Fractional-calculus-based electromagnetic tool to study pulse propagation in arbitrary dispersive dielectrics

Citation for published version (APA):

Mescia, L., Bia, P., & Caratelli, D. (2019). Fractional-calculus-based electromagnetic tool to study pulse propagation in arbitrary dispersive dielectrics. *Physica Status Solidi A : Applications and material science*, 216(3), [1800557]. <https://doi.org/10.1002/pssa.201800557>

DOI:

[10.1002/pssa.201800557](https://doi.org/10.1002/pssa.201800557)

Document status and date:

Published: 06/02/2019

Document Version:

Accepted manuscript including changes made at the peer-review stage

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Fractional-Calculus-Based Electromagnetic Tool to Study Pulse Propagation in Arbitrary Dispersive Dielectrics

Luciano Mescia^{*1}, Pietro Bia¹, Diego Caratelli²

¹ Department of Electrical and Information Engineering, Polytechnic University of Bari, Via E. Orabona 4, 70125 Bari-Italy

² The Antenna Company, High Tech Campus 41, 5656 AE Eindhoven, The Netherlands

Key words: Dielectric relaxation, dispersive media, fractional derivative, propagation model, computational algorithm

* Corresponding author: e-mail luciano.mescia@poliba.it

Fractional calculus is a fruitful field of research in science and engineering. The concept of fractional exponents is an outgrowth of exponents with integer value. In the same way, fractional order of integration is a generalization of the mathematical operations of differentiation and integration to arbitrary, general, noninteger order. Although it better models the higher complexity by nature, it is still fairly easy to physically represent its meaning. However, taking in consideration that the study of fractional calculus theory opens the mind to entirely new branches of thought, in this feature article we illustrate as such concept can be an interesting and useful tools in electromagnetic theory to solve specific electromagnetic problems regarding the wave propagation and radiation in arbitrary dispersive dielectric materials. In particular, time fractional Maxwell's equations for media with power-law frequency dispersions are illustrated focusing the attention on the mathematical and computational topics. Moreover, practical examples highlighting their usefulness for understanding a variety of electromagnetic phenomena are provided.

1 Introduction. The modeling of the pulsed electric field (PEF) propagation in disordered and dispersive dielectrics is a subject of increasing research activities since they are found in a growing number of applications [1–8]. The feasibility to change the dielectric response of a medium to generate unusual electromagnetic properties in artificially structured materials have been investigated in a variety of engineering applications including periodic and aperiodic subwavelength aperture arrays and left-handed materials [9–11]. Moreover, the estimation of dielectric response of an unknown material inside a lossy and dispersive structures is widely used in ground penetrating radar technique for the nondestructive diagnostic of buried targets and material characterization [12–14,8]. On the other hand, biological tissues are an important class of dispersive dielectric media playing a key role in a number of new research activities in bioelectrics, a new interdisciplinary field combining knowledge of electromagnetic principles and theory, modeling and simulations, physics, material

science, cell biology, and medicine [15]. The main goal of this discipline is the study of the interaction between electromagnetic radiation and biological tissues aimed at the investigation of nanopulse bioeffects for human safety, as well as at the use of ultrashort pulses in a variety of therapeutic and diagnostic applications such as hyperthermia, electroporation and the treatment of specific diseases [16–20]. Many research activities concerning microwave dielectric spectroscopy and imaging are in progress since they represent potential solutions for early-stage cancer detection and treatment, cancer investigations and diagnostic as well as to study the properties of materials occurring in production and processing of crops and food of agricultural origin [21–23]. Again, several studies are focused on the use of electromagnetic waves for discriminating malignant tissues from healthy ones. Moreover, PEFs are successfully used for reversible or irreversible electroporation to achieve selective killing of cancer cells, tissue ablation, gene therapy, and DNA based vaccination. Finally, recent

applications employing PEFs technology include medical implant communication service, wireless medical telemetry service, body area networks, nanonetworks in a living biological environment, in-body electromagnetic communications [24–27].

All these technologies involve the interaction of electromagnetic fields with complex dielectric materials. For instance, the cellular death mechanism due to the PEF treatment is strongly affected by electric pulse parameters such as amplitude, duration, number of pulses, and repetition frequency. Moreover, the right knowledge of the electromagnetic field distribution inside biological tissues is essential to demonstrate the effects of electromagnetic radiation absorption. Again, in the context of the in-body communications among the nano devices the right evaluation of the network performance in term of data rate and transmission range needs of an accurate modelling of the electromagnetic field propagation inside human tissues. In many applicative cases, the electromagnetic field cannot be easily measured. As a result, the development of theoretical models and computational techniques to determine the propagation properties of electromagnetic pulses is fundamental to gain insight into the several phenomena occurring within complex dielectric materials subject to an imposed electromagnetic field. In particular, numerical simulations under various conditions can be used to identify the fundamental parameters involved in noninvasive diagnosis and medical sensors as well as to provide guidance for computational dosimetry and for the development of specific therapeutic approaches.

The electromagnetic field distribution inside more complex or disordered dielectric materials such as polymers, biopolymers, emulsions and microemulsions, biological cells and tissues, porous materials mainly depends on their electric properties, but the lack of data and accurate models, over broad frequency ranges, has thus far been an obstacle for both theoretical and experimental studies. In fact, the complexity of the structure and composition of such matter produces anomalies in the dynamic dielectric properties resulting in a strong dispersion of dielectric susceptibility. This dispersion can be explained by considering that the disordered nature and microstructure of the systems yield multiple relaxation times resulting in a time-domain response generally nonsymmetric and markedly different from that of dielectric media modeled by the simple dielectric response relationships. As a consequence, an accurate representation of the experimental dielectric response in the frequency domain usually requires empirical models exhibiting fractional powers of the angular frequency $j\omega$. Due to this, the solution of the Maxwell's equations in the time domain is not trivial since it involves the concept of fractional derivatives. Just as fractional exponents may find their way into innumerable equations and applications, it will become apparent that integrations of fractional order can find practical use in many modern problems [28–32]. To this aim, in this feature article we

illustrate a generalization of the Maxwell's equations using fractional calculus theory and a general fractional function to model the multiple higher order dispersive relaxation processes defining the dielectric response. In particular, a novel FDTD algorithm for studying electromagnetic pulse propagation in arbitrary dispersive dielectric is presented from both mathematical and computational point of views. The resulting formulation is explicit, it has a second-order accuracy, and the need for additional storage variables is minimal. Examples showing the pulse propagation in a variety of dispersive media are examined with the aim to demonstrate the capability of the proposed method to solve complex electromagnetic problems.

