Incremental gain of LTI systems

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In this technical report we prove that for Linear Time Invariant (LTI) systems the $L_\infty$-gain and incremental gain are equivalent, whereas for nonlinear systems this is generally not the case [1]. Before we will give the proof, we first give the definitions of the $L_\infty$-gain and incremental gain.

Consider a dynamical system $\Sigma: \mathcal{L}_2^{nu} \to \mathcal{L}_2^{ny}$ given by

$$\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t); \\
y(t) &= Cx(t) + Bu(t); \\
x(t_0) &= x_0;
\end{align*}$$

where $x \in C_1^{nu}$ with $x_0 \in X \subseteq \mathbb{R}^{nu}$ is the state variable associated with the considered state-space representation of the system, $u \in \mathcal{L}_2^{nu}$ taking values in $U \subseteq \mathbb{R}^{nu}$ is the input, and $y \in \mathcal{L}_2^{ny}$ taking values in $Y \subseteq \mathbb{R}^{ny}$ is the output of the system.

**Definition I.1 ($L_\infty$-gain).** $\Sigma$, given by (1), is said to be $L_\infty$-gain stable if for all $u \in \mathcal{L}_2^{nu}$ and $x_0 \in X$, $\Sigma(u)$ exists and there is a finite $\gamma \geq 0$ and a function $\zeta(x) \geq 0$ with $\zeta(0,0) = 0$ such that

$$\|\Sigma(u)\|_2 \leq \gamma \|u\|_2 + \zeta(x_0).$$

(2)

The induced $L_\infty$-gain of $\Sigma$, denoted by $\|\Sigma\|_2$, is the infimum of $\gamma$ such that (2) still holds.

**Definition I.2 (Incremental gain [1], [2]).** $\Sigma$, given by (1), is said to be incrementally $L_\infty$-gain stable, from now on denoted as $L_{i2}$-gain stable, if it is $L_\infty$-gain stable and, there exist a finite $\eta \geq 0$ and a function $\zeta(x, \dot{x}) \geq 0$ with $\zeta(0,0) = 0$ such that

$$\|\Sigma(u) - \Sigma(\tilde{u})\|_2 \leq \eta \|u - \tilde{u}\|_2 + \zeta(x_0, \dot{x}_0),$$

(3)

for all $u, \tilde{u} \in \mathcal{L}_2^{nu}$ and $x_0, \dot{x}_0 \in X$. The induced $L_{i2}$-gain of $\Sigma$, denoted by $\|\Sigma\|_{i2}$, is the infimum of $\eta$ such that (3) holds.

II. MAIN RESULTS

**Theorem II.1.** For an (LTI) dynamical system given by (1) the $L_\infty$-gain and $L_{i2}$-gain as defined in Definition I.1 and Definition I.2 are equivalent.

**Proof.** For the proof we use Theorem 2.7 from [3]. Therefore, formulate the following augmented difference system for the LTI system in (1)

$$\begin{align*}
y_\Delta &= \Sigma(u) - \Sigma(\tilde{u}) = \Sigma_\Delta(u, \tilde{u}) \begin{bmatrix} \dot{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} - \begin{bmatrix} u(t) \\ \tilde{u}(t) \end{bmatrix}; \\
x(t_0) &= x_0; \\
\dot{x}(t_0) &= \tilde{x}_0.
\end{align*}$$

(4)

which has the state-space representation

$$\begin{bmatrix} \dot{x}_\Delta(t) \\ y_\Delta(t) \end{bmatrix} = \begin{bmatrix} A_\Delta & B_\Delta \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} x_\Delta(t) \\ u_\Delta(t) \end{bmatrix},$$

(5)

where

$$\begin{align*}
x_\Delta(t) &= \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix}, \\
u_\Delta(t) &= \begin{bmatrix} u(t) \\ \tilde{u}(t) \end{bmatrix}, \\
A_\Delta &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \\
B_\Delta &= \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}, \\
C_\Delta &= \begin{bmatrix} C & -C \end{bmatrix}, \\
D_\Delta &= \begin{bmatrix} D & -D \end{bmatrix}.
\end{align*}$$

