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Room acoustics modelling in the time-domain with the nodal discontinuous Galerkin method

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To solve the linear acoustic equations for room acoustic purposes, the performance of the time-domain nodal discontinuous Galerkin (DG) method is evaluated. A nodal DG method is used for the evaluation of the spatial derivatives, and for the time-integration an explicit multi-stage Runge-Kutta method is adopted. The scheme supports a high order approximation on unstructured meshes. To model frequency-independent real-valued impedance boundary conditions, a formulation based on the plane wave reflection coefficient is proposed. Semi-discrete stability of the scheme is analyzed using the energy method. The performance of the DG method is evaluated for four three-dimensional configurations. The first two cases concern sound propagations in free field and over a flat impedance ground surface. Results show that the solution converges with increasing DG polynomial order and the accuracy of the impedance boundary condition is independent on the incidence angle. The third configuration is a cuboid room with rigid boundaries, for which an analytical solution serves as the reference solution. Finally, DG results for a real room scenario are compared with experimental results. For both room scenarios, results show good agreements.

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I. INTRODUCTION

Computer simulation of the sound field in indoor environments has been investigated back in time since the publication of Manfred Robert Schroeder.¹ After all these years, prediction methods for room acoustic applications are still under development trying to improve efficiency of the calculations and accuracy and realism of the results, hand in hand with the advances in computer power. In acoustics, the computational techniques are mainly separated between wave-based, geometrical and diffuse field methods. Each of these methodologies has been amply presented in literature. Concepts, implementations and applications of room simulation methods are reviewed by Vorländer,² Savioja *et al.*,³ and Hamilton⁴ for geometrical and wave-based methods, while diffuse field methods are described for instance by Valeau⁵ or Navarro *et al.*⁶

In contrast with the high-frequency simplifications assumed in the geometrical and diffuse field methods, wave-based methodologies solve the governing physical equations, implicitly including all wave effects such as diffraction and interference. Among these methods, time-domain approaches to model wave problems have attracted significant attention in the last few decades, since they are favoured for auralization purposes over frequency-domain methods. The main wave-based time-domain numerical techniques employed in room acoustics problems are the finite-differences time-domain method (FDTD),^{7–10} finite-element (FEM)¹¹ and finite-volume (FVM) methods,¹² and Fourier spectral methods such

as the adaptive rectangular decomposition method (ARD)¹³ and the pseudospectral time-domain method (PSTD).^{14–16}

In the last few years, the discontinuous Galerkin time-domain method (DG)¹⁷ is another approach gaining importance, mainly in the aero-acoustic community.^{18,19} DG discretizes the spatial domain into non-overlapping mesh elements, in which the governing equations are solved elementwise, and uses the so-called numerical flux at adjacent elements interfaces to communicate the information between them. DG combines the favourable properties of existing wave-based time domain methods for room acoustics as it preserves high order accuracy, allows for local refinement by a variable polynomial order and element size, and therefore can deal with complex geometries. Also, because equations are solved elementwise, it allows for easy parallelization and massive calculation acceleration opportunities,²⁰ like other methods such as FDTD and FVM. DG can be seen as an extension to FV by using a polynomial basis for evaluating the spatial derivatives, leading to a higher order method. Also, DG can be seen as an extended FEM version by decoupling the elements without imposing continuity of the variables, thereby creating local matrices. Therefore, DG is a very suitable numerical method for acoustic propagation problems including, definitely, room acoustics. However, some developments towards room acoustic applications are still missing: although results for impedance boundary conditions with the DG method have been presented,²¹ a proper formulation of these boundary conditions in the framework of DG have not been published. In contrast, frequency-dependent impedance conditions have been extensively developed in other methodologies (FDTD, FVTD).¹² In the present work, a frequency-independent

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real-valued impedance boundary formulation, based on the plane wave reflection coefficient is proposed, following the idea first presented by Fung and Ju.²²

To the authors' best knowledge, no reference is found in the scientific literature about the application of DG to the room acoustics problems. The aim of this work is to address the positioning of DG as a wave-based method for room acoustics. The accuracy of the method for these type of applications is quantified and the developments needed to arrive at a fully fledged DG method for room acoustics are summarized as future work.

The paper is organized as follows. In Sec. II, the governing acoustic equations are introduced as well as the solution by the time-domain DG method. The formulations of impedance boundary conditions and its semi-discrete stability analysis are presented in Sec. III, and are in this work restricted to locally reacting frequency independent conditions. Section IV quantifies and discusses the accuracy of the implemented DG method for sound propagation in several scenarios: (1) free field propagation in a periodic domain; (2) a single reflective plane; (3) a cuboid room with acoustically rigid boundaries; (4) a real room. Finally, conclusions and outlook can be found in Sec. V.

II. LINEAR ACOUSTIC EQUATIONS AND NODAL DG TIME-DOMAIN METHOD

A. Linear acoustic equations

Acoustic wave propagation is governed by the linearized Euler equations (LEE), which are derived from the general conservation laws.²³ For room acoustics applications, we further assume that sound propagates in air that is completely at rest and constant in temperature. Under these assumptions, the LEE in primitive variables are simplified to the following homogeneous coupled system of linear acoustic equations:

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{\rho_0} \nabla p &= \mathbf{0}, \\ \frac{\partial p}{\partial t} + \rho_0 c_0^2 \nabla \cdot \mathbf{v} &= 0, \end{aligned} \quad (1)$$

where $\mathbf{v} = [u, v, w]^T$ is the particle velocity vector, p is the sound pressure, ρ_0 is the constant density of air, and c_0 is the constant adiabatic sound speed. The linear acoustic equations can be combined into one equation, the wave equation. Equation (1), completed with initial values or a force formulation on the right side, as well as a formulation of boundary conditions at all room boundaries, complete the problem definition. In this study, the linear acoustic equations are solved instead of the wave equation because it is beneficial for implementing impedance boundary conditions.

B. Spatial discretization with the nodal DG method

To numerically solve Eq. (1), the nodal discontinuous Galerkin method is used to discretize the spatial derivative operators. First of all, Eq. (1) is rewritten into the following linear hyperbolic system:

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{q}) = \frac{\partial \mathbf{q}}{\partial t} + \mathbf{A}_j \frac{\partial \mathbf{q}}{\partial x_j} = 0, \quad (2)$$

where $\mathbf{q}(\mathbf{x}, t) = [u, v, w, p]^T$ is the acoustic variable vector and $\mathbf{x} = [x, y, z]$ is the spatial coordinate vector with index $j \in [x, y, z]$. The flux is given as

$$\mathbf{F} = [\mathbf{f}_x, \mathbf{f}_y, \mathbf{f}_z] = [\mathbf{A}_x \mathbf{q}, \mathbf{A}_y \mathbf{q}, \mathbf{A}_z \mathbf{q}], \quad (3)$$

where the constant flux Jacobian matrix $\mathbf{A}_j$

$$\mathbf{A}_j = \begin{bmatrix} 0 & 0 & 0 & \frac{\delta_{xj}}{\rho_0} \\ 0 & 0 & 0 & \frac{\delta_{yj}}{\rho_0} \\ 0 & 0 & 0 & \frac{\delta_{zj}}{\rho_0} \\ \rho_0 c_0^2 \delta_{xj} & \rho_0 c_0^2 \delta_{yj} & \rho_0 c_0^2 \delta_{zj} & 0 \end{bmatrix}, \quad (4)$$

and δ_{ij} denotes the Kronecker delta function.