2 Dielectric function models. The interaction of heterogeneous mixtures of dielectric materials with electromagnetic radiation takes place in different relaxation processes including (i) reorientation of dipolar molecules, (ii) interfacial polarization arising by the presence of two or more regions with different electrical properties, (iii) ionic diffusion, (iv) motion of the molecules, (v) relaxations due to the nonspherical shape as well as in different resonant phenomena due to molecular, atomic or electronic vibrations. Their resulting behavior causes a frequency dispersion pattern of permittivity and conductivity [33,34]. Many efforts have been made to model this behavior for a multitude of different materials at a variety of frequency ranges and bandwidths. Within this framework, the Debye model has been widely used to describe a very simple dielectric response arising by the dipolar relaxation [35, 36]. It is a simple expression characterized by a single relaxation time that is capable of handling materials having high water content. Unfortunately, numerous experimental studies have demonstrated that the relaxation behavior of a wide range of dielectric materials deviates strongly from the Debye relaxation law. As a consequence, the demonstration of a number of phenomena such as broadness, asymmetry and excess in the dielectric dispersion has motivated the development of empirical response functions as Cole-Cole, Cole-Davidson, Havriliak-Negami, Kohlrausch-Williams-Watts, Jonscher, and Raicu [37–42].

Under the influence of a step-wise electric field, the material polarization occurs. In linear and isotropic medium, the effect of polarization is expressed in frequency domain by the expression [43]

$$\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega) \quad (1)$$

where $\mathbf{D}$, $\mathbf{E}$ and ε are the electric flux density, the electric field and the material relative permittivity, respectively. Moreover, the relaxation properties of different media (dielectrics, semiconductors, ferromagnetics, and so on) are normally expressed in terms of the response function $\phi(\omega)$ related to the frequency dependence of dielectric permittivity through the following relation

$$\phi(\omega) = \frac{\varepsilon(\omega) - \varepsilon_\infty}{\varepsilon_s - \varepsilon_\infty} \quad (2)$$

where ε_∞ is the asymptotic relative permittivity at high frequency and ε_s is the static relative permittivity. The classical Debye expression is

$$\phi(\omega) = \frac{1}{1 + j\omega\tau} \quad (3)$$

where τ is the relaxation time characterizing the Debye process. In this case, the time-domain response function turn out to be purely exponential since the inverse Fourier tranform gives

$$\phi(t) = \frac{1}{\tau} e^{-t/\tau} \quad (4)$$

An useful technique allowing the reproduction of the experimental spectra $\varepsilon(\omega)$ by adjusting free parameters of a mathematical expression is based on the use of an empirical dielectric function model in conjunction with non-linear least square optimization. Using this method, other empirical response functions exhibiting a broad distribution of relaxation times have been proposed. In particular, the Cole-Cole (C-C) function is

$$\phi(\omega) = \frac{1}{1 + (j\omega\tau)^\alpha} \quad (5)$$

the Cole-Davidson (C-D) function is

$$\phi(\omega) = \frac{1}{(1 + j\omega\tau)^\beta} \quad (6)$$

the Havriliak-Negami (H-N) function is

$$\phi(\omega) = \frac{1}{[1 + (j\omega\tau)^\alpha]^\beta} \quad (7)$$

and the Raicu (R) function is

$$\phi(\omega) = \frac{1}{[(j\omega\tau)^\gamma + (j\omega\tau)^\alpha]^\beta} \quad (8)$$

In Eq. (5)-(8) the adjusting parameters $0 \leq \alpha, \beta, \gamma \leq 1$ account for shape and behavioral features of the permittivity function. However, the resonant dispersion characteristics due to the fractional oscillator can modelled using Eq. (5) with $1 < \alpha \leq 2$ [44]. Moreover, tacking the inverse Fourier transform of Eq. (5)-(8) it can be inferred that the time-domain response function strongly deviate from the exponential Debye law.

The empirical relationships Eq. (5)-(8) work well for certain materials under specific conditions, but not for others. In fact, their effectiveness may fall when more complex or disordered systems having heterogeneous, inhomogeneous and disordered structure at both microscopic and mesoscopic scales are employed. To overcome this limitation and to provide an extended model flexibility enabling a better parametrization of the arbitrary dispersive media properties as well as a better fitting, over broad frequency ranges, of the experimental data, other models have been

discussed in literature. A modified version of the Havriliak-Negami model has been recently derived with the aim to fit some experimental data exhibiting a less typical two-power-law relaxation pattern with frequency power law exponents m and n satisfying $m < 1 - n$ [45,46]. In that connection, a more general empirical power law model has been proposed by the authors in [1]. In particular, using a general fractional polynomial series approximation, the characteristics of a general dispersive medium exhibiting multirelaxation processes is modeled by using the following relationship

$$\phi(\omega) = \sum_{i=1}^N \frac{\Delta\varepsilon_i}{\sum_{k=0}^K b_{k,i} (j\omega\tau_{k,i})^{\beta_{k,i}}} \quad (9)$$

where $b_{k,i}, \beta_{k,i}$ denote suitable real-valued parameters chosen so that the following conditions hold true

$$b_{i,k} > 0 \quad (10)$$

$$\lim_{\omega \rightarrow \infty} \frac{1}{\sum_{k=0}^K b_{k,i} (j\omega\tau_{k,i})^{\beta_{k,i}}} = 0 \quad (11)$$

$$\text{Im} \left\{ \sum_{k=0}^K b_{k,i} (j\omega\tau_{k,i})^{\beta_{k,i}} \right\} \geq 0 \quad (12)$$

Eq. (10) avoids model singularities and it reduces instabilities of the time-marching scheme adopted for the numerical solution of Maxwell's equations. Eq. (11) defines the consistency of the representation, and Eq. (12) ensures the passivity. Such power law modelling has been developed to overcome the limitations of the dielectric response presented in [7]. Moreover, it allows to derive the equations governing the evolution of the dielectric processes by using non-local pseudo-differential operators of non-integer order. On the other hand, the nature of the fractional order operators modelling the dielectric response enables its incorporation into time-domain Maxwell's equations. Finally, applying a dedicated optimization algorithm based on the enhanced weighted quantum particle swarm optimization and a suitable relative error function[47] the free parameters $b_{k,i}, \beta_{k,i}, \tau_{k,i}, K, N$ can be evaluated. This method is versatile because its capability to deal with every data, it can reproduce fine details, and it proved to feature superior effectiveness in terms of convergence rate and accuracy in comparison with alternative evolutionary stochastic search methods available in the scientific literature [48,49].

3 FDTD dispersive modelling. The FDTD technique, based on the direct time integration of Maxwell's equations using the central finite difference method, is a well-known numerically robust and appropriate method for the computer technology of today. It is relatively simple, can easily deal with a broadband response, has almost no limit in the description of geometrical and dispersive properties of the material being simulated [50]. However, one of the strengths of FDTD is that a single simulation can provide broadband results by using a pulsed excitation.

Maxwell's equations in dispersive media constitute a stiff problem and the development of FDTD methods to study the transient wave propagation in such media is an area of active interest. In particular, the main challenge of the FDTD algorithm is the numerical treatment of the constitutive relation (see Eq. (1)), which can be addressed in two alternative ways. The first approach uses a recursive convolution of dielectric susceptibility and the corresponding field components in the time-domain [51–53], and the second one is based on the discretization of an auxiliary differential equation [54–56]. To embed dispersion into the FDTD algorithm, the frequency dependent dielectric properties have often been described by Debye model (see Eq. (3)) [57–59] that requires the least computational overhead. Moreover, various methods have been suggested for modelling Lorentzian, modified Lorentzian and Drude media [60–64] as well as for approximating the relative permittivity using rational or polynomial functions [65–68].

