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Quick Estimation of Periodic Signal Parameters From 1-Bit Measurements

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Abstract—Estimation of periodic signals, based on quantized data, is a topic of general interest in the area of instrumentation and measurement. Although several methods are available, new applications require low-power, low-complexity, and adequate estimation accuracy. In this paper, we consider the simplest possible quantization, that is, binary quantization, and describe a technique to estimate the parameters of a sampled periodic signal, using a fast algorithm. By neglecting the possibility that the sampling process is triggered by some signal-derived event, sampling is assumed to be asynchronous, that is, the ratio between the signal and the sampling periods is defined to be an irrational number. To preserve enough information at the quantizer output, additive Gaussian input noise is assumed as the information encoding mechanism. With respect to the published techniques addressing the same problem, the proposed approach does not rely on the numerical estimation of the maximum likelihood function but provides solutions that are very close to this estimate. At the same time, since the main estimator is based on matrix inversion, it proves to be less time-consuming than the numerical maximization of the likelihood function, especially when solving problems with a large number of parameters. The estimation procedure is described in detail and validated using both simulation and experimental results. The estimator performance limitations are also highlighted.

Index Terms—Estimation, identification, nonlinear estimation problems, nonlinear quantizers, quantization.

I. INTRODUCTION

ESTIMATING the characteristic parameters of a system or a signal using quantized data is a central problem in instrumentation and measurement. Conventional procedures, such as least squares estimators, are shown to be suboptimal and perform increasingly worse when signals are more coarsely quantized [1]. The challenge to recover input signal information is maximum when a binary quantizer, for example, a comparator, is used. On the other hand, binary quantization is attractive, mainly because of the ease in generating and processing binary data, even at very large sampling rates. Clearly, if suitable modulation of the input sequence is possible, 1-bit quantization may be performed to reduce information loss, as in 1-bit ΔΣ analog-to-digital converters (ADCs) [2]. Conversely, if the input signal is not preprocessed and is directly converted by a 1-bit quantizer, processing needed to recover some or all of the signal parameters can only benefit from the signal encoding properties of the additive noise possibly affecting the comparator input. In fact, if the input signal is consistently above or below the comparator threshold, the output sequence can take only a single value, and any estimator would fail in providing meaningful information. In fact, if the input signal is unaffected by noise, the quantizer output only produces information about signal zero-crossings, possibly useful for estimating the signal frequency components but insufficient to allow the estimation of the amplitudes of these components. Thus, it is exactly the noise source at the input that acts as a sequence randomizer and that encodes information about the signal parameters so that it can be extracted at the output of the 1-bit quantizer. As an example, consider the quantization of a constant signal affected by white Gaussian noise. Moschitta et al. [3] show that the amount of information at the quantizer output vanishes when the noise standard deviation tends to zero.

Importance for the Instrumentation and Measurement Community

In general, advanced processing of signals based on complexity reduction is of interest for the measurement community [4]. Practical usage of the 1-bit quantization is the subject of 1-bit measurements of sparse signals [5]–[7]. Similarly, system identification based on binary quantized data [8]–[10] is a technically challenging problem with applications in the identification of both linear and nonlinear systems. At the same time, both measurements of ADCs’ performance and the impact ADC resolution has on the estimation of input signals’ parameters play a central role in instrumentation and measurement [11].

Applications in ADC Testing

In fact, when testing ADCs, test signals are sinusoidal or multitone signals [12]. The input signal parameters are estimated through the same data record used to find the ADC threshold levels. The accuracy with which these parameters are known affects the ADC test results. Many
published paper address this problem [13], [14]. A simple estimator, like the one described in this paper, may ease this task. Moreover, ADCs are among the first devices in the signal chain of most modern instrumentation. This paper shows how to process basic information such as binary data to estimate the parameters of a periodic input signal.

High-Frequency Applications

By observing that binary quantization may be performed using off-the-shelf high sample rate (5 GSa/s) comparators [15] or experimental (20 GHz) comparators [16], the analysis of techniques to estimate parameters of analog signals using very coarsely quantized data can lead to the use of simple instrumentation, also in that interval of frequencies traditionally characterized by the use of complex instruments.