Similar to the finite element method, the physical domain Ω is approximated by a computational domain Ω_h , which is further divided into a set of K non-overlapping elements D^k , i.e., $\Omega_h = \cup_{k=1}^K D^k$. In this work, the quadrature-free approach²⁴ is adopted and the nodal discontinuous Galerkin algorithm as presented in Ref. 25 is followed. The global solution is approximated by a direct sum of local piecewise polynomial solutions as

$$\mathbf{q}(\mathbf{x}, t) \approx \mathbf{q}_h(\mathbf{x}, t) = \bigoplus_{k=1}^K \mathbf{q}_h^k(\mathbf{x}, t). \quad (5)$$

The local solution $\mathbf{q}_h^k(\mathbf{x}, t)$ in element D^k is expressed by

$$\mathbf{q}_h^k(\mathbf{x}, t) = \sum_{i=1}^{N_p} \mathbf{q}_h^k(\mathbf{x}_i^k, t) l_i^k(\mathbf{x}), \quad (6)$$

where $\mathbf{q}_h^k(\mathbf{x}_i^k, t)$ are the unknown nodal values in element D^k and $l_i^k(\mathbf{x}_i^k)$ is the multi-dimensional Lagrange polynomial basis of order N based on the nodes $\mathbf{x} \in D^k$, which satisfies $l_i^k(\mathbf{x}_j^k) = \delta_{ij}$. The number of local basis functions (or nodes) N_p is determined by both the dimensionality of the problem d and the order of the polynomial basis N , which can be computed as $N_p = (N + d)!/N!d!$. In this work, the α -optimized nodes distribution²⁶ for tetrahedron elements are used over a wide range of polynomial order N . The locally defined basis functions constitute a function space as $V_h^k = \text{span}\{l_i^k(\mathbf{x})\}_{i=1}^{N_p}$. Then, the Galerkin projection is followed by choosing test functions equal to the basis functions. The solution is found by imposing an orthogonality condition: the local residual is orthogonal to all the test functions in V_h^k ,

$$\int_{D^k} \left(\frac{\partial \mathbf{q}_h^k}{\partial t} + \nabla \cdot \mathbf{F}_h^k(\mathbf{q}_h^k) \right) l_i^k d\mathbf{x} = 0. \quad (7)$$

Integration by parts and applying the divergence theorem results in the local weak formulation,

$$\int_{D^k} \left(\frac{\partial \mathbf{q}_h^k}{\partial t} l_i^k - \mathbf{F}_h^k(\mathbf{q}_h^k) \cdot \nabla l_i^k \right) d\mathbf{x} = - \int_{\partial D^k} \mathbf{n} \cdot \mathbf{F}^* l_i^k d\mathbf{x}, \quad (8)$$

where $\mathbf{n} = [n_x, n_y, n_z]$ is the outward normal vector of the element surface ∂D^k and $\mathbf{F}^*(\mathbf{q}_h^-, \mathbf{q}_h^+)$ is the so-called numerical flux from element D^k to its neighboring elements through their intersection ∂D^k . In contrast to the classical continuous Galerkin method, the discontinuous Galerkin method uses local basis functions and test functions that are smooth within each element and discontinuous across the element intersections. As a result, the solutions are multiply defined on the intersections ∂D^k , where the numerical flux $\mathbf{F}^*(\mathbf{q}_h^-, \mathbf{q}_h^+)$ should be defined properly as a function of both the interior and exterior (or neighboring) solution. In the remainder, the solution value from the interior side of the intersection is denoted by a superscript “−” and the exterior value by “+.” Applying integration by parts once again to the spatial derivative term in Eq. (8) yields the strong formulation

$$\begin{aligned} & \int_{D^k} \left(\frac{\partial \mathbf{q}_h^k}{\partial t} + \nabla \cdot \mathbf{F}_h^k(\mathbf{q}_h^k) \right) l_i^k d\mathbf{x} \\ &= \int_{\partial D^k} \mathbf{n} \cdot (\mathbf{F}_h^k(\mathbf{q}_h^k) - \mathbf{F}^*) l_i^k d\mathbf{x}. \end{aligned} \quad (9)$$

In this study, the flux-splitting approach²⁷ is followed and the upwind numerical flux is derived as follows. Let us first consider the case where the element D^k lies in the interior of the computational domain. As is shown in Eq. (9), the formulation of a flux along the surface normal direction $\mathbf{n}$, i.e., $\mathbf{n} \cdot \mathbf{F} = (n_x f_x + n_y f_y + n_z f_z)$ is of interest. To derive the upwind flux, we utilize the hyperbolic property of the system and decompose the normal flux on the interface ∂D^k into outgoing and incoming waves. Mathematically, an eigendecomposition applied to the normally projected flux Jacobian yields

$$\begin{aligned} \mathbf{A}_n &= (n_x \mathbf{A}_x + n_y \mathbf{A}_y + n_z \mathbf{A}_z) \\ &= \begin{bmatrix} 0 & 0 & 0 & \frac{n_x}{\rho_0} \\ 0 & 0 & 0 & \frac{n_y}{\rho_0} \\ 0 & 0 & 0 & \frac{n_z}{\rho_0} \\ \rho_0 c_0^2 n_x & \rho_0 c_0^2 n_y & \rho_0 c_0^2 n_z & 0 \end{bmatrix} \\ &= \mathbf{L} \mathbf{\Lambda} \mathbf{L}^{-1}, \end{aligned} \quad (10)$$

where

$$\begin{aligned} \mathbf{L} &= \begin{bmatrix} -n_z & n_y & n_x/2 & -n_x/2 \\ n_z & -n_x & n_y/2 & -n_y/2 \\ -n_y & n_x & n_z/2 & -n_z/2 \\ 0 & 0 & \rho_0 c_0/2 & \rho_0 c_0/2 \end{bmatrix}, \\ \mathbf{\Lambda} &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & c_0 & 0 \\ 0 & 0 & 0 & -c_0 \end{bmatrix}. \end{aligned} \quad (11)$$

The upwind numerical flux is defined by considering the direction of the characteristic speed, i.e.,

$$(\mathbf{n} \cdot \mathbf{F})^* = \mathbf{L} (\mathbf{\Lambda}^+ \mathbf{L}^{-1} \mathbf{q}_h^- + \mathbf{\Lambda}^- \mathbf{L}^{-1} \mathbf{q}_h^+), \quad (12)$$

where $\mathbf{\Lambda}^+$ and $\mathbf{\Lambda}^-$ contain the positive and negative entries of $\mathbf{\Lambda}$, respectively. Physically, $\mathbf{\Lambda}^+$ ($\mathbf{\Lambda}^-$) corresponds to the characteristic waves propagating along (opposite to) the normal direction $\mathbf{n}$, which are referred to as outgoing waves out of D_k (incoming waves into D_k). Therefore, the outgoing waves are associated with the interior solution $\mathbf{q}_h^-$ whereas the incoming waves are dependent on the exterior (neighboring) solution $\mathbf{q}_h^+$. The expression of the numerical flux on the impedance boundary will be discussed in Sec. III. Finally, the semi-discrete formulation is obtained by substituting the nodal basis expansion Eq. (6) and the upwind flux Eq. (12) into the strong formulation Eq. (9), which can be further recast into the following matrix form:

$$\mathbf{M}^k \frac{\partial \mathbf{u}_h^k}{\partial t} + \frac{1}{\rho} \mathbf{S}_x^k \mathbf{p}_h^k = \sum_{r=1}^f \mathbf{M}^{kr} \hat{\mathbf{F}}_u^{kr}, \quad (13a)$$

$$\mathbf{M}^k \frac{\partial \mathbf{v}_h^k}{\partial t} + \frac{1}{\rho} \mathbf{S}_y^k \mathbf{p}_h^k = \sum_{r=1}^f \mathbf{M}^{kr} \hat{\mathbf{F}}_v^{kr}, \quad (13b)$$

$$\mathbf{M}^k \frac{\partial \mathbf{w}_h^k}{\partial t} + \frac{1}{\rho} \mathbf{S}_z^k \mathbf{p}_h^k = \sum_{r=1}^f \mathbf{M}^{kr} \hat{\mathbf{F}}_w^{kr}, \quad (13c)$$

$$\mathbf{M}^k \frac{\partial \mathbf{p}_h^k}{\partial t} + \rho c^2 \mathbf{S}_x^k \mathbf{u}_h^k + \rho c^2 \mathbf{S}_y^k \mathbf{v}_h^k + \rho c^2 \mathbf{S}_z^k \mathbf{w}_h^k = \sum_{r=1}^f \mathbf{M}^{kr} \hat{\mathbf{F}}_p^{kr}, \quad (13d)$$