Thanks to the rapid development of computing and data storage technology, new FDTD algorithms have become feasible for modelling the dispersion characteristics described by Eq. (5)–(9) with high accuracy. However, the implementation of such models using the FDTD method is difficult since the corresponding time-domain polarization relation contains fractional order derivatives, whose discretization is troublesome. Various methods have been suggested capable of implementing the C-C dispersion relation in FDTD codes. Some researchers provided a transformation from the more accurate C-C dispersion relation, in frequency domain, to the multiple-pole Debye relation with a coarse accuracy [69, 70]. Other methods implementing the C-C dispersion in FDTD utilize the Z-transform that may require storing of the electric field and/or some auxiliary field quantities for a few previous time steps leading to large memory requirement [71, 72]. An interesting approach for incorporating the C-C dispersion in FDTD is presented in [73] where the distribution function of the relaxation times was sampled in a finite number of values. This method could be potentially very accurate but at the same time it requires a number of distinct polarization equations for every pole of the C-C model making the work very tedious for only one pole. To address such shortcoming, alternative FDTD implementations based on the sampling of the distribution of relaxation time and the convolution integral formulation have been presented [74]. This approach required storing of the electric field and some related auxiliary quantities for only the previous time-step. Other techniques for FDTD modeling of wave propagation in n th-order dispersive media based on Mobius transformation technique and the Padé representation in Z-transform space are illustrated in [75, 76].

Alternative approaches based on the application of fractional derivatives and recursive evaluated integrals are suggested in [77, 78]. The accuracy of the method illustrated in [77] was not high and the recursive update of

the convolution integral is provided by a sum of decaying exponentials with suitable coefficients ensuring reasonable accuracy. Using the auxiliary equations in conjunction with the idea of fractional derivatives, the incorporation of the multiterm Cole-Cole model in the FDTD scheme was illustrated in [76]. Such method demonstrated excellent accuracy with minimum auxiliary variables and thus minimum storage spaces. However, the main limit of such approaches is that they can only deal with n th-order Debye and Cole-Cole models. Considering this drawback and the requirement to model a more general permittivity function (see Eq. (5)–(9)), the authors proposed a scheme in which the double fractional derivative operator relevant to the general H-N response is directly incorporated in the FDTD basic algorithm, thus avoiding the use of ADEs [2, 7]. The developed numerical method was applied to the analysis of pulse-wave propagation in general dispersive media modeled by C-C, C-D, and H-N equations. The proposed formulation is based on the optimal truncation of the binomial series relevant to the H-N fractional derivative operator, in accordance with the Riemann-Liouville theory. The performed tests demonstrated that the developed method provides a reliable and very accurate results for $\omega\tau \leq 10$, $\alpha < 0.5$, and $\beta < 0.25$. On the other hand, it suffers of a reduced numerical accuracy for $\omega\tau \geq 10$, $\alpha > 0.5$, and $\beta > 0.25$. To overcome this limitation, the authors have extended the FDTD scheme by implementing a more general series representation of the fractional derivative operators [1, 79]. This study was motivated to seek for an extended model flexibility enabling a better parametrization of the dispersive media properties as well as a better fitting, over broad frequency ranges, of the experimental dielectric response.

3.1 Fractional calculus based FDTD scheme. For a nonmagnetic dispersive dielectric material with response function described by Eq. (9), the differential version of Maxwell's equations in frequency domain are given by

$$\begin{aligned}\nabla \times \mathbf{H} &= j\omega\epsilon_0 \left(\epsilon_\infty + \sum_{i=1}^N \frac{\Delta\epsilon_i}{Q_i} - j \frac{\sigma}{\omega\epsilon_0} \right) \mathbf{E} \\ &= j\omega\epsilon_0 \left(\epsilon_\infty - j \frac{\sigma}{\omega\epsilon_0} \right) \mathbf{E} + \sum_{i=1}^N \mathbf{J}_i\end{aligned}\quad (13)$$

and

$$\nabla \times \mathbf{E} = -j\omega\mu_0 \mathbf{H} \quad (14)$$

where σ is the static conductivity, $\mathbf{H}$ is the magnetic field, and the p th term of the auxiliary displacement current density $\mathbf{J} = \sum_{i=1}^N \mathbf{J}_i$ is given by

$$\mathbf{J}_p = j\omega\epsilon_0 \frac{\Delta\epsilon_p}{Q_p} \mathbf{E} \quad (15)$$

where

$$Q_p = \sum_{k=0}^K b_{k,p} (j\omega\tau_{k,p})^{\beta_{k,p}} \quad (16)$$

Taking the inverse Fourier transform of eq. (15), it can be found

$$\mathcal{D}_t^{(p)} \mathbf{J}_p = \varepsilon_0 \Delta \varepsilon_p \frac{\partial \mathbf{E}}{\partial t} \quad (17)$$

involving the fractional derivative operator

$$\mathcal{D}_t^{(p)} = \sum_{k=0}^K b_{k,p} \tau_{k,p}^{\beta_{k,p}} D_t^{\beta_{k,p}} \quad (18)$$

where $D_t^{\beta_{k,p}}$ is the time derivative of fractional order $\beta_{k,p}$. Denoting $\mathcal{B}_{k,p} = [\beta_{k,p}] + 1$, $[\cdot]$ being the integer part operator, and considering the Riemann-Liouville fractional derivative operator it follows:

$$\mathcal{D}_t^{(p)} \mathbf{J}_p = \sum_{k=0}^K \frac{b_{k,p} \tau_{k,p}^{\beta_{k,p}}}{\Gamma(\mathcal{B}_{k,p} - \beta_{k,p})} \frac{\partial^{\mathcal{B}_{k,p}}}{\partial t} \int_0^t \frac{\mathbf{J}_p \mathrm{d}u}{(t-u)^{\beta_{k,p} - \mathcal{B}_{k,p} + 1}} \quad (19)$$

Applying a second-order accurate finite-difference scheme at the time instant $t = m\Delta t$ and using the semi-implicit approximation:

$$\mathbf{E}|^m = \frac{\mathbf{E}|^{m-1/2} + \mathbf{E}|^{m+1/2}}{2} \quad (20)$$

$$\mathbf{J}|^m = \frac{\mathbf{J}|^{m-1/2} + \mathbf{J}|^{m+1/2}}{2} \quad (21)$$

and the numerical approximation

$$\left. \frac{\partial \mathbf{E}}{\partial t} \right|^m = \frac{\mathbf{E}|^{m+1/2} - \mathbf{E}|^{m-1/2}}{\Delta t}, \quad (22)$$

after some mathematical manipulations, well detailed in [1, 7, 26, 79], it is possible to find the following updating equations for the magnetic and electric fields as well as for the displacement current density:

$$\mathbf{H}|^{m+1} = \mathbf{H}|^m - \frac{\Delta t}{\mu_0} \nabla \times \mathbf{E}|^{m+1/2} \quad (23)$$