State of the Art

Previous research on this subject considered the estimation of systems parameters when the input sequence is known [17], [18] and when the input signal is synchronously sampled, that is, when the ratio between the sampling sequence period and the signal period is a rational number [19]. Several other papers have addressed a wide class of similar problems in the context of system identification [20]–[27]. In the areas of signal processing and communications, several contributions were made over the years to address signal and channel estimation problems and threshold value optimization. For a comprehensive list of these papers, see [28]. The topic of 1-bit quantization is the subject of extensive research covering the characteristics of scalar- and vector-signal parameter estimation problems, under the assumptions of both known and unknown noise distributions [29], [30]. In these papers, general properties were derived, such as the Cramér–Rao lower bound (CRLB) and the log-convexity of the likelihood function. This latter property also applies to the case of asynchronous, λ is irrational with probability 1. In practice, θ is a point that is very close to the final solution, in order to improve the MLE convergence speed.

In fact, when severe quantization occurs, as in the binary case, conventional estimation procedures, such as least squares estimators, either fail to provide meaningful results [1] or require substantial numerical processing when the problem complexity increases, for example, when the number of parameters to be estimated exceeds 50, as shown in Section VII in the case of the MLE [37]. This allows the solution of very large estimation problems, for which the approach based on the MLE would result in a larger computational time. This setting of the problem, which is important for the instrumentation and measurement community, is not covered in the literature.

II. PROBLEM STATEMENT

The problem analyzed in this paper can be described according to the following characteristics.

Measurement Setup

The signal chain considered in this paper is depicted in Fig. 1(a), where \( Q(\cdot) \) is a binary quantizer and \( \eta_j \) is a sequence of zero-mean independent random variables with variance \( \sigma^2 \). The function \( f(\cdot) \) is an invertible nonlinear function possibly distorting the signal prior to its binary quantization.

Signal

The sequence \( x_j \) is obtained by sampling a periodic signal \( x(\cdot) \), asynchronously. Assuming that \( N \) samples are processed, when \( j = 0, \ldots, N - 1 \), we can write

\[
x_j = \theta_1 + \sum_{m=1}^{P} [\theta_{2m} \sin(2\pi \lambda mj) + \theta_{2m+1} \cos(2\pi \lambda mj)]
\]

where \( P \) is the known number of harmonic components in the periodic signal, \( \theta_m \) is the signal parameters to be estimated, and \( \lambda \) is the normalized sampling rate. Since sampling is asynchronous, \( \lambda \) is irrational with probability 1. In practice, however, digital signal generators and clock sources have a finite-frequency resolution that may result in a rational value of \( \lambda \), especially if the generators are synchronized.

Under suitable...
conditions, this number can be treated as being close to an irrational number, as shown in Section III-B. Thus, the case when $\lambda$ is a rational number is also included in the procedure described in the following.

**Noise Assumptions**

The input noise is a zero-mean white Gaussian random sequence.

**Goal of This Paper**

It is shown how to achieve a fast estimation of the amplitude $\theta_m$ of each component in $x_j$ by processing the quantizer output samples $y_j$. The resulting estimator is called binary quantile-based estimator (BQBE). In addition, a procedure is proposed to validate the assumption about the noise probability distribution.

**III. Estimation Procedure**

The description of the working principle of BQBE is based on the mathematical modeling of the three major entities involved in this problem which are as follows:

1) the signal chain;
2) the process of asynchronously sampling a periodic signal with known frequency;
3) the actual parametric estimator.

**A. Modeling the Signal Chain**

The quantizer input–output characteristic can be described as

$$Q(x) = \begin{cases} 1, & x \geq T_0 \\ 0, & x < T_0 \end{cases}$$

where $T_0$ is the quantizer threshold. Thus, $Q(x)$ models the behavior of a simple comparator. It can be observed that the cascade of the nonlinear function $f(x)$ and the binary quantizer is equivalent to a new binary quantizer having a possibly different threshold. In fact

$$Q(f(x)) = \begin{cases} 1, & f(x) \geq T_0 \\ 0, & f(x) < T_0 \end{cases}$$

where $f^{-1}(\cdot)$ is the inverse function of $f(\cdot)$ and $T = f^{-1}(T_0)$. Accordingly, the resulting signal chain is shown in Fig. 1(b). Observing that without the noise source, information about the input sequence would be poor and possibly insufficient to completely identify the signal parameters [3].

**B. Modeling the Asynchronous Sampling of a Periodic Signal**

Data processing by the BQBE requires knowledge about the structure and the density of time instants generated by the irrational frequency $\lambda$ multiplied by increasing time indices in (1). In addition, it requires suitable reordering of the measured binary samples, according to their amplitudes, before processing. To address both issues, this section contains a description of the mathematical tools useful to describe and predict the behavior of the sequence of generated samples.

