf \lambda \\
\text{subject to:} & \quad t_{ij}^{*}(k) - t_{ij}(k) = t_{ij}(k) & i=1, ..., n_i \\
& \quad t_{ij}^{*}(k), t_{ij}(k) \geq 0 & j=1, ..., n_w \\
& \quad k=0, 1, 2, ... \\
& \quad \sum_{j=1}^{n_w} \sum_{k=0}^{\infty} [t_{ij}^{*}(k) + t_{ij}(k)] \leq \lambda \\
& \quad G_{11} - T_{ci} \in S.
\end{align*}
\]

Because (LP) has an in infinite number of variables and, in the bad rank case, constraints it cannot be solved directly using general linear programing techniques. Therefore, instead of (OPT) we study the following family of related finite dimensional problems which we call truncated problems, and which can be indexed by the non negative integer \( \delta \).

\[
\text{(OPT}_\delta\text{):} \quad \mu_\delta = \min \| G_{11} - M \| \quad \text{OPT}_\delta
\]

where \( S_\delta := \{ M \text{ stable :} G_{11} - M \text{ is a polynomial of degree } \leq \delta \} \).

This is equivalent to the following linear program:

\[
\text{(LP}_\delta\text{):} \quad \inf \lambda
\]
subject to:
\[ t'_{ij}(k) - t'_{ij}(k) = t_{ij}(k) \quad i = 1, \ldots, n_z \]
\[ t'_{ij}(k), t'_{ij}(k) \geq 0 \quad j = 1, \ldots, n_w \]
\[ k = 0, \ldots, \delta \]

\[
\sum_{j=1}^{n_w} \sum_{k=0}^{\delta} [t'_{ij}(k) + t'_{ij}(k)] \leq \lambda
\]

\[ G_{11} - T_{cl} \in S_\delta \]

The problem is that the set \( S_\delta \) may be empty. In this case \( \mu_8 \) of \( (\text{OPT}_8) \) is not defined. In order to address this problem, we state the following condition:

Condition 1. There exists \( \delta^* \) such that \( S_{\delta^*} \) is non-empty or equivalently, such that \( (\text{OPT}_{\delta}) \) has a feasible point for which \( T_{cl} = G_{11} - M \) is a polynomial of degree \( \leq \delta^* \).

If and only if this condition is satisfied, we can define a monotonically increasing integer sequence \( (\delta(i))_{i=0}^{\infty} \) for which by taking \( \delta(0) \geq \delta^* \) \( (\text{OPT}_8) \) will have well defined norms \( \mu_{\delta(i)} \), and consequently we will able to find a procedure for finding arbitrarily good approximate minimizers by solving a sequence of problems \( (LP_{\delta}) \).

Indeed it is quite obvious that:

\[
\lim_{i \to \infty} \mu_{\delta(i)} = \mu_{\text{opt}} \quad (\text{OPT}_{\delta(i)})
\]

For the proof, see [7].

In the bad rank case we need to make use of the following theorem which characterize when condition is satisfied. (\( G_{11} \) is partitionned as \( M \) was in theorem VI-3 and to simplify the notations let us define \( H \) as follows \( H = G_{11} \)).

**Theorem VI-4.** Given \( G_{12} \) and \( G_{21} \) with their polynomial factorization seen in the previous section, Condition 1 is satisfied if and only if the transfer matrices \( T_{12H} \) and \( T_{H21} \) defined:

\[
T_{12H} := (D_{12} - N_{12}) \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & Hb \end{pmatrix}
\]

\[
T_{H21} := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & Hb \end{pmatrix} \begin{pmatrix} D_{21} \\ -N_{12} \end{pmatrix}
\]

are both polynomials.

It is not the case for our problem.
Indeed, in our case $H_{12} = H_{21} = H_b = 0$. Thus

$$T_{12H} = (D_{12}H_{11} \ 0)$$

and $T_{H21} = \begin{pmatrix} H_{11} & D_{21} \\ 0 & 0 \end{pmatrix}$

$$D_{12} = \Delta(z) (0.25488z - 0.25668) (0.26594z^2 - 0.49951z + 0.2345)$$

(note that it is a scalar).

$$H_{11} = \begin{pmatrix} S_1(z) \\ S_2(z) \\ S_3(z) \end{pmatrix}$$

Also $T_{12H}$ is not polynomial. (The greatest common divisor of the three denominators of $S_1(z)$, $S_2(z)$ and $S_3(z)$ does not divide $D_{12}(z)$). Actually it is because of the pole which comes from the input filter on $q_0 (0.25498z - 0.25458)$.