$$\mathbf{E}|^{m+1/2} = \frac{2\varepsilon_0\varepsilon_\infty - \sigma\Delta t}{2\varepsilon_0\varepsilon_\infty + \sigma\Delta t} \mathbf{E}|^{m-1/2} + \frac{2\Delta t}{2\varepsilon_0\varepsilon_\infty + \sigma\Delta t} \left[\nabla \times \mathbf{H}|^m - \frac{1}{2} \sum_{i=1}^N \left(\mathbf{J}_i|^{m-1/2} + \mathbf{J}_i|^{m+1/2} \right) \right] \quad (24)$$

$$\begin{aligned} & \left(\varepsilon_\infty + \frac{\sigma\Delta t}{2\varepsilon_0} \right) \frac{C^{(\beta_{k,p})}}{\Delta \varepsilon_p} \mathbf{J}_p|^{m+1/2} + \frac{1}{2} \sum_{i=1}^N \mathbf{J}_i|^{m+1/2} \\ &= \nabla \times \mathbf{H}|^m - \sigma \nabla \times \mathbf{E}|^{m-1/2} + \\ & - \frac{1}{2} \sum_{i=1}^N \mathbf{J}_i|^{m-1/2} - \frac{1}{\Delta \varepsilon_p} \left(\varepsilon_\infty + \frac{\sigma\Delta t}{2\varepsilon_0} \right) \\ & \left[\sum_{k=0}^{K_i} \sum_{s=1}^{\nu} \xi_{k,s}^{(\beta_{k,p})} \mathbf{J}_p|^{m-s+1/2} + \right. \\ & \left. + \sum_{k=0}^{K_i} \sum_{s=1}^{\nu} \sum_{q=1}^{Q_{k,\alpha p}} \eta_{k,s,q}^{(\beta_{k,p})} \psi_q^{(\beta_{k,p})} \right]^{m-s} \end{aligned} \quad (25)$$

In the above equations, the curl of magnetic and electric fields are approximated by the standard Yee algorithm in a uniform space-time grid.

3.2 UPML boundary conditions. As well known, no computer can store an unlimited amount of data, and therefore, the computational domain has to be limited in size. For such problems, and especially for FDTD method, an absorbing boundary condition (ABC) should be used with the aim to complete the interior numerical scheme and to simulate the expression of the lattice to infinity. The perfect matched layer (PML) introduced by Bérenger is a effective ABC to terminate domains employing free space, lossy, dispersive, inhomogeneous, anisotropic and nonlinear media [80–82]. In particular, the innovation of the PML is that plane waves of arbitrary incidence, polarization, and frequency are matched at the boundary. Lots of papers appeared to validate PML technique and among the various implementations of PMLs the uniaxial anisotropic PML (UPML), presented by Gedney [83], appears very attractive in view of the fact that Maxwell's equations maintain their familiar physical form as well as its application to the FDTD method is more computationally efficient and ideal for parallel computation.

In the fractional-calculus-based FDTD algorithm the numerical implementation of the UPML conditions requires special treatments due to the approximation of fractional derivatives into the simulator. To this aim, the authors derived dedicated UPML boundary conditions in combination with the basic time-marching scheme accounting for the electrical conductivity and the multirelaxation characteristics of the dielectric material under analysis [1, 7]. In particular, the developed approach combines the stretched auxiliary electric field and density current vectors with the fractional derivative equation describing the dispersion properties of the medium. The discretization of the resulting equations on the Yee lattice was carried out by adopting the usual leapfrog scheme in time, wherein the loss terms was averaged according to the semiimplicit approximation. In particular, in accordance with the complex coordinate stretching approach [50], by introducing the auxiliary electric field vector, $\mathbf{e}$, and by following a mathematical procedure well detailed in [1, 7], the update equations for both electric and magnetic field within the

UPML termination can be written as

$$\begin{aligned} \mathbf{E}|^{m+1/2} &= \frac{2\varepsilon_0\kappa_x - \sigma_x\Delta t}{2\varepsilon_0\kappa_x + \sigma_x\Delta t} \mathbf{E}|^{m-1/2} + \\ &+ \frac{2\varepsilon_0}{2\varepsilon_0\kappa_x + \sigma_x\Delta t} \left(\mathbf{e}|^{m+1/2} - \mathbf{e}|^{m-1/2} \right) \end{aligned} \quad (26)$$

and

$$\begin{aligned} \mathbf{H}|^{m+1} &= \frac{2\varepsilon_0\kappa_x - \sigma_x\Delta t}{2\varepsilon_0\kappa_x + \sigma_x\Delta t} \mathbf{H}|^m + \\ &- \frac{2Y_0\Delta t}{2\varepsilon_0\kappa_x + \sigma_x\Delta t} \nabla \times \mathbf{E}|^{m+1/2} \end{aligned} \quad (27)$$

where σ_x, κ_x are the UPML material parameters, and $Y_0 = (\varepsilon_0/\mu_0)^{1/2}$ is the wave admittance in free space. However, the maximum value assumed by the conductivity at the truncation of the UPML region, the UPML thickness, and the polynomial function describing the UPML conductivity profile have been optimized to enhance the absorption of electromagnetic waves thus minimizing the spurious reflection level in the solution domain.

3.3 Numerical stability Considering that the FDTD algorithm samples the electromagnetic field at discrete points both in time and space, the choice of the sampling periods has to comply with specific restrictions to guarantee the stability of the solution. In fact, these parameters affect the accuracy of the solution and if their choosing is not appropriate the computed electric and magnetic field components will increase without limit as the simulation progresses. The stability analysis of the FDTD scheme is usually performed following the von Neumanns spectral approach [50, 84, 85]. In particular, to ensure numerical stability the time step needs to satisfy the Courant-Friedrich-Lewy (CFL) criterion. This condition is generally derived with the assumption of homogeneous lossless dielectrics and linear dispersive materials. However, to the best of the authors knowledge, there is still no satisfactory theory to explain the stability of FDTD formulation containing media characterized by a power-law dielectric function. To this aim, in [1] the authors derived the characteristic equation for the fractional-calculus-based FDTD algorithm demonstrating that the corresponding eigenvalues remain bounded. In particular, assuming the time-harmonic dependence of the electromagnetic field the first-order derivatives in time and space degenerate in unnormalized cardinal sine function. Then, by introducing the numerical amplification factor, z , it is possible to derive, after quite lengthy calculations, the following characteristic polynomial equation

$$[c_0\Delta t\Xi(\xi)z]^2 \tilde{D}_r(z) + (z^2 - 1)^2 \tilde{N}_r(z) = 0, \quad (28)$$

where Ξ is the algebraic operator corresponding to the transformation of the first-order time derivative, $\tilde{D}_r, \tilde{N}_r$ are two polynomials defining the transformed permittivity function as a rational function. Using the Jenkins-Traub

Figure 1 Location of the zeros z_i of the stability polynomial associated with the FDTD scheme used to characterize an H-N dielectric material. © 2016 IEEE. Reprinted, with permission, from IEEE Transactions on Antennas and Propagation, Fractional-Calculus-Based FDTD Algorithm for Ultrawideband Electromagnetic Characterization of Arbitrary Dispersive Dielectric Materials.

root-finding method for polynomials with complex-valued coefficients it is possible to calculate the eigenvalues z_i satisfying Eq. (28). In particular, to check the numerical stability of the proposed algorithm the location of z_i has been analyzed as a function of the Courant factor and the normalized spatial frequency of the general plane-wave mode propagating within the FDTD lattice. Fig. 1 shows the location of the zeros z_i for the test case illustrated in [1] and pertaining the single-layered H-N dielectric slab in air. The obtained results highlighted that the spectral radius of the polynomial equation (28) is always smaller than unity, thus verifying the numerical stability of the proposed time-marching scheme.