We assume that the sequence $x_j$ in (1) is obtained by sampling the periodic continuous time signal $x(\cdot)$, having period $T_P$ with a constant sampling period $T_S$.

Any real number $\alpha$ can be written as the sum $[\alpha] + \langle \alpha \rangle$, where $[\alpha]$ is the largest integer lower than or equal to $\alpha$, and $\langle \alpha \rangle$ is its fractional part, we can write

$$x_j = x(f/T_S) = x \left( \left\lfloor \frac{j T_S}{T_P} \right\rfloor T_P + \frac{j T_S}{T_P} T_P \right)$$

where the last equality does not change when an integer number of periods is added to its argument.

The properties of the sampled sequence depend on the properties of the fractional map

$$u_j = \langle \lambda j \rangle, \quad j = 0, \ldots, N-1$$

where $\lambda = T_S/T_P$ is assumed to be known. Two cases are of interest: when $\lambda$ is an irrational and a rational number.

1) **Irrational $\lambda$**: In this case, the map (5) produces the orbit $0, \langle \lambda \rangle, \langle 2\lambda \rangle, \ldots, \langle (N-1)\lambda \rangle$ of unique values that divides the interval $[0, 1)$ into $N$ distinct intervals, whose magnitude can take at most three possibly different values. This result is known as the three gap theorem or the three distance theorem [38]-[40]. These intervals can be evaluated by sorting the elements in the sequence (5) and by calculating the distance $\delta_n$ between neighboring values. By recalling that the Farey series of order $N$ is the sequence of increasing irreducible rationals $n/d$ such that both $n$ and $d$ are not greater than $N$ [41], the three gap theorem states that $\delta_n$ belongs to the three-element set

$$\{ \lambda d_1 - n_1, n_2 - \lambda d_2, n_1 - \lambda (d_2 - d_1) \}$$

where the rationals $n_1/d_1$ and $n_2/d_2$ are the consecutive elements of the Farey series of order $N - 1$, such that

$$\frac{n_1}{d_1} < \lambda < \frac{n_2}{d_2}.$$

Moreover, each element in the set (6) occurs $N - d_2, N - d_1$, and $d_1 + d_2 - N$ times, respectively. Since $\lambda$ is irrational, at least two out of three elements in the set (6) always occur.

2) **Rational $\lambda$**: When $\lambda$ is the ratio $D/M$ in its lowest terms, results presented in Section III-B still hold true provided that $N < M$. When $N \geq M$, values generated by (5) are no longer distinct and the only possible gap is $1/M$ [42].

3) **Sorting the Sequence**: The application of the procedure described in the following requires ordering of the samples according to their amplitudes. Clearly, given the chaotic nature of the map (5), ordering of $u_j$ requires a permutation of the indices $j$. The set of indices $p_j, j = 0, \ldots, N-1$ producing an increasingly sorted version $u_{p_j}$ of the sequence $u_j$ is obtained through the following algorithm [43]:

$$p_0 = 0$$

$$p_{j+1} - p_j = \begin{cases} d_1, & 0 \leq p_j < N - d_1 \\ d_1 - d_2, & N - d_1 \leq p_j < d_2 \\ -d_2, & d_2 \leq p_j < N. \end{cases}$$
The estimator working principle is based on the estimation of the probability with which, for a given value of $x_j$, the quantizer outputs a logical 0, that is [19],

$$p_j = P(x_j + \eta_j \leq T) = \Phi \left( \frac{T - x_j}{\sigma} \right)$$ (11)

where $\Phi(\cdot)$ is the cumulative distribution function (CDF) of a standard normal random variable. The estimation of $p_j$ in (11) requires the percentage count of the binary quantizer output, based on a possibly large number of samples, obtained under the same given constant value $x_j$. However, an incoherent sampling of a periodic signal results in time samples that neither remain constant over a varying time index nor repeat themselves, as stated in Section III-B1. To overcome this limitation, input samples are grouped by selecting close enough time indices so that corresponding signal amplitudes provided by (1) may be considered sufficiently close. Clearly, closeness among time samples does not imply proximity among signal amplitudes. A needed requirement is that the signal derivative has a locally small magnitude.