Moreover the other condition is not satisfied neither:

$$D_{21} = \Delta(z) (0.25488z - 0.25668)$$

And $S_1(z)D_{21}(z)$ is not polynomial because of the poles which come from the output filter on the vertical acceleration $(0.26594z^2 - 0.49951z + 0.2345)$.

**Therefore our problem with input and output weighting filters cannot be solved using this method. Indeed we proved that $S_5$ is empty $\forall \delta$.**

It should be possible to make the $l_1$ optimum control problem feasible for our suspension system if the output filter on the vertical acceleration is suppressed and if the numerator of the output filter on $u$ is the same as the denominator of the input filter on $q_0$ which means $\omega_4 = \omega_b$. ($\omega_4$ and $\omega_b$ have been defined in section III-4).

**VI-2-6-The case with requirements on the weighting filters.**

Now we consider the same problem but with the following output filter matrix: ($W_1(s)$, $W_2(s)$, $W_3(s)$ and $W_4(s)$ have been defined in section III-4-2).

$$W_1(s) := \rho \omega_1^2 \omega_{10}$$

$W_2(s)$ and $W_3(s)$ remain the same.

$$W_4(s) := \frac{s}{\omega_0} + 1$$

We have kept the static gain for $W_1(s)$ and $\omega_4$ has been replaced by $\omega_b$. 

We obtain:

\[
G_{12} = \begin{pmatrix}
96 (609z^4 - 2417z^3 + 3599z^2 - 2383z + 5892) \\
\Delta(z)10^9 \\
-0.002(z+1)^2(1183z^2 - 2400z + 1217) \\
\Delta(z)10^9 \\
-0.0002(z+1)^2(13657z^2 - 2797z + 13946) \\
\Delta(z)10^9 \\
0.25498z - 0.25458 \\
0.81847z + 0.18153
\end{pmatrix}
\]

\[
G_{21} = \begin{pmatrix}
0.96(z+1)^2(31z^3 - 91z^2 + 89z - 29) \\
\Delta(z).10^8 (0.25498z - 0.25458) \\
-2400(z+1)^3(8017z^2 - 16000z + 7983) \\
\Delta(z).10^9 (0.25498z - 0.25458)
\end{pmatrix}
\]

with \( \Delta(z) = 0.14689z^4 - 0.58152z^3 + 0.86373z^2 - 0.57048z + 0.14137 \)

In this case it is easy to check that the theorem VI-4 is reached. That means that there exist \( \delta \) such that \( S_\delta \) is not empty.

Therefore we can define a sequence \( l_{0(i)} \) such that \( l_{0(i)} - l_{0\text{opt}} \) is as small as possible.

The problem is that there were no time left to implement the corresponding linear problem.

Actually I don’t know how to translate in terms of linear programing the fact that \( t_{ij}(k) \) and \( t_{ij}(k) \) can not be non zero at the same time: \( t_{ij}(k) \cdot t_{ij}(k) = 0 \), which is not linear.

My research ends with this problem.
Chapter VII-Comparaisons

VII-1-Active and passive suspension.

The input are the same as in section IV-3, V-3, and VI-3.

VII-1-1-Vertical acceleration

![Graph of H2 control](image1)

![Graph of H control](image2)

![Graph of passive suspensions](image3)

The $H_2$ optimal control seems to be the most suitable one for the vertical acceleration because it is more effective to reduce the noise and because the amplitude of the output signal is less large in comparison to the $H_\infty$ optimal controller.

We can also see how much the actuator that means the active suspensions, has improved the performances of the system.
VII-1-2-Suspension deflections $q_2 - q_1$

For this output signal, the $H_2$ optimum control is the best one again to reduce the influence of the noise sensor, but the $H_{\infty}$ is better if we want to have an amplitude as small as possible which is in fact the most important objective. We notice again the improvements in comparison to the passive suspension.
Tire deflections $q_1 - q_0$

Fig VII-1-3-1 $H_2$ control

Fig VII-1-3-2 $H_\infty$ control

Fig VII-1-3-4 passive suspensions

The same conclusions as previously
VII-1-3-Force of the actuator u

The maximum amplitude of this signal is a little bit smaller for the $H_2$ control than for the $H_\infty$ control. Therefore we can say that the $H_2$ control reduces the risks of overheat.