where the second superscript r denotes the r th faces ∂D^{kr} of the element D^k and f is the total number of faces of the element D^k , which is equal to 4 for tetrahedra elements. For brevity, the subscript 0 in ρ_0 and c_0 are omitted from here. $\mathbf{u}_h^k$, $\mathbf{v}_h^k$, $\mathbf{w}_h^k$, and $\mathbf{p}_h^k$ are vectors representing all the unknown nodal values $u_h^k(\mathbf{x}_i^k, t)$, $v_h^k(\mathbf{x}_i^k, t)$, and $p_h^k(\mathbf{x}_i^k, t)$ respectively, e.g., $\mathbf{u}_h^k = [u_h^k(\mathbf{x}_1^k, t), u_h^k(\mathbf{x}_2^k, t), \dots, u_h^k(\mathbf{x}_{N_p}^k, t)]^T$. $\hat{\mathbf{F}}_u^{kr}$, $\hat{\mathbf{F}}_v^{kr}$, $\hat{\mathbf{F}}_w^{kr}$, and $\hat{\mathbf{F}}_p^{kr}$ are flux terms associated with the integrand $\mathbf{n} \cdot (\mathbf{F}_h^k(\mathbf{q}_h^k) - \mathbf{F}^*)$ over the element surface ∂D^{kr} in the strong formulation Eq. (9). The element mass matrix $\mathbf{M}^k$, the element stiffness matrices $\mathbf{S}_j^k$ and the element face matrices $\mathbf{M}^{kr}$ are defined as

$$\mathbf{M}_{mn}^k = \int_{D^k} l_m^k(\mathbf{x}) l_n^k(\mathbf{x}) d\mathbf{x} \in \mathbb{R}^{N_p \times N_p}, \quad (14a)$$

$$(\mathbf{S}_j^k)_{mn} = \int_{D^k} l_m^k(\mathbf{x}) \frac{\partial l_n^k(\mathbf{x})}{\partial x_j} d\mathbf{x} \in \mathbb{R}^{N_p \times N_p}, \quad (14b)$$

$$\mathbf{M}_{mn}^{kr} = \int_{\partial D^{kr}} l_m^{kr}(\mathbf{x}) l_n^{kr}(\mathbf{x}) d\mathbf{x} \in \mathbb{R}^{N_p \times N_{fp}}, \quad (14c)$$

where j is the j th Cartesian coordinates and N_{fp} is the number of nodes along one element face. When the upwind flux is used, the flux terms for each acoustic variable read as

$$\begin{aligned}\hat{F}_u^{kr} = & -\frac{cn_x^{kr2}}{2}[u_h^{kr}] - \frac{cn_x^{kr}n_y^{kr}}{2}[v_h^{kr}] - \frac{cn_x^{kr}n_z^{kr}}{2}[w_h^{kr}] \\ & + \frac{n_x^{kr}}{2\rho}[p_h^{kr}],\end{aligned}\quad (15a)$$

$$\begin{aligned}\hat{F}_v^{kr} = & -\frac{cn_y^{kr2}}{2}[v_h^{kr}] - \frac{cn_x^{kr}n_y^{kr}}{2}[u_h^{kr}] - \frac{cn_y^{kr}n_z^{kr}}{2}[w_h^{kr}] \\ & + \frac{n_y^{kr}}{2\rho}[p_h^{kr}],\end{aligned}\quad (15b)$$

$$\begin{aligned}\hat{F}_w^{kr} = & -\frac{cn_z^{kr2}}{2}[w_h^{kr}] - \frac{cn_x^{kr}n_z^{kr}}{2}[u_h^{kr}] - \frac{cn_y^{kr}n_z^{kr}}{2}[v_h^{kr}] \\ & + \frac{n_z^{kr}}{2\rho}[p_h^{kr}],\end{aligned}\quad (15c)$$

$$\hat{F}_p^{kr} = \frac{c^2\rho n_x^{kr}}{2}[u_h^{kr}] + \frac{c^2\rho n_y^{kr}}{2}[v_h^{kr}] + \frac{c^2\rho n_z^{kr}}{2}[w_h^{kr}] - \frac{c}{2}[p_h^{kr}],\quad (15d)$$

where $[u_h^{kr}] := u_h^{kr} - u_h^{ls}$, $[v_h^{kr}] := v_h^{kr} - v_h^{ls}$, $[w_h^{kr}] := w_h^{kr} - w_h^{ls}$, and $[p_h^{kr}] := p_h^{kr} - p_h^{ls}$ are the jump differences across the shared intersection face ∂D^{kr} or, equivalently, ∂D^{ls} , between neighboring elements D^k and D^l , u_h^{kr} , etc., are the nodal value vectors, over the element surface ∂D^{kr} .

In this work, flat-faced tetrahedra elements are used so that each tetrahedron can be mapped into a reference tetrahedron by a linear transformation with a constant Jacobian matrix. As a consequence, the integrals in the above element matrices, i.e., M^k , S_j^k , and M^{kr} , need to be evaluated only once. The reader is referred to Ref. 25 for more details on how to compute the matrices locally and efficiently.

1. Numerical dissipation and dispersion properties

For a discontinuous Galerkin scheme that uses polynomial basis up to order N , it is well known that generally the rate of convergence in terms of the global L^2 error is $h^{N+1/2}$ (h being the element size).²⁸ The dominant error comes from the representations of the initial conditions, while the additional dispersive and dissipative errors from the wave propagations are relatively small and only visible after a very long time integration.²⁵ The one-dimensional eigenvalue problem for the spatially propagating waves is studied in Ref. 29 and it is reported that the dispersion relation is accurate to $(\kappa h)^{2N+2}$ locally, where κ is the wavenumber. Actually, when the upwind flux is used, the dissipation error has been proved to be of order $(\kappa h)^{2N+2}$ while the dispersion error is of order $(\kappa h)^{2N+3}$.³⁰ When the centered numerical flux is used, the dissipation rate is exactly zero, but the discrete dispersion relation can only approximate the exact one for a smaller range of the wavenumber.³¹ Extensions to the two-dimensional hyperbolic system on triangle and quadrilateral mesh are studied in Ref. 32 and the same numerical dispersion relation as the one-dimensional case are reported. In Ref. 30, a rigorous mathematical proof of the above numerical dispersion relation and error behavior is provided for a general multi-dimensional setting (including 3D).

C. Time integration with the optimal Runge-Kutta method

After the spatial discretization by the nodal DG method, the semi-discrete system can be expressed in a general form of ordinary differential equations (ODE) as

$$\frac{dq_h}{dt} = \mathcal{L}(q_h(t), t),\quad (16)$$

where q_h is the vector of all discrete nodal solutions and $\mathcal{L}$ the spatial discretization operator of DG. Here, a low-storage explicit Runge-Kutta method is used to integrate Eq. (16), which reads

$$\begin{aligned}q_h^{(0)} &= q_h^n, \\ \begin{cases} k^{(i)} = a_i k^{(i-1)} + \Delta t \mathcal{L}(t^n + c_i \Delta t, q_h^{(i-1)}), \\ q_h^{(i)} = q_h^{(i-1)} + b_i k^{(i)}, \end{cases} & \text{for } i = 1, \dots, s, \\ q_h^{n+1} &= q_h^{(s)}, \end{aligned}\quad (17)$$

where $\Delta t = t^{n+1} - t^n$ is the time step, q_h^{n+1} and q_h^n are the solution vectors at time t^{n+1} and t^n , respectively, s is the number of stages of a particular scheme. In this work, the coefficients a_i , b_i , and c_i are chosen from the optimal Runge-Kutta scheme reported in Ref. 33.

III. IMPEDANCE BOUNDARY CONDITIONS AND NUMERICAL STABILITY

A. Numerical flux for frequency-independent impedance boundary conditions

The numerical flux F^* plays a key role in the DG scheme. Apart from linking neighboring interior elements, it serves to impose the boundary conditions and to guarantee stability of the formulation. Boundary conditions can be enforced weakly through the numerical flux either by reformulating the flux subject to specific boundary conditions or by providing the exterior solution q_h^+ .³⁴ In both cases, the solutions from the interior side of the element face (equivalent to boundary surface) q_h^- are readily used, whereas, for the second case, the exterior solutions q_h^+ need to be suitably defined as a function of interior solution q_h^- based on the imposed conditions. In the following, the impedance boundary condition is prescribed by reformulating the numerical flux. It should be noted that throughout this study, only plane-shaped reflecting boundary surfaces are considered. Furthermore, only locally reacting surfaces are considered, whose surface impedance is independent of the incident angle. This assumption is in accordance with the nodal DG scheme, since the unknown acoustic particle velocities on the boundary surface nodes depend on the pressure at exactly the same positions.