1) Partitioning the Input Samples: In practice, measured data can be partitioned in $H$ sets such that the values provided by $x_j$ in the same set are close together. This can be done, for instance, by sorting, at first, $\lambda_j$ in increasing order and by partitioning the corresponding indices in subsets $I_h$ defined as

$$I_h = \{ i \in \mathbb{H} | \epsilon_i \leq \lambda_i < (h + 1)\epsilon \} = \left\{ i \left| \left\lfloor \frac{\lambda_i}{\epsilon} \right\rfloor = h \right\} \right.$$

$$h = 0, \ldots, H - 1$$ (13)

where $\epsilon$ is a small interval length, $H = \lceil 1/\epsilon \rceil$, and because of (10) $h = \lfloor (\lambda_j)/\epsilon \rfloor$. Thus, if $h$ and $k$ are in $I_h$, $|\lambda_h - \lambda_k| < \epsilon$. Accordingly, indices in $I_h$ select amplitudes in the sequence $x_j$ that are close enough to allow estimation of $p_j$. The sorting operation can be done as described in Section III-B3.

2) Estimating the Probability of the Quantizer Outcomes: The probability $p_j$ refers to the input signal being approximately equal to $x_j$. As described in Section III-C1, indices in $I_h$ select amplitudes in the sequence $x_j$ that are close enough to allow estimation of $p_j$. Consequently, the percentage count over the indices in $I_h$ can be written as

$$\hat{p}_h = \frac{1}{|I_h|} \sum_{i \in I_h} [x_i + \eta_i \leq T], \quad h = \left\lfloor \frac{(\lambda_j)}{\epsilon} \right\rfloor$$ (14)

where $|I_h|$ is the cardinality of $I_h$, and $[\cdot]$ is the indicator function of the event $\mathcal{E}$. Since each term in the summation in (14) is a Bernoulli random variable with success probability depending on $i$, $\sum_{i \in I_h}[x_i + \eta_i \leq T]$ is a Poisson binomial random variable with expected value $\sum_{i \in I_h} p_i$, such that [44]

$$E(\hat{p}_h) = \frac{1}{|I_h|} \sum_{i \in I_h} \Phi \left( \frac{T - x_i}{\sigma} \right) \simeq \Phi \left( \frac{T - \overline{x}_h}{\sigma} \right)$$ (15)

where $E(\cdot)$ is the expectation operator, $\overline{x}_h = 1/|I_h| \sum_{i \in I_h} x_i$ and where the right-most approximation is proven in Appendix A, where also the Taylor series expansion of the error term is described. The error in (15) is in the order of $1/|I_h| \sum_{i \in I_h} \epsilon_i^2$, where $\epsilon_i = x_i - \overline{x}_h$ (Appendix A).
Thus, \( \hat{\varphi}_h \) can be considered as an estimator of the right-most term in (15), so that \( \sigma \Phi^{-1}(\hat{\varphi}_h) \) estimates \( T - \bar{x}_h \). Appendix A shows that

\[
\bar{x}_h = \theta_1 + \sum_{m=1}^{p} \frac{\theta_{2m}}{|I_h|} \sum_{i \in I_h} \sin(2\pi m \lambda_i) + \frac{\theta_{2m+1}}{|I_h|} \sum_{i \in I_h} \cos(2\pi m \lambda_i)
\]

from which the parameters embedded in \( \bar{x}_h \) can be linearly related to \( \sigma \Phi^{-1}(\hat{\varphi}_h) \) that, in turn, is obtained using measurement results.

3) Building and Solving the Model: Using matrix notation, we can write the observation model (12), shown at the top of this page. This is in the form \( Y = A\Theta \), where

\[
Y = \begin{bmatrix} \sigma \Phi^{-1}(\hat{\varphi}_0) \\ \sigma \Phi^{-1}(\hat{\varphi}_1) \\ \vdots \\ \sigma \Phi^{-1}(\hat{\varphi}_{H-1}) \end{bmatrix}, \quad \Theta = \begin{bmatrix} T - \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \\ \theta_{2p} \\ \theta_{2p+1} \end{bmatrix}
\]

and where \( Y \) is a \( H \times 1 \) vector, \( A \) is a \( H \times (2P + 1) \) matrix, and \( \Theta \) a \( (2P + 1) \times 1 \) vector. For each \( h = 0, 1, \ldots, H - 1 \), the variance of \( \hat{\varphi}_h \) can be calculated as [44]

\[
\sigma_{\hat{\varphi}_h}^2 = \frac{1}{|I_h|^2} \sum_{i \in I_h} p_i (1 - p_i).
\]