VII-2-Robustness

In order to test the robustness of each type of optimal control, I have performed the real time simulations for each input road profil which has been defined in section II-2. We obtain the following table:

| size of the bump | type of the controller | $\frac{d^2q}{dt^2}$ [m.s$^{-2}$] min max | $q_2$-$q_1$ [m] min max | $q_1$-$q_0$ [m] min max | $|u|$ [N] min max |
|------------------|------------------------|---------------------------------|----------------|----------------|----------------|
| huge             | $H_2$                  | -3.2  3.5                        | -0.15  0.12    | -0.025  0.04  | 3.5 10$^4$     |
|                  | $H_\infty$             | -3    4                          | -0.15  0.08    | -0.015  0.035 | 5 10$^6$       |
| large            | $H_2$                  | -3    4                          | -0.13  0.08    | -0.023  0.031 | 2.5 10$^4$     |
|                  | $H_\infty$             | -3    4.5                        | -0.12  0.06    | -0.023  0.032 | 4 10$^4$       |
| medium           | $H_2$                  | -2.2  4.5                        | -0.12  0.04    | -0.04  0.03  | 1.3 10$^4$     |
|                  | $H_\infty$             | -3    5.5                        | -0.10  0.04    | -0.04  0.03  | 2.5 10$^4$     |
| small            | $H_2$                  | -0.8  1.3                        | -0.09  0.03    | -0.025  0.02  | 0.4 10$^4$     |
|                  | $H_\infty$             | -1    1.5                        | -0.08  0.03    | -0.025  0.02  | 1 10$^4$       |

The maximum value of the extension of the tyres which is ($q_1$-$q_0$), is the only one which does not conform to the requirements of section II-3, and even this by only a little, which is not significant.

We can conclude that thoses types of control are quite robust, and especially the $H_\infty$ optimal control, because the results still remain relevant if we increase the size of the input bump.
Chapter VIII-Conclusions

Results.
The final results for $H_2$ and $H_\infty$ optimal control are very encouraging, for they show well the improvements brought by the active suspension especially at the level of the absorption of the bump. Moreover they concord with the requirements that we fixed, although the specifications of the model is not exact. In fact the requirements provide an order of size. Finally, they show, and this is the most important thing as far as I am concerned, the advantages of these controls which aim to minimize the $H_2$ and $H_\infty$ norm of a system: quality of the simulations (damping, noise reducing) and robustness.

About the $l_1$ optimal control, the final results (real time simulation) have not been found yet. Actually we proved that in our case it was not possible to find a stabilizing and minimizing controller. But if we change a little bit the beginning data (in particular cancellation of one output filter), a linear programming problem can be solved in order to determine a $l_1$ optimal controller.

Future perspectives.
This project has to be continued in order to solve the $l_1$ optimal control problem and then it will be very relevant to compare it with the two other which have been described in this report. Anyway, the fact to minimize the amplitude of the output signals whatever the input signal (which has just to be bounded), is a very good idea, and this leads to a lot of applications (in the research field of medical, biology, robotic engineering...).

Personal feelings.
I confess to being very satisfied with having worked within the measurement and control group. This project has benefitted me enormously: I have had to manage on my own by searching for such and such piece of information in specialized works: I have had to speak English, I have learned about new and modern types of control, etc...

I would like to add that I have found this subject very interesting, because it is both very theoretical (significant mathematic background) and on the other hand, it can be currently technically applied: active suspension system.

Finally, I want to thank Dr. Siep Weiland for his explanations and his advice and Pr. Van Damn who has given to me the possibility to work in the T.U.E for the second time.
Appendix A

Signal spaces and norms

A-I Signals and signal norms.

For each time instant, a signal takes values in a set, say W, called the signal space.

Definition A.I.1 A signal is a function s: T → W where T ⊆ R is the time set and W is the set called signal space.