To reformulate the numerical flux at an impedance boundary, we take advantage of the characteristics of the underlying hyperbolic system and utilize the reflection coefficient R for plane waves at normal incidence. First, the same eigendecomposition procedure is performed for the projected flux Jacobian on the boundary as is shown in Eq. (10).

Second, by pre-multiplying the acoustic variables $\mathbf{q}$ with the left eigenmatrix $\mathbf{L}^{-1}$, the characteristics corresponding to the acoustic waves^{35,36} read

$$\begin{bmatrix} \omega_o \\ \omega_i \end{bmatrix} = \begin{bmatrix} p/\rho c + un_x + vn_y + wn_z \\ p/\rho c - un_x - vn_y - wn_z \end{bmatrix}, \quad (18)$$

where ω_o corresponds to the outgoing characteristic variable that leaves the computational domain and ω_i is the incoming characteristic variable.

The general principle for imposing boundary conditions of hyperbolic systems is that the outgoing characteristic variable should be computed with the upwind scheme using the interior values, while the incoming characteristic variable are specified conforming with the prescribed behaviour across the boundary. The proposed real-valued impedance boundary formulation is accomplished by setting the incoming characteristic variable as the product of the reflection coefficient and the outgoing characteristic variable, i.e., $\omega_i = R\omega_o$. Finally, the numerical flux on the impedance boundary surface can be expressed in terms of the interior values $\mathbf{q}_h^-$ as follows:

$$(\mathbf{n} \cdot \mathbf{F}^*) = \mathbf{L}\mathbf{A} \begin{bmatrix} 0 \\ 0 \\ \mathbf{p}_h^-/\rho c + \mathbf{u}_h^- n_x + \mathbf{v}_h^- n_y + \mathbf{w}_h^- n_z \\ R \cdot (\mathbf{p}_h^-/\rho c + \mathbf{u}_h^- n_x + \mathbf{v}_h^- n_y + \mathbf{w}_h^- n_z) \end{bmatrix}. \quad (19)$$

For given constant values of the normalized surface impedance Z_s , the reflection coefficient can be calculated from $R = (Z_s - 1)/(Z_s + 1)$, which is consistent with the fact that the numerical flux from the nodal DG scheme is always normal to the boundary surface. When the reflection coefficient is set to zero it can be easily verified that the proposed formulation reduces to the characteristic non-reflective boundary condition, which is equivalent to the first-order Engquist-Majda absorbing boundary condition.³⁷

B. Numerical stability of the DG scheme

In this section, the stability properties of the DG scheme are discussed. First, the semi-boundedness of the spatial DG operator together with the proposed impedance boundary conditions is analyzed using the energy method. Second, the fully discrete stability is discussed and the criterion for choosing the discrete time step is presented.

1. Stability of the semi-discrete formulation

Under a certain initial condition and impedance boundary condition, the governing linear acoustic equations (1) constitute a general initial-boundary value problem. For real-valued impedance boundary conditions, the classical von Neumann (or Fourier) stability analysis can no longer be applied, because the necessary periodic boundary conditions for the Fourier components do not exist. To analyze the stability or boundedness of the semi-discrete system, the energy method³⁸ is adopted here. The principle is to construct a

norm and to demonstrate it does not grow with increasing time. This technique has also been applied in other acoustic simulation methods,^{4,39} even in the fully discrete case.

For the numerical solution of the acoustic variables, e.g., $u_h(\mathbf{x}, t)$, the local inner product and its associated L^2 norm in function space V_h^k are defined as

$$(u_h^k, u_h^k)_{D^k} = \int_{D^k} u_h^k(\mathbf{x}, t) u_h^k(\mathbf{x}, t) d\mathbf{x} = \|u_h^k\|_{D^k}^2. \quad (20)$$

Similarly, over the element surface ∂D^{kr} , define

$$(u_h^{kr}, u_h^{kr})_{\partial D^{kr}} = \int_{\partial D^{kr}} u_h^{kr}(\mathbf{x}, t) u_h^{kr}(\mathbf{x}, t) d\mathbf{x} = \|u_h^{kr}\|_{\partial D^{kr}}^2, \quad (21)$$

where u_h^{kr} is the numerical solution on the element surface ∂D^{kr} . Now, the discrete acoustic energy norm inside single element D^k can be defined

$$E_h^k = \frac{1}{2} \rho \|u_h^k\|_{D^k}^2 + \frac{1}{2} \rho \|v_h^k\|_{D^k}^2 + \frac{1}{2} \rho \|w_h^k\|_{D^k}^2 + \frac{1}{2\rho c^2} \|p_h^k\|_{D^k}^2. \quad (22)$$

This definition is in complete analogy with the continuous acoustic energy, denoted as E , throughout the whole domain Ω , i.e., $E = \int_{\Omega} (1/2\rho c^2)p^2 + (\rho/2)|\mathbf{v}|^2 d\mathbf{x}$.

By summing all the local discrete acoustic energies over the volume and the boundaries, it can be proved in the Appendix that the total discrete acoustic energy, which is denoted as $E_h = \sum_{k=1}^K E_h^k$, is governed by

$$\begin{aligned} \frac{d}{dt} E_h = & - \sum_{\partial D^{mt} \in \mathcal{F}_B} \left(\frac{1 - R^{mt}}{2\rho c} \|p_h^{mt}\|_{\partial D^{mt}}^2 \right. \\ & \left. + \frac{\rho c}{2} (1 + R^{mt}) \|v_{hn}^{mt}\|_{\partial D^{mt}}^2 \right) \cdots \\ & - \sum_{\partial D^{kr} \in \mathcal{F}_I} \left(\frac{1}{2\rho c} \|p_h^{kr}\|_{\partial D^{kr}}^2 + \frac{\rho c}{2} \|n_x^{kr}\| [u_h^{kr}] \right. \\ & \left. + n_y^{kr} [v_h^{kr}] + n_z^{kr} [w_h^{kr}] \right) \|u_h^{kr}\|_{\partial D^{kr}}^2, \end{aligned} \quad (23)$$

where $\mathcal{F}_I$ and $\mathcal{F}_B$ denote the union set of interior elements and elements with at least one surface collocated with a physical boundary. $[\cdot]$ denotes the jump differences across the element surfaces. $v_{hn}^{mt} = n_x^{mt} u_h^{mt} + n_y^{mt} v_h^{mt} + n_z^{mt} w_h^{mt}$ denotes the outward velocity component normal to the impedance boundary. R^{mt} is the normal incidence plane-wave reflection coefficient along the t th boundary surface of element ∂D^{mt} . ∂D^{kr} and ∂D^{ls} refers to the same element intersection surface between neighboring elements D^k and D^l . Since each norm is non-negative and $R \in [-1, 1]$ holds for a passive impedance boundary,⁴⁰ it is proved that the semi-discrete acoustic system resulting from the DG discretization is unconditionally stable for passive boundary conditions with a real-valued impedance.

It is worth mentioning that the second sum term of Eq. (23) is related to the energy dissipation inside the

computational domain due to the use of the upwind scheme. This dissipation will converge to zero when the jump differences across the shared element interfaces converge to zero at a rate corresponding to the approximation polynomial order. The first sum of Eq. (23) is associated with the energy flow through the impedance boundary. One advantage of using the reflection coefficient to impose the impedance boundary condition is that the following singular cases can be considered without the need for exceptional treatments.

- Hard wall case. As $Z_s \rightarrow \infty$ or $R \rightarrow 1$, $v_{hn} \rightarrow 0$, then the boundary energy term converges to 0, meaning that the energy is conserved.
- Pressure-release condition. As $Z_s \rightarrow 0$ or $R \rightarrow -1$, $p_h \rightarrow 0$, then the boundary energy term once again converges to 0, and the energy is conserved as well.