The variance of each component of \( Y \) can be approximated using a Taylor series expansion of \( \Phi^{-1}(\hat{\varphi}_h) \) about the mean value of \( \hat{\varphi}_h \). In [19], it is shown that this calculation provides

\[
\sigma_{\hat{\varphi}_h}^2 = \text{var}(\sigma \Phi^{-1}(\hat{\varphi}_h)) \simeq 2\pi \sigma^2 \frac{\hat{\varphi}_h (1 - \hat{\varphi}_h)}{|I_h|} e^{-\Phi^{-1}(\hat{\varphi}_h)^2}. \]

Finally, an estimate \( \hat{\Theta} \) of \( \Theta \) is obtained by using the weighted least squares estimator [45]

\[
\hat{\Theta} = (A^T WA)^{-1} A^T W Y
\]

where \( W \) is a \( H \times H \) diagonal matrix that contains the reciprocal of the variance of each estimator, \( 1/\sigma_{\hat{\varphi}_h}^2 \), on the principal diagonal [45], estimated according to (19). An estimate of the estimator covariance matrix is given by [45]

\[
C_{\hat{\Theta}} = \text{Cov}(\hat{\Theta}) = (A^T WA)^{-1}.
\]

D. Some Remarks

The following remarks can be made about the devised estimator.

1) A requirement is that \( H \geq (2P + 1) \), that is, the number of unknown parameters.

2) The observation model (12) shows that (20) estimates \( T - \theta_1 \). Thus, either \( \theta_1 \) or \( T \) must be known if the other parameter needs to be estimated.

3) When \( \sigma \) is unknown, both left-hand and right-hand terms in (12) can be divided by \( \sigma \); in this case, a new estimation problem can be setup if \( T \) is known and if the reciprocal of \( \sigma \) is defined as a new parameter to be estimated [19].

4) The observation model (12) assumes that \( 0 < \hat{\varphi}_h < 1 \), \( \forall h \). In practice, when \( \sigma \) is small compared to the input signal span, \( \hat{\varphi}_h \) may be 0 or 1 for several values of \( h \). The estimator still provides meaningful results provided \( H \geq (2P + 1) \). Clearly, the dimensions of the left-hand vector and of the matrix in (12) scale accordingly.

IV. ESTIMATOR PROPERTIES

In this section, at first, the estimator bias is analyzed. Then, an analysis is done to show how the BQBE can be seen as the discrete Fourier transform (DFT) of preprocessed data.

A. BQBE Bias

Because of the approximation in (15), BQBE is a biased estimator even when \( N \) is large. Using (15) and the approximation

\[
E(\Phi^{-1}(\hat{\varphi}_h)) \simeq \Phi^{-1}(E(\hat{\varphi}_h))
\]

the bias vector is given by

\[
B(\epsilon, \sigma) = (A^T WA)^{-1} A^T WP(\epsilon, \sigma)
\]

where an expression for \( P(\epsilon, \sigma) \) is shown at the bottom of this page and where it is assumed that \( H \) is the number of rows in \( A \). It will be shown in Section V that an estimate of the bias vector is needed to find a value of \( \epsilon \) that optimizes the estimator performance. To this aim, an estimate \( \hat{x}_i \) of the \( i \)th sample in the input signal is first obtained by using the estimated parameters in (9). Then, an estimate \( \hat{P}(\epsilon, \sigma) \) of \( P(\epsilon, \sigma) \) is provided by (25), by substituting each occurrence of \( x_i \) with \( \hat{x}_i \). To appreciate the behavior both of \( B(\epsilon, \sigma) \) and of its estimator with respect to \( \epsilon \), consider the graphs of \( \| B(\epsilon, \sigma) \| \) and \( \| (\hat{B}(\epsilon, \sigma)) \| \), where \( \langle \cdot \rangle \) is the mean value
operator and $\| \cdot \|$ is the Euclidean norm. Both norms, obtained by Monte Carlo simulations, are graphed in Fig. 3 as a function of $\epsilon$, for $P = 1.7$ and $N = 5 \cdot 10^3$. To guarantee that $H$ is always sufficiently large to allow estimation of all $2P + 1$ parameters, $\epsilon$ must be upper bounded by $\epsilon_B(P) = 1/2P + 1$. This is the value that normalizes the $x$-axis shown in Fig. 3.