3.4 Test cases The reliability of the FDTD procedure based on Riemann-Liouville fractional operator theory has been assessed for modeling the ultrawideband electromagnetic pulse propagation in single and multilayered systems employing various dielectrics featuring the Debye, C-C, C-D, H-N, R dispersion relationships as well as to address complex electromagnetic problems involving spatial-dependent permittivity modelled by Maxwell-Garnet and Bruggeman approaches [1, 2, 7, 79]. All the tests have been done implementing the conventional total field/scattered field formulation and a sinusoidally time-modulated Gaussian pulse source [50]. Moreover, just for the 1-D problems, the results computed using our fractional-calculus-based FDTD scheme have been compared with those derived using a rigorous fully analytical technique based on

Figure 2 Reflectance and transmittance spectra of a three-layered H-N dielectric slab with multiple relaxation times. Adapted from [1].

the transfer matrix approach [86]. In [7], single layer C-C, C-D, and H-N media was investigated in detail focusing the attention on the UPML characteristics and checking that the numerical electromagnetic field distribution does not feature any long time divergence. In particular, it was inferred that UPML having thickness $d_{\text{UPML}} = 30\Delta z$, and polynomial order $m = 7$ has optimal absorbing properties. In [1], we analyzed the short-pulse wave propagation in general layered dispersive materials characterized by multiple relaxation times giving emphasis to the relevant numerical stability and to the optimization procedure adopted to derive the fractional power series representation of the permittivity function and to perform the nonlinear fitting of the exponential terms appearing in the finite-difference approximation of the fractional derivative operator. A relevant test regarded the electromagnetic characterization of a three-layered H-N dispersive dielectric slab over an ultrawide frequency band. In fact, because the multiple and very different relaxation times characterizing each layer, it was a stress test aimed to demonstrate the reliability and robustness of the proposed algorithm. Fig. 2 shows the reflectance and transmittance spectra as evaluated by using the methodology illustrated in [1] and [7] as well as a rigorous analytical approach. The excellent agreement with the analytical technique validates the enhancement of the numerical accuracy of the new methodology illustrated in [1] in comparison with the previous one introduced in [7]. However, the numerical simulations was performed over a time interval more than 300 times the duration of the source pulse showing a well-behaved distribution and asymptotically decreasing energy over time.

The extended FDTD scheme was also applied to evaluate the pulse propagation in multilayered slab involving dielectrics with higher order dispersion characteristics such as those exhibited by R equation with multiple relaxation

Figure 3 Reflectance and transmittance spectra of a three-layered R dielectric slab with multiple relaxation times. Adapted from [1].

times. Considering the higher order fractional-power-law, the R representation poses nontrivial problems in the embedding of the model into the core of an existing FDTD scheme. Thus, the capability of the proposed methodology to easily overcome such drawback providing accurate results further confirms its usefulness for addressing of complex electromagnetic problems. Just to justify this statement, Fig. 3 shows the reflectance and transmittance versus frequency as computed by means of the developed time domain technique and the analytical transfer-matrix-based method in frequency domain for layered R dielectric slab with multiple relaxation times. Clearly, the agreement between the two approaches is very good.

To prove just how meddlesome the developed FDTD technique is, we also analyzed the pulse propagation inside dielectrics characterized by a spatial and power law frequency dependence of the electric permittivity. To this aim, both Maxwell-Garnet and Bruggeman models was taken into account [79]. In particular, a binary mixture composed by a bulk material and inclusions having ellipsoidal shape was considered. In these materials, further parameters as the depolarization factor and the space-dependent filling factors function define the dielectric permittivity. So, a number of investigations was performed considering linear and exponential filling factor functions as well as by changing both the depolarization factor and the mixing model. The comparison between Maxwell-Garnet and Bruggeman models shows differences in the propagation characteristics especially at low frequency. Such result highlights a critical point regarding the choice of the suitable mixing model since for denser composites Bruggemans formula is better suited than the Maxwell-Garnett one. Fig. 4 and 5 show the space-time distribution of the electric field pertaining a growing and decreasing piecewise filling factor functions, respectively. The transmission and reflection

Figure 4 Space-time electric field distribution inside a dispersive media modelled by a Maxwell-Garnet relationship characterized by a growing piecewise linear filling factor function. Data from [79].

Figure 5 Space-time electric field distribution inside a dispersive media modelled by a Maxwell-Garnet relationship characterized by a decaying piecewise linear filling factor function. Data from [79].

phenomena occurring at the air-dielectric interface is clear as well as the pulse spreading inside the dispersive material and the spatial variation of the group velocity are evident.

4 Special Applications. The developed FDTD technique provides a general-purpose tool useful to address different kind of complex electromagnetic problems. It could be useful in the field of the computational dosimetry to study the temperature elevation in biological tissues subject to PEF excitation. It could be successfully applied in the field of remotely powered implantable devices for

optimizing the trade-off between the received power and tissue absorption as well as in the field of the biological nanonetworks to provide a detailed channel characterization in terms of transmission rate and communication distance. Moreover, it could be used in the field of dielectric spectroscopy to estimate the thickness and/or the dielectric response of an unknown material inside a lossy and dispersive multilayer structures or in the field of ground penetrating radar for the nondestructive diagnostic of buried targets and material characterization.

4.1 Computational dosimetry. In general, when considering the interaction of electromagnetic fields with biological systems, it is important to take into account that part of the electromagnetic energy is absorbed by tissues. Such interaction leads to a temperature improvement which could cause deleterious thermal damage due to Joule heating. Thus, to achieve a quantitative understanding of biological responses, the induced electric field, power density and the derived dosimetric quantity of specific absorption rate (SAR) as well as the temperature rise inside the tissues have to be quantified and correlated with any observed phenomena. In view of these requirements and considering that the dielectric response of a broad variety of biological media can be described by power-law in the frequency domain, our fractional calculus-based FDTD algorithm was extended to incorporate further physics outlined by the Pennes bioheat equation, modelling the heat conduction, metabolism and blood perfusion effects, and the Gaggles two-node model, which takes into account the thermoregulation mechanisms as shivering, regulatory sweating and vasomotion [87]. A multi-layered human head structure consisting of skin, fat, bone, dura, CSF and brain was considered. So, the space-time temperature distribution inside the biological layers forming the human head was calculated as well as the temperature profiles with and without thermoregulation phenomenon was compared. Moreover, other investigations was carried out by changing electric parameters such as the amplitude and duration of the electric pulses. In particular, Fig.6 displays the spatial distribution of the SAR when the system is irradiated with a PEF burst having different hold times. It is clear that the SAR increases for longer pulse durations as well as, for shorter PEF signals, it is higher in the outer layer because the electromagnetic wave is strongly attenuated by the skin. On the other hand, for longer pulse duration the electromagnetic wave propagates deeper thus generating higher values of SAR inside the CSF layer.