Let us define for a discrete time signal s:T→R the following norms:

\[ \| s \|_\infty := \max_j \sup_{t \in T} |s_j(t)| \]

\[ \| s \|_1 := \left\{ \sum_{t \in T} |s(t)|^i \right\}^{1/i} \quad i \in \mathbb{N} \]

And in continuous time:

\[ \| s \|_1 := \left\{ \int_{t \in T} |s(t)|^i dt \right\}^{1/i} \quad i \in \mathbb{N} \]

\[ \| s \|_\infty := \max_j \sup_{t \in T} |s_j(t)| \]

Finite norm signals are of special interest and define the following normed signal spaces

\( l_\infty = \{ s: T \to W / \| s \|_\infty < \infty \} \)

\( l_1 = \{ s: T \to W / \| s \|_1 < \infty \} \quad i \in \mathbb{N} \)

for discrete time signals and for continuous time signals:

\( \mathcal{L}_\infty = \{ s: T \to W / \| s \|_\infty < \infty \} \)

\( \mathcal{L}_1 = \{ s: T \to W / \| s \|_1 < \infty \} \quad i \in \mathbb{N} \)

A.II Systems and system norms.
A system set $S$ of signals. A system specifies the relations between the input signals and the output signals and most of the time using transfer functions, state space representations or differential equations.

In continuous time, an input signal is transformed into an output signal $y=H(u)$ according to the convolution:

$$y(t) = (Hu)(t) = h * u = \int_{-\infty}^{\infty} h(t-\tau) u(\tau) d\tau$$ \hspace{1cm} A-1

$h$ is actually the impulse response of the system. Indeed if $u(t)=\delta(t)$, Dirac impulse, then $y(t)=h(t)$. Obviously $H$ defines a linear map and the corresponding input-output system is also called linear.

We need to specify the class $U$ of signals $u$ which can be taken as input for $H$. It can be for instance $L^\infty$ or $L^2$. Then if $H$ defines a map $U \rightarrow Y$ where $U$ is the input class and $Y$ is the output class, we call $H$ bounded if there is a constant $M$ such that:

$$\|H(u)\| \leq M \|u\| \quad \text{pour tout } u \in U$$

And we can define the norm $\|H\|$ of the system in several equivalent ways:

$$\|H\| = \inf_{M} \{ M / \|Hu\| \leq M \|u\|, \text{pour tout } u \in U \}$$

$$\|H\| = \sup_{u \in U, \|u\| \leq 1} \|Hu\|$$ \hspace{1cm} A-2

$$\|H\| = \sup_{u \in U, \|u\|=1} \|Hu\|$$

**A-III-Multivariable generalization.**

We want to generalize the measures of the relative size of a single input single output system for multivariable systems.

Throughout this section we will consider an input-output system with $m$ inputs and $p$ outputs. The equation A.1 remains the same but now $h(t)$ is a real matrix of dimension $p \times m$.

$$y(t) = (Hu)(t) = h * u = \int_{-\infty}^{\infty} h(t-\tau) u(\tau) d\tau$$

Thus the associated transfer function $H(s) = \int_{-\infty}^{\infty} h(t)e^{st} dt$ has also dimension $p \times m$.

Let us again assume that the system is stable, that means all entries $[H(s)]_{ij}$ of $H(s)$ ($i=1,\ldots,p$) and ($j=1,\ldots,m$) have their poles in the left half plane or equivalently the impulse responses
[h(t)]_{ij} belong to \mathcal{L}_1. A bounded input signal gives a bounded output signal.

The singular value decomposition.

**Definition** A singular value decomposition (SVD) of a matrix \( H \in \mathbb{R}^{p \times m} \) is a decomposition \( H = Y \Sigma U \) where \( Y \in \mathbb{R}^{p \times p} \) is orthogonal
\[
Y \in \mathbb{R}^{p \times p}
\]
is orthogonal
\[
U \in \mathbb{R}^{m \times m}
\]
is diagonal.

\[
\Sigma = \begin{pmatrix}
\Sigma'_{1,1} & 0 & \cdots & 0 \\
0 & \Sigma'_{2,2} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \Sigma'_{r,r}
\end{pmatrix}
\]

with \( \Sigma'_{1,1} = \begin{pmatrix}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \sigma_r
\end{pmatrix} \)

and \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0 \). Every matrix \( H \) has such a decomposition and the ordered positive number \( \sigma_1, \sigma_2, \ldots, \sigma_r \) are unique and are called singular values of \( H \).(They correspond to the square root of the eigenvalues of the symmetric matrix \( H^T H \)).