Observe that, for a given value of $\epsilon$ and $N$, the bias norm increases with $P$. Conversely, for given values of $\epsilon$ and $P$, increasing $N$ does not affect $\mathbf{B}(\epsilon, \sigma)$ significantly. Also, observe that the norm of the estimated vector provides a reasonable approximation of $\| \mathbf{B}(\epsilon, \sigma) \|$ for most values of $\epsilon$.

### B. Interpretation of BQBE as a DFT Processor

To better appreciate how the BQBE process data, in this section, the properties of the unweighted estimator will be analyzed by assuming $N \to \infty$. This estimator has the following expression:

$$\mathbf{\Theta} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Y}. \quad (26)$$

With respect to (26), (20) weighs the results by also including the effects of the noise covariance. The simple interpretation of the estimation properties of (26) will, however, provide some insights also into the properties of (20). Because of Kolmogorov’s strong law of large numbers, when $N \to \infty$, $\hat{\mathbf{p}}_h$ in (14) converges almost surely to $E(\mathbf{p}_h)$ in (15). Because of Weyl’s equidistribution theorem, $\lambda_j$ in (10) becomes an equidistributed sequence modulo 1 when $N \to \infty$. Thus, $\mathbf{T}_h$ contains an infinite number of samples, equidistributed in $[h\epsilon, (h+1)\epsilon]$ and (15) becomes a Riemann sum such that [46]

$$p_{h, \infty} = \lim_{N \to \infty} \frac{1}{|I_h|} \sum_{i \in I_h} \Phi \left( \frac{T - x_i}{\sigma} \right)$$

$$= \lim_{N \to \infty} \frac{1}{|I_h|} \sum_{i \in I_h} \Phi \left( \frac{T}{\sigma} - \frac{x_i}{\sigma} \right)$$

$$- \sum_{m=1}^{P} \left[ \theta_{2m} \sin \left( 2\pi m \lambda_j + \theta_{2m+1} \cos \left( 2\pi m \lambda_j \right) \right) \right]$$

$$= \frac{1}{\epsilon} \int_{h \epsilon}^{(h+1)\epsilon} \Phi \left( \frac{T}{\sigma} - \frac{x}{\sigma} \right)$$

$$- \sum_{m=1}^{P} \left[ \theta_{2m} \sin \left( 2\pi m x + \theta_{2m+1} \cos \left( 2\pi m x \right) \right) \right] dx$$

$$= \int_{0}^{1} \Phi \left( \frac{T}{\sigma} - \frac{x}{\sigma} - \sum_{m=1}^{P} \left[ \theta_{2m} \sin \left( 2\pi m (x \epsilon + h) \right) \right. \right.$$

$$+ \theta_{2m+1} \cos \left( 2\pi m (x \epsilon + h) \right) \right) dx. \quad (27)$$

Similarly, when $m = 1, \ldots, P$ and $h = 0, \ldots, H - 1$, entries in $\mathbf{A}$ become

$$a_{h,m} = \lim_{N \to \infty} \left( -\frac{1}{|I_h|} \sum_{i \in I_h} \sin \left( 2\pi m \lambda_j \right) \right)$$

$$= \frac{1}{\epsilon} \int_{h \epsilon}^{(h+1)\epsilon} \sin \left( 2\pi m x \right) dx$$

$$= \frac{2\pi m \epsilon}{\pi} \left[ \cos \left( 2\pi m (h + 1) \epsilon \right) - \cos \left( 2\pi m h \right) \right]$$

$$= -\sin \left( m \epsilon \right) \sin \left( \pi m (2h + 1) \epsilon \right) \quad (28)$$

where $\sin \left( x \right) = \lim_{\epsilon \to 0} \sin \left( \pi x \epsilon / \pi \epsilon \right)$ and

\[
\begin{bmatrix}
\sigma \Phi^{-1}(\mathbf{p}_{0,\infty}) \\
\sigma \Phi^{-1}(\mathbf{p}_{1,\infty}) \\
\sigma \Phi^{-1}(\mathbf{p}_{H-1,\infty}) \\
\end{bmatrix}
\]