Fig.7 and Fig.8 show the temperature versus the time for three different PEF durations at the air-skin boundary and at the outer surface of the brain layer, respectively. It is worth to notice that the temperature rise exhibits a non-linear dependance respect to the PEF duration. For shorter pulses, a brain temperature raise lower than 0.5°C occurs because the high frequency spectral components exhibit a strong attenuation before reaching the brain. On the other hand, for longer pulses the lower frequency spectral com-

Figure 6 SAR profile inside the multilayered structure for a PEF burst characterized by three different hold times. Data from [87].

Figure 7 Temperature evolution at the air-skin boundary for a PEF burst characterized by three different hold times. Data from [87].

ponents can reach the brain layer thus resulting in a higher temperature improvement.

4.2 Non-destructive diagnostic Ground-penetrating radar (GPR) is a popular non-destructive method used in many applications such as aquifer and soil studies, civil engineering, glaciology, archeology, and waste disposal programs. Numerous research efforts have been devoted to the numerical modelling of GPR. However, the main challenges to address for a successful application of GPR regard the prediction and understanding of the electromagnetic field propagation in heterogeneous environments. The investigation of this problem is the primary motivation for the computational modelling of GPR and considering that many soils and rocks can be simulated using

Figure 8 Temperature evolution at the outer surface of the brain layer for a PEF burst characterized by three different hold times. Data from [87].

Figure 9 Sketch of the investigated electromagnetic scattering problem.

empirical relationships based on fractional power law, our FDTD model could be a useful solution. Fig. 9 displays the considered electromagnetic problem. In particular, the electromagnetic pulse launched from the air half space spreads into the dispersive soil interacting with a lossy cylindrical cylinder. The frequency-dependent permittivity of the soil and the cylinder conductivity as those defined in [9]. To solve this problem, a 2-D FDTD formulation based on fractional calculus theory has been derived and expanded by using the general theory detailed in [1, 7]. Moreover, 2-D UPML boundary conditions have been implemented in combination with the basic time-marching scheme. The simulations have been carried out by considering as electromagnetic source a current density wire covering the

Figure 10 B-scan GPR image when the electromagnetic source is placed 15 cm above the air-dispersive soil surface and the pipe is buried 50 cm under the same surface.

frequency range $10 \text{ MHz} \div 1 \text{ GHz}$. Fig. 10 shows the B-scan GPR image obtained by collecting the scattered electric field along a straight path on the top of the air-dispersive soil surface, when the electromagnetic source and the pipe are placed 15 cm above and 50 cm under such surface. The obtained preliminary results demonstrate that the developed FDTD technique could be used in the field of the geophysical inspection of the soil, the mapping of buried utilities, the surveys of roads, buildings, and bridges as well as the landmine detection.

4.3 In-body electromagnetic communications

Recently, the Internet of Nano Things (IoNT) paradigm has transformed the use of the Internet since it supports new networked systems referred to as Body Area Nano-NETwork (BANNET) [88,89]. In these systems, a multitude of nanosensors and nanomachines can interact to form a pervasive network for monitoring of biological functionalities of people and for supporting of advanced healthcare services, ambient assisted living, wearable health monitoring, and telemedicine applications. Electromagnetic-based communications are a viable technique to support data exchange in the nanonetworks and, in particular, THz-band can provide users with unprecedentedly large bandwidths. Moreover, the nanonetworks can benefit from the compact size of THz transceivers and antennas as well as such band open a new interests due to its less susceptibility to some propagation effects as the Rayleigh scattering.

The design, implementation and deployment of nanonetworks in a biological environment poses considerable challenges since the propagation of electromagnetic waves inside the human body is drastically impacted by the absorption of liquid water molecules and by the noise due to the strong molecular absorption. As critical parameters, the path loss and molecular absorption need to be investigated in-depth for the evaluation of nanonetworks performance since they can strongly affect both the transmission rates

Figure 11 Modulus of the normalized space-time distribution of the electric field. Reprinted from Nano Communication Networks, vol. 10, G. Piro, P. Bia, G. Boggia, D. Caratelli, L.A. Grieco, L. Mescia, Terahertz electromagnetic field propagation in human tissues: A study on communication capabilities, pages 51-59, Copyright (2016), with permission from Elsevier.

and communication ranges. Moreover, the channel characterization in a realistic scenario requires the study of the PEF propagation in biological media. Based on these premises, we applied our fractional-calculus-based FDTD algorithm to formulate a more accurate study of in-body electromagnetic communications [26]. It provided an important step forwards for the state of the art because it was developed a sophisticated channel model quantifying the impact of both the frequency and spatial dependence of the skin permittivity on the channel performance. The scenario under study was composed by a source node located outside of the arm, but directly attached to the human body, and a destination device implanted inside the body. The non-homogeneous dielectric properties of the skin tissue, due to the spatial variation of the water concentration, was modelled by using a stratified media stack consisting of stratum corneum, epidermis, dermis and fat. Fig. 11 shows the modulus of the normalized electric field as a function of the time and the distance between source and destination nodes. The multiple reflected waves generated by the stratified media stack and the reflection phenomenon occurring at the air-skin interface can be clearly observed. The total path loss due to the spreading path loss, generated by the expansion of waves, and the absorption path loss, concerning to the molecular absorption in the biological tissue, was numerically calculated using our FDTD algorithm. Fig. 12 displays the total path loss versus both the frequency and distance between source and destination nodes. By the obtained results we inferred that the total path loss is mainly influenced by absorption phenomena

Figure 12 Total path loss versus both frequency and distance between source and destination nodes. Reprinted from Nano Communication Networks, vol. 10, G. Piro, P. Bia, G. Boggia, D. Caratelli, L.A. Grieco, L. Mescia, Terahertz electromagnetic field propagation in human tissues: A study on communication capabilities, pages 51-59, Copyright (2016), with permission from Elsevier.

which is 6 times higher than the one due by the waves expansion.

The noise power spectral density due to the internal molecular vibrations was also evaluated in terms of the equivalent noise temperature. In particular, the obtained results highlighted that noise temperature is not extremely high at the level of millimeters allowing acceptable signal to noise ratio within the human tissue in the Terahertz band. Moreover, to quantify the upper bound of channel capacity and the related transmission ranges the flat, pulse-based and optimal communication schemes was investigated. From the obtained results, was observed that the channel capacity decreases with the distance between source and destination node and that a physical data rate in the order of Tbps can be only reached for transmission ranges less than 2 mm. On the other hand, the communication capabilities are extremely injured for distances higher than 9 mm.