The induced norm of \( H \) is related to the singular value decomposition as follows:

\[
\|H\|_2 = \sup_{u \in \mathbb{R}^m} \frac{\|Hu\|}{\|u\|} = \frac{\|Hu_1\|}{\|u_1\|} = \sigma_1
\]

**Proof:** see [2].

In other words, the largest singular value \( \sigma_1 \) of \( H \) equals to the induced norm of \( H \) (viewed as a function from \( \mathbb{R}^m \) to \( \mathbb{R}^p \)). This value can be considered as the maximal 'gain' of the matrix \( H \).
Appendix B

Zeros of a multivariable system

B-I- The Smith and MacMillan form of a matrix

see [3].

**Theorem B-1** Let $G(s)$ be a rational matrix $m \times p$ of normal rank $r$. Then $G(s)$ may be transformed by a series of elementary row and column operations into a pseudo-diagonal rational matrix $M(s)$ of the form (assuming $m < p$):

\[
M(s) = \begin{pmatrix}
\frac{e_1(s)}{\Psi_1(s)} & 0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \frac{e_2(s)}{\Psi_2(s)} & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \frac{e_r(s)}{\Psi_r(s)} & 0 & \ldots & 0 \\
\vdots & \ldots & \ldots & 0 & 0_{(m-r) \times r} & \ldots & 0_{(m-r) \times (p-r)}
\end{pmatrix}
\]

in which the monic polynomial \{\(e_i(s)/\Psi_i(s)\)\} are coprime for each $i$ (they have no common factor) and satisfy the divisibility properties

- $e_i(s)$ is a divisor of $e_{i+1}(s)$
- $\Psi_{i+1}(s)$ is a divisor of $\Psi_i(s)$ \(i = 1, \ldots, r-1\).

$M(s)$ is the Smith-MacMillan form of $G(s)$.

The $e_i(s)$ are called the invariant factors of $G(s)$.

For instance for a full row rank matrix (obviously $p > m$)

\[
M(s) = \begin{pmatrix}
\frac{e_1(s)}{\Psi_1(s)} & 0 & 0 & \ldots & \ldots & \ldots & 0 \\
0 & \frac{e_2(s)}{\Psi_2(s)} & 0 & \ldots & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \frac{e_m(s)}{\Psi_m(s)} & 0 & \ldots & 0 \\
\end{pmatrix}
\]

B-II- Poles and zeros.

**Definition B-1** Let $G(s)$ be a rational transfer function matrix with Smith-MacMillan form $M(s)$, as previously, and define the pole polynomial and zero polynomial

\[
p(s) = e_1(s) \ldots e_r(s)
\]

\[
z(s) = \Psi_1(s) \ldots \Psi_r(s)
\]

The roots of $p(s)$ are called the poles of $G(s)$ and the roots of $z(s)$ are called the zeros of $G(s)$. 

Appendix C

The filter parameters

see [6]

C-I-Influence of $\theta$

Let us compare the real time simulation of the vertical acceleration and of $q_1 - q_0$ for $\theta=10^{-2}$ and $\theta=10^{-4}$.

left column $\theta=10^{-2}$
right column $\theta=10^{-4}$

When $\theta$ increases, the system is less sensitive to noise but it absorbs the shock of the deterministic bump less.

Note: For a too high level for $\theta$, ($\theta=10$), there is no stable solution.

It is possible to check it out by comparing for instance the two Bode diagrams of the transfer
functions which map $d_1 \rightarrow q_1 - q_0$ for the same values of $\theta$.
The dotted lines correspond to the transfer function in an open loop, the unbroken lines to
that in a closed loop.

The graph for $\theta = 10^{-2}$ attenuates the high frequencies less, on the other hand it attenuates the
frequencies between $10^2$ and 1 more which means that the vehicle is less influenced by the
deterministic road profile: see the amplitude of the first oscillation on figure C.I.3 and C.I.4.
By increasing the weight of $\theta$ to $10^{-1}$, the importance of the noise input $d_2$ grows up in the
computation of the optimal controller, that's why the noise is reduced.

C-II Theory about the use of weighting filters.

see [6] and [2].
Bibliography