2. Stability of the fully discrete formulation and time step choices

The above analysis is devoted to the stability analysis of the semi-discrete formulation Eq. (16), which in matrix form reads

$$\frac{dq_h}{dt} = L_h q_h, \quad (24)$$

where L_h is the matrix representation of the spatial operator $\mathcal{L}$. Ideally, the fully discrete approximation should be stable, at least under a reasonable upper bound on the time step size. Unfortunately, the theoretical ground for stability of a discretized PDE system is not very complete,³⁸ particularly for high order time integration methods. A commonly used approach based on the von Neumann analysis is to choose the time step size Δt small enough so that the product of Δt with the full eigenvalue spectrum of L_h falls inside the stability region of the time integration scheme. It should be noted that this is only a necessary condition for a general initial-boundary value problem, with the sufficient condition being more restrictive and complex.^{41,42} However, for real world problems, this necessary condition serves as a useful guideline.

It is computationally infeasible to compute the eigenvalue of L_h before the simulation is started for various unstructured mesh, polynomial order and boundary conditions. For the DG method, it is found that for the first order system Eq. (16), the gradients of the normalized N th order polynomial basis are of order $\mathcal{O}(N^2/h)$ near the boundary part of the element,²⁵ consequently the magnitude of the maximum eigenvalue λ_N scales with the polynomial order N as: $\max(\lambda_N) \propto N^2$, indicating that $\Delta t \propto N^{-2}$. This severe time step size restriction limits the computational efficiency of high polynomial order approximations. In all the numerical experiments presented in this work, the temporal time steps are determined in the following way:³³

$$\Delta t = C_{CFL} \min(r_D) \frac{1}{c}, \quad (25)$$

where r_D is the radius of the inscribed sphere of the tetrahedral elements. As a reference, the tabulated maximum allowable Courant number C_{CFL} of the current used *RKF84*

scheme for each polynomial order N can be found in Ref. 33. In each of the following numerical tests, the exact value of C_{CFL} are explicitly stated for completeness.

IV. APPLICATIONS

To investigate the applicability of the nodal DG time-domain method as described in Sec. II and Sec. III for room acoustics problems, various 3-D numerical tests are designed and compared in this section. The first test is a free field propagation of a single frequency plane wave under periodic boundary conditions. In this case, the dissipation error in terms of the wave amplitude and the dispersion error are investigated. The second configuration is a sound source over an impedance plane. The accuracy of the proposed DG formulation to simulate frequency-independent impedance boundary conditions is verified. The third configuration is a sound source in a cuboid room with rigid boundary conditions, embodying an approximation to a real room including multiple reflections. The modal behaviour of the space is investigated for different polynomial order N of the basis functions when compared with the analytical solution, together with an analysis of the sound energy conservation inside the room to quantify the numerical dissipation. Finally, the fourth configuration is adopted to demonstrate the applicability of the method to a real room. The configuration is a room with complex geometry and a real-valued impedance boundary condition. In this configuration, the pressure response functions in the frequency domain are compared with the measured results at several receiver locations. For the acoustic speed and the air density, $c = 343$ m/s and $\rho = 1.2$ kg/m³ are used in all calculations. Due to the fact that there are duplicated nodes along the element interfaces, in this work, the number of degrees of freedom per wavelength λ (DPW) is used to give a practical indication of the computational cost. It is computed as

$$DPW = \lambda \sqrt[3]{\frac{N_p \times K}{V}}, \quad (26)$$

where $N_p \times K$ is the number of degrees of freedom for a single physical variable in the computational domain, V is the volume of the whole domain.

A. Free field propagation in periodic domain

To verify the accuracy of the free field propagation, we consider a cubic computational domain of size $[0, 1]^3$ in meters, which is discretized with six congruent tetrahedral elements. $10 \times 10 \times 10$ receivers are evenly spaced in all directions throughout the domain. The domain is initialized with a single frequency plane wave propagating in the x-direction only,

$$p(\mathbf{x}, t = 0) = \sin(-2\pi x), \quad (27a)$$

$$u(\mathbf{x}, t = 0) = \frac{1}{\rho c} \sin(-2\pi x), \quad (27b)$$

$$v(\mathbf{x}, t = 0) = 0, \quad w(\mathbf{x}, t = 0) = 0. \quad (27c)$$

The wavelength λ is chosen to be equal to 1 m such that periodic boundary conditions can be applied in all directions. As mentioned in Sec. II B, when an initial value problem is

simulated, the approximation error associated with the representations of the initial conditions is a dominant component. In order to rule out this approximation error and to assess the dissipation and dispersion error accumulated from the wave propagation alone, the solution values at receiver locations recorded during the first wave period T of propagation are taken as the reference values. The solutions sampled during later time period interval $t = [(n - 1)T, nT]$ are compared with these reference values, where $n = 10, 20, 30, \dots, 100$. The amplitude and phase values of the single frequency wave at each of the receiver locations are obtained from a Fourier transform of the recorded time signals without windowing. The dissipation error ϵ_{amp} in dB and the phase error ϵ_ϕ in % are calculated as follows:

$$\epsilon_{amp} = \max \left(20 \log_{10} \frac{|P_{ref}(\mathbf{x})|}{|P_{nT}(\mathbf{x})|} \right), \quad (28a)$$

$$\epsilon_\phi = \max \left(\frac{|\phi(P_{nT}(\mathbf{x})) - \phi(P_{ref}(\mathbf{x}))|}{\pi} \times 100\% \right), \quad (28b)$$

where $P_{ref}(\mathbf{x})$ and $P_{nT}(\mathbf{x})$ are the Fourier transform of the recorded pressure values at different locations, during the first time period and the n th period, respectively. $\phi(\cdot)$ extracts the phase angle of a complex number.

Simulations for $N=5, 6, 7$ corresponding to $DPW=6.9, 7.9, 8.9$ have been carried out and a single time step size $\Delta t = T/100 = 1/(100 \times 343)$ is used for all simulations to make sure the time integration error is much smaller than the spatial error.

The dissipation and the phase error from the explicit Runge-Kutta time integration is calculated based on the descriptions presented in Ref. 43 and shown as dashed lines in Fig. 1. As can be seen, both the dissipation error and the phase error grow linearly with respect to the propagation distance. For the 5th order polynomial basis ($DPW=6.9$), the averaged dissipation error is approximately 0.035 dB when the wave travels one wavelength distance while the phase error is 0.095%. Both error drop to 0.002 dB and 0.005%, respectively, when the DPW increases to 8. When the DPW

is equal to 8.9, the dissipation error is 1.1×10^{-4} dB per wavelength of propagation and the phase error is less than $3 \times 10^{-4}\%$.

B. Single reflective plane

To verify the performance of the proposed frequency-independent impedance boundary condition, a single reflection scenario is considered and the reflection coefficient obtained from the numerical tests is compared with the analytical one based on a locally reacting impedance. The experiment consists of two simulations. In the first simulation, we consider a cubic domain of size $[-8, 8]^3$ in meters, where the source is located at the center $[0, 0, 0]$ m, and two receivers are placed at $\mathbf{x}_{r1} = [0, 0, -1]$ m and $\mathbf{x}_{r2} = [0, 4, -1]$ m. In this case, the free field propagation of a sound source is simulated, and sound pressure signals are recorded at both receiver locations. In the second simulation, a plane reflecting surface is placed 2 m away from the source at $z = -2$ m. The measured sound pressure signals not only contain the direct sound but also the sound reflected from the impedance surface. In both cases, initial pressure conditions are used to initiate the simulations:

$$p(\mathbf{x}, t = 0) = e^{-\ln(2)/b^2((x-x_s)^2 + (y-y_s)^2 + (z-z_s)^2)}, \quad (29a)$$

$$\mathbf{v}(\mathbf{x}, t = 0) = \mathbf{0}, \quad (29b)$$

which is a Gaussian pulse centered at the source coordinates $[x_s, y_s, z_s] = [0, 0, 0]$. The half-bandwidth of this Gaussian pulse is chosen as $b = 0.25$ m. Simulations are stopped at around 0.0321 s in order to avoid the waves reflected from the exterior boundaries of the whole domain. In order to eliminate the effects of the unstructured mesh quality on the accuracy, structured tetrahedra meshes are used for this study, which are generated with the meshing software GMSH.⁴⁴ The whole cuboid domain is made up of structured cubes of the same size, then each cube is split into six tetrahedra elements. The length of each cube is 0.5 m.