\[=
\begin{bmatrix}
1 & \frac{1}{2\pi \epsilon} \cos(2\pi \epsilon) & -\frac{1}{2\pi \epsilon} \sin(2\pi \epsilon) & \cdots & \frac{1}{2\pi \epsilon} \cos(2\pi P \epsilon) - 1 \\
1 & \frac{1}{2\pi \epsilon} \cos(2\pi \epsilon) - \cos(2\pi \epsilon) & -\frac{1}{2\pi \epsilon} \sin(2\pi \epsilon) - \sin(2\pi \epsilon) & \cdots & \frac{1}{2\pi \epsilon} \cos(2\pi P \epsilon) - \cos(2\pi P \epsilon) - 1 \\
1 & \frac{1}{2\pi \epsilon} \cos(2\pi H \epsilon) - \cos(2\pi (H - 1) \epsilon) & -\frac{1}{2\pi \epsilon} \sin(2\pi H \epsilon) - \sin(2\pi (H - 1) \epsilon) & \cdots & \frac{1}{2\pi \epsilon} \cos(2\pi H P \epsilon) - \cos(2\pi (H - 1) P \epsilon) - 1 \\
\end{bmatrix}
\times
\begin{bmatrix}
T - \theta_1 \\
\theta_2 \\
\vdots \\
\theta_P \\
\theta_{2P+1} \\
\end{bmatrix}
\]

\[
\mathbf{P}(\epsilon, \sigma) \approx
\begin{bmatrix}
\sigma \Phi^{-1} \left( \frac{1}{|I_0|} \sum_{i \in I_0} \Phi \left( \frac{T - x_i}{\sigma} \right) \right) - T + \frac{1}{|I_0|} \sum_{i \in I_0} x_i \\
\sigma \Phi^{-1} \left( \frac{1}{|I_1|} \sum_{i \in I_1} \Phi \left( \frac{T - x_i}{\sigma} \right) \right) - T + \frac{1}{|I_1|} \sum_{i \in I_1} x_i \\
\vdots \\
\sigma \Phi^{-1} \left( \frac{1}{|I_{H-1}|} \sum_{i \in I_{H-1}} \Phi \left( \frac{T - x_i}{\sigma} \right) \right) - T + \frac{1}{|I_{H-1}|} \sum_{i \in I_{H-1}} x_i \\
\end{bmatrix}
\quad (25)
\]
and where $P$ and $\epsilon$, norms of corresponding vectors have similar values.

\[ b_{h,m} = \lim_{N \to \infty} \left( -\frac{1}{|I_h|} \sum_{i \in I_h} \cos(2\pi m \lambda_i) \right) \]

\[ = -\frac{1}{\epsilon} \int_{(-h+1)\epsilon}^{(h+1)\epsilon} \cos(2\pi mx) dx \]

\[ = -\frac{1}{2\pi m \epsilon} [\sin(2\pi m(h+1)\epsilon) - \sin(2\pi mh\epsilon)] \]

\[ = -\sin(c(m\epsilon)) \cos(p m(2h+1)\epsilon). \] \tag{29}

Thus, when $N \to \infty$, the measurement model in (12) becomes as shown in (22), shown at the bottom of the previous page.

The evaluation of the pseudoinverse of $A$ in (26) requires the calculation of the following summations:

\[ S_1(\epsilon, m, n) = \sum_{h=0}^{H-1} \cos(\pi m (2h+1)\epsilon) \sin(\pi n(2h+1)\epsilon) \]

\[ S_2(\epsilon, m, n) = \sum_{h=0}^{H-1} \cos(\pi m (2h+1)\epsilon) \cos(\pi n(2h+1)\epsilon) \]

\[ S_3(\epsilon, m, n) = \sum_{h=0}^{H-1} \sin(\pi m (2h+1)\epsilon) \sin(\pi n(2h+1)\epsilon) \] \tag{30}

where $m, n$ are two integers in $1, \ldots, P$. In Appendix B, it is shown that, when $\epsilon \to 0^+$, all summations approximately vanish when $m \neq n$ and $n = m$

\[ S_1(\epsilon, m, n) \simeq 0, \quad S_2(\epsilon, m, m) \simeq S_3(\epsilon, m, m) \simeq \frac{H}{2} \] \tag{31}

as expected on the basis of the orthogonality of the sine and cosine functions. Under these approximations, that is, $N \to \infty$ and $\epsilon \to 0^+$

\[ A^T A \simeq \frac{H}{2} \begin{bmatrix} 2 & 0 & \cdots & \cdots & 0 \\ 0 & \sin^2(\epsilon) & 0 & \cdots & 0 \\ 0 & 0 & \sin^2(\epsilon) & \cdots & 0 \\ 0 & 0 & 0 & \sin^2(2\epsilon) & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \sin^2(P\epsilon) \end{bmatrix} \] \tag{32}