The differential dissipation inequality (DDI) is given by

$$\partial_x S(x(t), f(x(t), u(t))) \leq w(u(t), y(t)), \quad \text{for all } x(t), u(t), y(t),$$

(6)

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where $S(x)$ is a storage function, $w(u, y)$ a supply function and $f(x, u)$ the state equation. In our case, per Theorem 2.7 from [3], as storage function we take (omitting time dependence for brevity)

$$S(x, \dot{x}) = S(x_\Delta) = (x - \dot{x})^TP(x - \dot{x}) = x_\Delta^T\begin{bmatrix} P & -P \\ -P & P \end{bmatrix}x_\Delta,$$

and as supply function we take

$$w_\Delta(u, \tilde{u}, y_\Delta) = \eta^2 \|u - \tilde{u}\|^2 - \|y_\Delta\|^2.$$  

The state equation, based on (5), is given by

$$f(x_\Delta, u_\Delta) = A_\Delta x_\Delta + B_\Delta u_\Delta.$$  

Combining (6)-(9) results in

$$2x_\Delta^T\hat{P}(A_\Delta x_\Delta + B_\Delta u_\Delta) \leq \eta^2 \|u - \tilde{u}\|^2 - \|y_\Delta\|^2,$$

which can be rewritten as

$$\begin{bmatrix} x_\Delta^T \\ u_\Delta^T \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} 0 & \hat{P} \\ I & B_\Delta \end{bmatrix} \begin{bmatrix} x_\Delta \\ u_\Delta \end{bmatrix} \leq \begin{bmatrix} 0 & I \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} H \\ 0 \\ -I \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} x_\Delta^T \\ u_\Delta^T \end{bmatrix},$$

which needs to hold for all $x_\Delta$ and $u_\Delta$ values over all $t$, with

$$H = \begin{bmatrix} \eta^2 I & -\eta^2 I \\ -\eta^2 I & \eta^2 I \end{bmatrix}.$$  

Next, (11) holds if and only if

$$\begin{bmatrix} I & 0 \\ A_\Delta & B_\Delta \end{bmatrix} \begin{bmatrix} 0 & \hat{P} \\ I & B_\Delta \end{bmatrix} = \begin{bmatrix} 0 & I \\ C_\Delta & D_\Delta \end{bmatrix} \begin{bmatrix} H \\ 0 \\ -I \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\Delta & D_\Delta \end{bmatrix} \leq 0.$$  

Collapsing (12) gives

$$\begin{bmatrix} M_{11} & -M_{11} & M_{12} & -M_{12} \\ -M_{11} & M_{11} & -M_{12} & M_{12} \\ M_{12} & -M_{12} & M_{22} & -M_{22} \\ -M_{12} & M_{12} & -M_{22} & M_{22} \end{bmatrix} \leq 0,$$

where

$$M_{11} = A^T P + PA + C^T C,$$

$$M_{12} = PB + C^T D,$$

$$M_{22} = D^T D - \eta^2 I.$$  

Introduce the non-singular

$$\mathcal{I} = \begin{bmatrix} I_n & I_n & 0 & 0 \\ 0 & -I_n & 0 & 0 \\ 0 & 0 & I_{n_u} & 0 \\ 0 & 0 & -I_{n_u} & -I_{n_u} \end{bmatrix}.$$  

By using $\mathcal{I}$ as a congruence transformation, (13) can equivalently be written as

$$\mathcal{I} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & M_{11} & M_{12} & 0 \\ 0 & M_{12} & M_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mathcal{I}^T \leq 0.$$  

We can reduce (16) to

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & M_{11} & M_{12} & 0 \\ 0 & M_{12}^T & M_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leq 0,$$

and to

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \eta^2 I \end{bmatrix} \leq 0,$$

which is equivalent with the bounded real lemma [4]. This shows that the $\mathcal{L}_2$-gain and $\mathcal{L}_{12}$-gain are equivalent for LTI systems.  

$\square$
REFERENCES