Let p_d denote the direct sound signal measured from the first simulation, then the reflected sound signal $p_r(t)$ is

FIG. 1. (Color online) Amplitude error ϵ_{amp} (a) and phase error ϵ_ϕ (b) for the periodic propagation of a single frequency plane wave Eq. (27).

obtained by eliminating $p_d(t)$ from the solution of the second simulation. Let R_1 denote the distance between the source and the receiver and R_2 is the distance between the receiver and the image source (located at $[0, 0, -4]$ m) mirrored by the reflecting impedance surface. The spectra of the direct sound and the reflected sound, denoted as $P_d(f)$ and $P_r(f)$ respectively, are obtained by Fourier transforming p_d and p_r without windowing. The numerical reflection coefficient Q_{num} is calculated as follows:

$$Q_{num}(f) = \frac{P_r(f) \cdot G(\kappa R_1)}{P_d(f) \cdot G(\kappa R_2)}, \quad (30)$$

where

$$G(\kappa R) = e^{i\kappa R}/R \quad (31)$$

is the Green function in 3D free space. κ is the wavenumber. The analytical spherical wave reflection coefficient Q reads⁴⁵

$$Q = 1 - 2 \frac{\kappa R_2}{Z_s e^{i\kappa R_2}} \int_0^\infty e^{-q\kappa/Z_s} \frac{e^{i\kappa \sqrt{r_p^2 + (z+z_s+iq)^2}}}{\sqrt{r_p^2 + (z+z_s+iq)^2}} dq, \quad (32)$$

where Z_s is the normalized surface impedance, $z = 1$ is the distance between the receiver and the reflecting surface, $z_s = 2$ is the distance between the source and the surface, r_p is the distance between the source and the receiver projected on the reflecting surface.

Simulations with polynomial order $N = 5$ up to $N = 8$ are carried out with the corresponding C_{CFL} and time step Δt presented in Table I.

The results of the numerical tests for $Z_s = 3$ are illustrated in Fig. 2. The DPW is calculated based on the frequency of 500 Hz. The comparison of the magnitudes of the spherical wave reflection coefficient for both the normal incidence angle $\theta = 0^\circ$ and the oblique incidence ($\theta = 53^\circ$) are shown in Figs. 2(a) and 2(b), respectively. The phase angle comparison is presented in Figs. 2(c) and 2(d). It can be seen that with increasing polynomial order N (or DPW), the numerical reflection coefficient converges to the analytical one in terms of the magnitude and the phase angle. Also, the accuracy is rather independent on the two angles of incidence θ . In order to achieve a satisfactory accuracy, at least 12 DPW are needed. Many tests are performed with different impedances ($Z_s \in [1, \infty]$) and receiver locations ($\theta \in [0^\circ, 90^\circ]$), the same conclusion can be reached.

TABLE I. C_{CFL} number and time step Δt for single reflection case ($h = 0.5$ m).

N	C_{CFL}	Δt [s]
5	0.185	9.721×10^{-5}
6	0.144	7.550×10^{-5}
7	0.114	5.993×10^{-5}
8	0.094	4.908×10^{-5}

C. Cuboid room with rigid boundaries

In this section, the nodal DG method is applied to sound propagation in a 3-D room with rigid boundaries ($R = 1$). In contrast to the previous applications, sound propagation inside the room is characterized by multiple reflections and sound energy is conserved. The domain of the room is $[0, L_x] \times [0, L_y] \times [0, L_z]$ m, with $L_x = 1.8$, $L_y = 1.5$, $L_z = 2$. Initial conditions are given as in Eqs. (29), with $b = 0.2$ m. The source is positioned at $[0.9, 0.75, 1]$ m, and a receiver is positioned at $[1.7, 1.45, 1.9]$ m. Similar as in the previous test case, the room is discretized using structured tetrahedral elements of size 0.4 m. The analytical pressure response in a cuboid domain can be obtained by the modal summation method, and can in the 3-D Cartesian coordinate system be written as⁴⁶

$$p(\mathbf{x}, t) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \hat{p}_{lmn}(t) \psi_{lmn}(\mathbf{x}) \cos(\omega_{lmn} t), \quad (33a)$$

$$\psi_{lmn}(\mathbf{x}) = \cos\left(\frac{l\pi x}{L_x}\right) \cos\left(\frac{m\pi y}{L_y}\right) \cos\left(\frac{n\pi z}{L_z}\right), \quad (33b)$$

$$\omega_{lmn} = c \sqrt{\left(\frac{l\pi}{L_x}\right)^2 + \left(\frac{m\pi}{L_y}\right)^2 + \left(\frac{n\pi}{L_z}\right)^2}, \quad (33c)$$

with ψ_{lmn} the modal shape function; $\hat{p}_{lmn}$ the modal participation factor; ω_{lmn} the natural angular frequency; and l, m, n the mode indices. Since reflections from the room boundaries occur without energy loss, the modal participation factors are constant over time. To obtain $\hat{p}_{lmn}(0)$, the initial pressure distribution is projected onto each modal shape as

$$\hat{p}_{lmn}(0) = \frac{1}{\Lambda_{lmn}} \int_{\Omega} p(\mathbf{x}, t=0) \psi_{lmn}(\mathbf{x}) d\mathbf{x}, \quad (34a)$$

$$\Lambda_{lmn} = \int_{\Omega} \psi_{lmn}^2(\mathbf{x}) d\mathbf{x}. \quad (34b)$$

The integration in Eq. (34a) can be calculated separately for each coordinate. For example, in the x coordinate, the indefinite integration can be expressed in terms of the error function as

$$\begin{aligned} & \int e^{(-a_0(x-x_0)^2)} \cos(b_0 x) dx \\ &= \frac{\sqrt{\pi}}{4\sqrt{a_0}} e^{((-b_0^2/4a_0) - ib_0 x_s)} [\text{erf}(B) + e^{(i2b_0 x_s)} \text{erf}(B^*)] + C, \end{aligned} \quad (35)$$

with $B = \sqrt{a_0}(x - x_0) + ib_0/2\sqrt{a_0}$, $a_0 = \ln(2)/b^2$, $b_0 = l\pi/L_x$, and C is a constant. Equation (33a) is used as the reference solution with modal frequencies up to 8 kHz. Furthermore, to show the applicability of the nodal DG method for a long time simulation, 10 s is taken as the simulation duration.

To solve for this configuration, the C_{CFL} numbers and time steps for the approximating polynomial orders of $N = 3$ up to $N = 7$ are presented in Table II.

The sound pressure level is computed as

FIG. 2. (Color online) Numerical reflection coefficient calculated by Eq. (30) with different polynomial orders, compared with the theoretical result according to Eq. (32) (black dashed line): (a) magnitude for receiver 1, $\theta = 0^\circ$, (b) magnitude for receiver 2, $\theta = 53^\circ$, (c) phase angle for receiver 1, $\theta = 0^\circ$, (d) phase angle for receiver 2, $\theta = 53^\circ$.

$$L_p = 20 \log_{10} \left| \frac{P(f)}{\sqrt{2}P_0} \right|, \quad (36)$$

with $P_0 = 2 \times 10^{-5}$ Pa, and $P(f)$ the spectrum of recorded pressure time signal $p(t)$ at the receiver location. The end of the time signal is tapered using a Gaussian window with a length of 3.5 s to avoid the Gibbs effect.

Figure 3 shows the sound pressure level at the receiver location. The numerical solutions show an excellent agreement with the reference solution, with the accuracy of the numerical solution increasing as the approximating polynomial order increases.

TABLE II. C_{CFL} number and time step Δt for a rigid cuboid room ($h = 0.4$ m).