5 Conclusion. The theory of the fractional calculus has been developed mainly as an interesting, but purely mathematical, concept for many years. However, starting from the 20th century it has been used in numerous mathematical models involving different physical phenomena. Recently, the fractional-order differentiation and integration have been successfully implemented into electrodynamics, leading to the generalisations of the Maxwell's equations. These investigations have started with an attempts to explain the frequency-dependent electromagnetic effects in dielectrics. It appears that these effects may be successfully modelled with the generalised Maxwell's equations incorporating fractional derivatives. The research in this field is currently in its early stage. As a result,

the development of specific numerical techniques for solving Maxwell's equations with fractional order derivatives could provide interesting tools for understanding the electromagnetic wave propagation and radiation in dielectrics characterized by power law response in the frequency domain, as well as to predict new electromagnetic phenomena. Moreover, they may be an essential tool to engineer novel artificial and structured materials for the realization of new electromagnetic components and systems. In view of such fascinating picture, this feature article is devoted to illustrate the recent advantages on the dielectric function modelling and fractional calculus based FDTD schemes as well as to outline the main research results obtained by the authors in the field of the fractional electrodynamics. In particular, with the purpose to analyze the short-pulse wave propagation in general dispersive materials a novel FDTD technique based on the fractional derivative operator theory is presented. Specific studies directed towards the establishment of efficient and effective UPML boundary conditions and computing algorithm have been carried out to generate a stable FDTD scheme. To this aim, the numerical stability of the proposed procedure has been analyzed by deriving a relevant numerical dispersion equation. The soundness and reliability of the developed FDTD scheme has been assessed by several test cases involving various types of dispersive materials and systems in both 1-D and 2-D geometries. Considering the performed test cases and the illustrated special applications, we demonstrated that the FDTD code can be applied for investigating the pulse wave propagation and temperature rise in a multi-layered human head structure, to study the electromagnetic scattering by object buried in dispersive soils, and for the channel characterization of the in-body electromagnetic communications.

References

- [1] L. Mescia, P. Bia, and D. Caratelli, *IEEE Trans. Antennas Propag.*, **64**, 3533 (2016).
- [2] P. Bia, D. Caratelli, L. Mescia, R. Cicchetti, G. Maione, and F. Prudeniano, *Signal Process.* **107**, 312 (2015).
- [3] A. Agrawal, Z. Valy Vardeny, and A. Nahata, *Opt. Express*, **16**, 9601 (2008).
- [4] D. Kissinger and J.-C. Chiao, *IEEE Microwave Magazine*, **16**, 34 (2015).
- [5] J. Chiao and D. Kissinger, *IEEE Microwave Magazine*, **16**, 12 (2015).
- [6] Joao M.L.P. Caldeira, Joel J.P.C. Rodrigues, and P. Lorenz, *IEEE Communications Magazine*, **50**, 108 (2012).
- [7] L. Mescia, P. Bia, and D. Caratelli, *IEEE Trans. Microw. Theory Techn.*, **62**, 1920 (2014).
- [8] L. Mescia, P. Bia, D. Caratelli, *Proc. IEEE International Symposium on Antennas and Propagation & USNC/URSI National Radio Science Meeting*, San Diego, California USA, (2017), pp. 815–816.
- [9] T. W. Ebbesen, H. J. Lezec, H. F. Ghaemi, T. Thio, and P. A. Wolff, *Nature*, **391**, 667 (1998).
- [10] H. Liu and P. Lalanne, *Nature*, **452**, 728 (2008).