N	C_{CFL}	Δt [s]
3	0.355	1.400×10^{-4}
4	0.248	9.810×10^{-5}
5	0.185	7.322×10^{-5}
6	0.144	5.687×10^{-5}
7	0.114	4.514×10^{-5}

Figure 4 displays the results for $f = 950$ – 1000 Hz. From this figure, we can see that the resonance frequencies are not well represented for $N \leq 5$, for which DPW varies between 4.5 and 6.6 in this frequency range. On the other hand, the resonance frequencies are correctly represented for $N \geq 6$, where the minimum number of DPW is 7.2. The correct representation of the room resonance frequencies indicates that the numerical dispersion is low in the DG solution. The numerical dispersion aspect is essential with regards to auralization as shown by Saarelma *et al.*,⁴⁷ where the audibility of the numerical dispersion error from the finite difference time domain simulation is investigated. Furthermore, Fig. 4 clearly shows that the DG results have reduced peak amplitudes when the approximating polynomial orders are low.

D. Real room with real-valued impedance boundary conditions

The final scenario is a comparison between experimental and numerical results of a real room. The room is located in the Acoustics Laboratory building (ECHO building) at the campus of the Eindhoven University of Technology. Geometrical data of the room, including the dimensions and

FIG. 3. (Color online) Sound pressure level at receiver position in the configuration of the 3-D rigid cuboid room.

the location of the source and microphone positions are presented in Fig. 5. The room has a volume of $V = 89.54 \text{ m}^3$ and a boundary surface area of $S = 125.08 \text{ m}^2$.

The source is located at $[1.7, 2.92, 1.77] \text{ m}$ and microphones (M) are located at $[3.8, 1.82, 1.66] \text{ m}$ for M1 and $[4.75, 3.87, 1.63] \text{ m}$ for M2. The height (z -coordinate) of the sound source location is measured at the opening (highest point) of the used sound source (B&K type 4295, OmniSource Sound Source). The measurements were performed using one free-field microphone B&K type 4189 connected to a Triton USB Audio Interface. The impulse responses were acquired with a sampling frequency of 48 KHz with a laptop using the room acoustics software DIRAC (B&K type 7841). The input channel is calibrated before starting the measurements using a calibrator (B&K type 4230). The sound signal used for the excitation of the room is the DIRAC built-in e-Sweep signal with a duration of 87.4 s connected to an Amphion measurement amplifier. At each microphone position, three measurement repetitions were performed. The results presented in this section for M1 and M2 represent the average of the three repetitions.

The room is discretized in 9524 tetrahedral elements by using GMSH and the largest element size is 0.5 m. A detail of the mesh is shown in Fig. 5(b). The same initial pressure

distribution as for the 3-D cuboid room of Sec. IV C is used. The polynomial order used in the calculations is $N = 4$ with a CFL number of $C_{CFL} = 0.25$. The computed impulse responses have a duration of 15 s. The model uses a DPW of 13 for the frequency of 400 Hz. All the boundaries of the model are computed using a uniform real-valued reflection coefficient of $R = 0.991$. The coefficient is calculated from the experimental results at M1 by computing the Q-value of the resonance at $f_0 = 97.9 \text{ Hz}$, using $R = 1 - \delta_r 8V/cS$ with, $\delta_r = 2\pi f_0/2Q$ the decay constant of the room's resonance.

Both impulse responses from the measurements and simulations were transformed to the frequency domain by using a forward Fourier transform. The end of the time signals is tapered by a single-sided Gaussian window with a length of 500 samples (approximately, 5.6 ms) to avoid the Gibbs effect. Furthermore, the time function of the numerical source is obtained from the following analytical expression: $p_{s,ana}(t) = [(r_{sr} - ct)/2r_{sr}]e^{(-\ln(2)/b^2)(r_{sr} - ct)^2} + [(r_{sr} + ct)/2r_{sr}]e^{(-\ln(2)/b^2)(r_{sr} + ct)^2}$ (with r_{sr} the source-receiver distance). This function is transformed to the frequency domain to normalize the calculated impulse responses in DG by the source power spectrum. Likewise, the experimental results have been normalized by the B&K 4295 sound power spectra. The source spectra of an equivalent source B&K 4295 has

FIG. 4. (Color online) Receiver sound pressure response level between 950 and 1000 Hz.

FIG. 5. (Color online) Graphical data of the room under investigation: (a) isometric view; (b) isometric view with surface elements; (c) plan view; (d) section view; (e) picture during the measurements.

been obtained by measurements in the anechoic room of the Department of Medical Physics and Acoustics at Carl von Ossietzky Universität Oldenburg. The corrected results should be taken with care at frequencies below 50 Hz, due to limitations of the anechoic field in the determination of the power spectra of the source. The numerical and experimental results have been normalized at 100 Hz, using the results of position M1.

The comparison between numerical and experimental solutions is shown in Fig. 6 for narrow and 1/3 octave frequency bands. The results are quite satisfactory considering that only one uniform real-valued impedance has been used for the whole frequency range of interest. The biggest deviation, 3.6 dB, is found at position M2 in the 63 Hz 1/3 octave band, while for position M1 the maximum deviation is 2.8 dB in the 250 Hz 1/3 octave band. The average deviation for the 1/3 octave band spectra is 1.2 dB for M1 and 2.3 dB for M2. Overall, the deviations shown in Fig. 6 are within a reasonable range. Factors like the geometrical mismatches between the real room and the model or the uncertainty in the location of the source and microphone positions are influencing the deviations.

V. CONCLUSIONS

In this paper, the time-domain nodal Discontinuous Galerkin (DG) method has been evaluated as a method to solve the linear acoustic equations for room acoustic purposes. A nodal DG method is used for the evaluation of the spatial derivatives, and for time-integration a low-storage optimized eight-stage explicit Runge-Kutta method is adopted. A new formulation of the impedance boundary condition, which is based on the plane wave reflection coefficient, is proposed to simulate the locally

reacting surfaces with frequency-independent real-valued impedances and its stability is analyzed using the energy method.

The time-domain nodal Discontinuous Galerkin (DG) method is implemented for four configurations. The first test case is a free field propagation, where the dissipation error and the dispersion error are investigated using different polynomial orders. Numerical dissipation exists due to the upwind numerical flux. The benefits of using high-order basis are demonstrated by the significant improvement in accuracy. When DPW is around 9, the dissipation error is 1.1×10^{-4} dB and the phase error is less than $3 \times 10^{-4}\%$ under propagation of one wavelength. In the second configuration, the validity and convergence of the proposed impedance boundary formulation is demonstrated by investigating the single reflection of a point source over a planar impedance surface. It is found that the accuracy is rather independent on the incidence angle. As a third scenario, a cuboid room with rigid boundaries is used, for which a long-time (10 s) simulation is run. By comparing against the analytical solution, it can be concluded again that with a sufficiently high polynomial order, the dispersion and dissipation error become very small. Finally, the comparison between numerical and experimental solutions shows that DG is a suitable tool for acoustic predictions in rooms. Taking into account that only one uniform real-valued impedance has been used for the whole frequency range of interest, the results are quite satisfactory. In this case, the implementation of frequency dependent boundary conditions will clearly improve the precision of the numerical results.

In this study, the performance of the time-domain nodal DG method is investigated by comparing with analytical solutions and experimental results, without comparing with

FIG. 6. (Color online) Sound pressure level L_p in the real room configuration for the experimental and the DG results in narrow frequency bands (black broken line and red solid line, respectively) and 1/3 octave bands (black dot and red dot, respectively) for the microphone positions (a) M1 and (b) M2.

other commonly used room acoustics modelling techniques such as FDTD and FEM. The aim of this work is to demonstrate the viability of the DG method to room acoustics modelling, where high-order accuracy and geometrical flexibility are of key importance. With the opportunity to massively parallelize the DG method, it has great potential as a wave-based method for room acoustic purposes. Whereas the results show that high accuracy can be achieved with DG, some issues remain to be addressed. The improvements in accuracy using high-order schemes come at a cost of smaller time step size for the sake of stability. There is a trade-off between a high-order scheme with a small time step and fewer spatial points and low-order methods, where a larger time step is allowed but a higher number of spatial points are needed to achieve the same accuracy. Further investigations are needed to find out the most cost-efficient combination of the polynomial order and the mesh size under a given accuracy requirement. Also, when the mesh configuration is fixed by the geometry, the local adaptivity of polynomial orders and time step sizes could be a feasible approach to improve the computational efficiency of DG for room acoustics applications. Furthermore, general frequency-dependent impedance boundary conditions as

well as extended reacting boundary conditions are still to be rigorously developed in DG.

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APPENDIX: DERIVATIONS OF THE TOTAL DISCRETE ACOUSTIC ENERGY OF THE SEMI-DISCRETE SYSTEM

It can be seen that the local energy can be recovered from the product of the element mass matrix $\mathbf{M}^k$ and the nodal vectors $\mathbf{u}_n^k$ as follows:

$$\begin{aligned} (\mathbf{u}_h^k)^T \mathbf{M}^k \mathbf{u}_h^k &= \int_{D^k} \sum_{i=1}^{N_p} u_h^k(\mathbf{x}_i^k, t) l_i^k(\mathbf{x}) \sum_{j=1}^{N_p} u_h^k(\mathbf{x}_j^k, t) l_j^k(\mathbf{x}) d\mathbf{x} \\ &= \|\mathbf{u}_h^k\|_{D^k}^2. \end{aligned} \quad (\text{A1})$$

Furthermore, it can be verified that

$$\begin{aligned} (\mathbf{u}_h^k)^T \mathbf{S}_x^k \mathbf{p}_h^k &= \int_{D^k} \sum_{i=1}^{N_p} p_h^k(\mathbf{x}_i^k, t) l_i^k(\mathbf{x}) \sum_{j=1}^{N_p} u_h^k(\mathbf{x}_j^k, t) \frac{\partial l_j^k(\mathbf{x})}{\partial x} d\mathbf{x} \\ &= \int_{D^k} p_h^k(\mathbf{x}, t) \frac{\partial u_h^k(\mathbf{x}, t)}{\partial x} d\mathbf{x} = \left(p_h^k, \frac{\partial u_h^k}{\partial x} \right)_{D^k} \end{aligned} \quad (\text{A2})$$

and

$$\begin{aligned} (\mathbf{u}_h^k)^T \mathbf{M}^{kr} \mathbf{p}_h^{kr} &= \int_{\partial D^{kr}} \sum_{i=1}^{N_p} u_h^k(\mathbf{x}_i^k, t) l_i^k(\mathbf{x}) \sum_{j=1}^{N_{fp}} p_h^{kr}(\mathbf{x}_j^{kr}, t) l_j^{kr}(\mathbf{x}) d\mathbf{x} \\ &= \int_{\partial D^{kr}} \sum_{i=1}^{N_{fp}} u_h^{kr}(\mathbf{x}_i^{kr}, t) l_i^{kr}(\mathbf{x}) \sum_{j=1}^{N_{fp}} p_h^{kr}(\mathbf{x}_j^{kr}, t) l_j^{kr}(\mathbf{x}) d\mathbf{x} \\ &= \int_{\partial D^{kr}} u_h^{kr}(\mathbf{x}, t) p_h^{kr}(\mathbf{x}, t) d\mathbf{x} = (u_h^{kr}, p_h^{kr})_{\partial D^{kr}}. \end{aligned} \quad (\text{A3})$$

Now, the total discrete acoustic energy E_h of the semi-discrete formulation Eq. (13) can be calculated. By pre-multiplying Eq. (13a) with $\rho(\mathbf{u}_h^k)^T$, pre-multiplying Eq. (13b) with $\rho(\mathbf{v}_h^k)^T$, pre-multiplying Eq. (13c) with $\rho(\mathbf{w}_h^k)^T$, pre-multiplying Eq. (13d) with $(1/\rho c^2)(\mathbf{p}_h^k)^T$ and sum them together, using the relations mentioned in Eqs. (A1), (A2), yields

$$\begin{aligned} \frac{d}{dt} E_h^k &= - \sum_{r=1}^f \left(n_x^{kr} (u_h^{kr}, p_h^{kr})_{\partial D^{kr}} + n_y^{kr} (v_h^{kr}, p_h^{kr})_{\partial D^{kr}} \right. \\ &\quad \left. + n_z^{kr} (w_h^{kr}, p_h^{kr})_{\partial D^{kr}} \right) \dots + \rho (\mathbf{u}_h^k)^T \sum_{r=1}^f \mathbf{M}^{kr} \hat{\mathbf{F}}_u^{kr} \\ &\quad + \rho (\mathbf{v}_h^k)^T \sum_{r=1}^f \mathbf{M}^{kr} \hat{\mathbf{F}}_v^{kr} + \rho (\mathbf{w}_h^k)^T \sum_{r=1}^f \mathbf{M}^{kr} \hat{\mathbf{F}}_w^{kr} \\ &\quad + \frac{1}{\rho c^2} (\mathbf{p}_h^k)^T \sum_{r=1}^f \mathbf{M}^{kr} \hat{\mathbf{F}}_p^{kr}, \end{aligned} \quad (\text{A4})$$

where the divergence theorem is used to obtain the surface integral term, that is

$$\begin{aligned} &\left(u_h^k, \frac{\partial p_h^k}{\partial x} \right)_{D^k} + \left(v_h^k, \frac{\partial p_h^k}{\partial y} \right)_{D^k} + \left(w_h^k, \frac{\partial p_h^k}{\partial z} \right)_{D^k} \\ &\quad + \left(p_h^k, \frac{\partial u_h^k}{\partial x} \right)_{D^k} + \left(p_h^k, \frac{\partial v_h^k}{\partial y} \right)_{D^k} + \left(p_h^k, \frac{\partial w_h^k}{\partial z} \right)_{D^k} \\ &= \sum_{r=1}^f \left(n_x^{kr} (u_h^{kr}, p_h^{kr})_{\partial D^{kr}} + n_y^{kr} (v_h^{kr}, p_h^{kr})_{\partial D^{kr}} \right. \\ &\quad \left. + n_z^{kr} (w_h^{kr}, p_h^{kr})_{\partial D^{kr}} \right). \end{aligned} \quad (\text{A5})$$

Substitute the numerical flux Eqs. (15) into Eq. (A4) and use Eq. (A3), after some simple algebraic manipulations, the semi-discrete acoustic energy balance on element yields

$$\frac{d}{dt} E_h^k = \sum_{r=1}^f \mathcal{R}_h^{kr}, \quad (\text{A6})$$

where

$$\begin{aligned} \mathcal{R}_h^{kr} &= (p_h^{kr}, v_{hn}^{kr})_{\partial D^{kr}} - \frac{1}{2} (\omega_o^{kr}, p_h^{kr} + \rho c v_{hn}^{kr})_{\partial D^{kr}} \\ &\quad + \frac{1}{2} (\omega_i^{ls}, p_h^{kr} - \rho c v_{hn}^{kr})_{\partial D^{kr}} \end{aligned} \quad (\text{A7})$$

is the discrete energy flux through the shared surface ∂D^{kr} or equivalently ∂D^{ls} between the neighboring elements D^k and D^l in the interior of the computation domain. ω_o and ω_i are the characteristic waves defined in Eq. (18). By using the condition that the outward normal vector of neighboring elements are opposite, the final form of energy contribution from the coupling across one shared interface reads

$$\begin{aligned} \mathcal{R}_h^{kr} + \mathcal{R}_h^{ls} &= - \left(\frac{1}{2\rho c} \|[p_h^{kr}]\|_{\partial D^{kr}}^2 + \frac{\rho c}{2} \|n_x^{kr} [u_h^{kr}] \right. \\ &\quad \left. + n_y^{kr} [v_h^{kr}] + n_z^{kr} [w_h^{kr}]\|_{\partial D^{kr}}^2 \right), \end{aligned} \quad (\text{A8})$$

which is non-positive. This ends the discussion for the interior elements. Now, for elements that have at least one surface lying on the real-valued impedance boundary, e.g., element D^m with surface $\partial D^{mt} \in \partial \Omega_h$, the numerical flux is calculated using Eq. (19). After some algebraic operations, the energy flux through the reflective boundary surface becomes

$$\mathcal{R}_h^{mt} = - \left(\frac{1 - R^{mt}}{2\rho c} \|p_h^{mt}\|_{\partial D^{mt}}^2 + \frac{\rho c}{2} (1 + R^{mt}) \|v_{hn}^{mt}\|_{\partial D^{mt}}^2 \right). \quad (\text{A9})$$

Finally, by summing the energy flux through all of the faces of the mesh, we get the total acoustic energy of the whole semi-discrete system as in Eq. (23).

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