- [11] T. Matsui, A. Agrawal, A. Nahata, and Z. V. Vardeny, *Nature*, **446**, 517 (2006).
- [12] C. Ponti, and S. Vellucci, *IEEE Trans. Microw. Theory Techn.*, **63**, 30 (2015).
- [13] I. Giannakis, A. Giannopoulos, and C. Warren, *IEEE J. Sel. Topics Appl. Earth Observ. Remote Sens.*, **9**, 37 (2016).
- [14] D. Caratelli, A. Yarovoy, L. P. Ligthart, *Proc. 5th Eur. Radar Conf.*, Amsterdam, The Netherlands, (2008), pp. 204–207.
- [15] H. Akiyama, and R. Heller, *Bioelectrics* (Springer, Berlin, 2016).
- [16] S. J. Beebe, *J. Nanomed. Nanotechol.*, **4**, Art. ID 1000163 (2013).
- [17] J. Teissi, J.M. Escoffre, M. P. Rols, M. Golzio, *Radiol. Oncol.*, **42**, 196 (2008).
- [18] A. T. Esser, K. C. Smith, T. R. Gowrishankar, and J. C. Weaver, *Technol. Cancer Res. Treatment*, **8**, 289 (2009).
- [19] M. Breton, L. M. Mir, *Bioelectromagnetics*, **33**, 106 (2012).
- [20] L. Mescia, M.A. Chiapperino, P. Bia, J. Gielis, and D. Caratelli, *IEEE Trans. Biom. Eng.*, **65**, 414 (2018).
- [21] K. Greiner, D. Dubuc, T. Chen, F. Ertis, T. Chretiennot, M. Poupot, and J. J. Fourni, *IEEE Trans. Microw. Theory Techn.*, **61**, 2023 (2013).
- [22] M. Lazebnik, C. Zhu, G. M. Palmer, J. Harter, S. Sewall, N. Ramanujam, and S. C. Hagness, *IEEE Trans Biomed Eng.*, **55**, 2444 (2008).
- [23] W. Skierucha, A. Wilczek, and A. Szyplowska, *Int. Agrophys.*, **26**, 187 (2012).
- [24] S. Movassaghi, M. Abolhasan, J. Lipman, D. Smith, and A. Jamalipour, *IEEE Comm. Surveys and Tutorials*, **16**, 1658 (2014).
- [25] T. Nakano, M. J. Moore, F. Wei, A. V. Vasilakos, and J. Shuai, *IEEE Trans. Nanobioscience*, **11**, 135 (2012).
- [26] G. Piro, P. Bia, G. Boggia, D. Caratelli, L.A. Grieco, L. Mescia, *J. Nano Comm. Netw.*, **10**, 51 (2016).
- [27] S. Canovas-Carrasco, A. J. Garcia-Sanchez, J. Garcia-Haro, *Nano Commun. Netw.*, **15**, 17 (2018).
- [28] L. Debnath, *Int. J. Mathematics and Mathematical Sciences*, **2013**, 3413 (2013).
- [29] M. Dalir, *Appl. Mathematical Sciences*, **4**, 1021 (2010).
- [30] J. F. Douglas, *Adv. Chemical Physics*, **102**, 121 (1997).
- [31] V. V. Kulish, and J. L. Lage, *J. Fluids Engineering, Trans. of the ASME*, **124**, 803 (2002).
- [32] R. Magin, *Fractional Calculus in Bioengineering* (Begell House, 2006).
- [33] C. Polk, and E. Postow, *Biological Effects of Electromagnetic Fields* (Crc Press, Boca Raton, USA, 1996).
- [34] G. G. Raju, *Dielectrics in Electric Fields* (Marcel Dekker, New York, NY, USA, 2003).
- [35] P. Debye, *Polar molecules* (Dover, New York, 1954).
- [36] W. D. Hurt, *IEEE Trans. Biom. Eng.*, **32**, 60 (1985).
- [37] K. S. Cole, and R. H. Cole, *J. Chem. Phys.*, **9**, 341 (1941).
- [38] D. W. Davidson, and R. H. Cole, *J. Chem. Phys.*, **18**, 1417 (1950).
- [39] S. Havriliak, and N. Negami, *J. Polymer Science Part C: Polymer Symposia*, **14**, 99 (1966).
- [40] G. Williams, and D.C. Watts, *Trans. Faraday Soc.*, **66**, 80 (1970).
- [41] A.K. Jonscher, *Nature*, **250**, 191 (1974).
- [42] V. Raicu, *Phys. Rev. E*, **70**, 4677 (1999).
- [43] R. E. Collin, *Field Theory of Guided Waves* (Wiley-IEEE Press, New York, NY, USA, 1990).
- [44] A. A. Stanislavsky, *Phys. Rev. E*, **70**, 051103 (2004).
- [45] A. Stanislavsky, K. Weron, J. Trzmiel, *EPL*, **91**, 40003 (2010).
- [46] R. Garrappa, F. Mainardi, G. Maione, *Fract. Calc. Appl. Anal.*, **19**, 1105 (2016).
- [47] P. Bia, D. Caratelli, L. Mescia, and J. Gielis, *IET Microw., Antennas Propag.*, **9**, 1497 (2015).
- [48] D. Ashlock, *Evolutionary Computation for Modeling and Optimization* (Springer, New York, NY, USA, 2005).
- [49] M. Avriel, *Nonlinear Programming: Analysis and Methods* (Prentice-Hall, Englewood Cliffs, NJ, USA, 1976).
- [50] A. Taflov S. C. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method* (Artech House, Norwood, MA, USA, 2005).
- [51] D. F. Kelley, and R. J. Luebbers, *IEEE Trans. Antennas Propag.*, **44**, 792 (1996).
- [52] J. L. Young, and R. O. Nelson, *IEEE Antennas Propag. Mag.*, **43**, 61 (2001).
- [53] F. L. Teixeira, *IEEE Trans. Antennas Propag.*, **56**, 2150 (2008).
- [54] Q. Chen, M. Katsurai, and P. H. Aoyagi, *IEEE Trans. Antennas Propag.*, **46**, 1739 (1998).
- [55] M. Okoniewski, M. Mrozowski, and M. A. Stuchly, *IEEE Microw. Guided Wave Lett.*, **7**, 121 (1997).
- [56] Y. Takayama, and W. Klaus, *IEEE Microw. Wireless Compon. Lett.*, **12**, 102 (2002).
- [57] M. Mrozowski, and M. A. Stuchly, *IEEE Trans. Antennas Propag.*, **45**, 1438 (1997).
- [58] O. P. Gandhi, B.-Q. Gao, and J.-Y. Chen, *IEEE Trans. Microw. Theory Tech.*, **41**, 658 (1993).
- [59] M. Lazebnik, M. Okoniewski, John H. Booske, and Susan C. Hagness, *IEEE Microw. Wireless Comp. Lett.*, **17**, 822 (2007).
- [60] Z. Lin, Y. Fang, J. Hu, and C. Zhang, *IEEE Trans. Antennas Propag.*, **59**, 1338 (2011).
- [61] B. H. Jung, Z. Mei, and T. K. Sarkar, *Progress In Electromagnetics Research*, **118**, 135 (2011).
- [62] K. P. Prokopidis, E. P. Kosmidou, and T. D. Tsiboukis, *J. of Electromagn. Waves and Appl.*, **18**, 1171 (2004).
- [63] Guo-Xin Fan, and Qing Huo Liu, *IEEE Trans. Antennas Propag.*, **48**, 637 (2000).
- [64] K. P. Prokopidis, and D. C. Zografopoulos, *IEEE Microw. Wireless Compon. Lett.*, **21**, 283 (2011).
- [65] L. Han, D. Zhou, K. Li, X. Li, and Wei-Ping Huang, *J. Lightw. Technol.*, **30**, 2216 (2012).
- [66] Z. Lin, C. Zhang, P. Ou, Y. Jia, and L. Feng, *J. Lightw. Technol.*, **28**, 2843 (2010).
- [67] A. Mock, *Opt. Express*, **2**, 771 (2012).
- [68] O. Ramadan, *IEEE Microw. Wireless Compon. Lett.*, **21**, 513 (2011).
- [69] M. Mrozowski, and M. A. Stuchly, *IEEE Trans. Antennas Propag.*, **45**, 1438 (1997).
- [70] T. Wuren, T. Takai, M. Fujii, and I. Sakagami, *IEEE Microw. Wireless Comp. Lett.*, **17**, 483 (2007).
- [71] D. M. Sullivan, *IEEE Trans. Antennas Propag.*, **44**, 28 (1996).
- [72] S. Su, W. Dai, D. T. Haynie, and N. Simicevic, *Bioelectromagnetics*, **26**, 389 (2005).

- [73] T. Kashiwa, Y. Ohtomo, and I. Fucui, *Microw. Opt. Technol. Lett.*, **3**, 416 (1990).
- [74] Mohammad-Reza Tofighi, *IEEE Trans. Microw. Theory Tech.*, **57**, 2588 (2009).
- [75] J. A. Pereda, A. Vegas, and A. Prieto, *IEEE Trans. Microw. Theory Tech.*, **50**, 1689 (2002).
- [76] W. H. Weedon, and C. M. Rappaport, *IEEE Trans. Antennas Propag.*, **45**, 401 (1997).
- [77] F. Torres, P. Vaudon, and B. Jecko, *Microw. Opt. Technol. Lett.*, **13**, 300 (1996).
- [78] H. H. Abdullah, H. A. Elsadek, H. E. ElDeeb, and N. Bagherzadeh, *IEEE Microw. Wireless Compon. Lett.*, **11**, 281 (2002).
- [79] P. Bia, L. Mescia, and D. Caratelli, *Math. Probl. Eng.* **2016**, ID 5676903 (2016).
- [80] J. P. Bernger, *J. Comput. Phys.*, **114**, 185 (1994).
- [81] L. Bernard, R. R. Torrado, and L. Pichon, *IEEE Trans. Magn.*, **46**, 3492 (2010).
- [82] I. Udagedara, M. Premaratne, I. D. Rukhlenko, H. T. Hattori, and G. P. Agrawal, *Opt. Express*, **17**, 21179 (2009).
- [83] S. D. Gadney, *IEEE Trans. Antennas Propag.*, **44**, 1630 (1996).
- [84] J. A. Pereda, O. Garcia, A. Vegas, and A. Prieto, *IEEE Microw. Guided Wave Lett.*, **8**, 245 (1998).
- [85] J. A. Pereda, L. A. Vielva, A. Vegas, and A. Prieto, *IEEE Trans. Microw. Theory Techn.*, **49**, 377 (2001).
- [86] P. Yeh, *Optical Waves in Layered Media* (Wiley, New York, NY, USA, 1988).
- [87] L. Mescia, P. Bia, M. A. Chiapperino, and D. Caratelli, *Electronics*, **6**, 106 (2017).
- [88] S. Balasubramaniam, and J. Kangasharju, *Computer*, **46**, 62 (2013).
- [89] I. F. Akyildiz, M. Pierobon, and S. Balasubramaniam, *IEEE Commun. Mag.*, **53**, 32 (2015).