Consequently, when $N \to \infty$, $(A^T A)^{-1} A^T$ in (26) becomes approximately equal to (33), shown at the top of the next page. Observe that if $\epsilon = 1/H$, apart from a phase and a scaling factor, (33) is a DFT matrix. In fact, by defining

\[ \chi_m = \sum_{h=0}^{H-1} x_h e^{-j m \frac{2\pi h}{H}}, \quad m = 0, \ldots, H - 1 \] \tag{34}

as the $m$th element of the DFT of the sequence $x_h$, it can be recognized that when $m = 1, \ldots, P$

\[ \hat{\theta}_1 \simeq \frac{\chi_0}{H} \]

\[ \hat{\theta}_{2m} \simeq \frac{2}{H \sin(c(m\epsilon))} \sum_{j=1}^{H/2} \left\{ e^{-j \frac{2\pi j}{H} m \chi_m} \right\} \]

\[ \hat{\theta}_{2m+1} \simeq \frac{-2}{H \sin(c(m\epsilon))} \sum_{j=1}^{H/2} \left\{ e^{-j \frac{2\pi j}{H} m \chi_m} \right\} \tag{35} \]

where $\chi_m$ is the $m$th element of the DFT of $Y$, in (26). Thus, apart from an amplitude and a phase factor, the asymptotic behavior of (26) is based on the DFT of the statistics $\sigma \Phi^{-1}(\hat{p}_{h,\infty})$ that is on data preprocessed by the CDF inverse operation decoding the quantizer input signal. Since the correcting factor removes the product by a $\sin c(\cdot)$ function in the frequency domain, we can conclude that the equivalent amplitude domain effect is the convolution with a boxcar function. This is coherent with the impact of 1-bit quantization on input signals which is often modeled as the summation of independent noise, having uniform probability density function. A similar behavior was highlighted in [47] when modeling the effect of jitter in waveform digitizers.

V. SELECTING THE ESTIMATOR’ S PARAMETERS

Being based on matrix inversion, the practical implementation of BQBE is rather straightforward. It requires, however, the choice of $\epsilon$ and, if allowed by the adopted system setup, the selection of a suitable value of $\sigma$. In fact, depending on the application requirements, the noise magnitude may not be user selectable, for example, when the noise is entirely due to selfdithering sources inside the used circuits [48]. When this is the case, $\sigma$ can only be estimated beforehand. Alternatively, the BQBE can be reparametrized to account for the joint estimation of noise and signal parameters.

A. Selecting $\epsilon$

The user always needs to choose $\epsilon$. In this section, it is shown how to select it by using two methods. Consider that the optimal choice is a tradeoff choice. In fact, large values of $\epsilon$ allow for the inclusion of a large number of signal samples in each interval $I_h$ and thus reduce the estimation variance but contribute to a larger value of the estimation bias and a reduced number $H = \lceil 1/\epsilon \rceil$ of rows in the observation matrix. On the contrary, when $\epsilon$ is a small value, the bias is reduced, at the expense of an increase in the estimation variance, possibly counterbalanced by the increase of $H$. 

as a function of the minimization problem, consider the behavior of the minimization procedure. To appreciate the characteristics of the solution to (36) can be found by a constrained numerical processing aimed at the computation of the estimate variance and squared bias, 

\[(A^T A)^{-1} A^T \simeq -\frac{2}{H}\]

\[
\begin{bmatrix}
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\
\sin(\pi \epsilon) & \sin(3\pi \epsilon) & \sin(5\pi \epsilon) & \cdots & \sin(\pi (2H-1)\epsilon) \\
\sin(\epsilon) & \sin(3\epsilon) & \sin(5\epsilon) & \cdots & \sin(\epsilon) \\
\sin(2\pi \epsilon) & \sin(6\pi \epsilon) & \sin(10\pi \epsilon) & \cdots & \sin(2\pi (2H-1)\epsilon) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\cos(\pi P\epsilon) & \cos(3\pi P\epsilon) & \cos(5\pi P\epsilon) & \cdots & \cos(P\pi (2H-1)\epsilon) \\
\sin(P\epsilon) & \sin(P\epsilon) & \sin(P\epsilon) & \cdots & \sin(P\epsilon)
\end{bmatrix} \tag{33}
\]

1) Selection Based on Minimum Mean Square Error: To select the optimal value for \(\epsilon\), an expression for the mean square estimation error (MSE) is first derived. By recalling that, in the scalar parameter case, the MSE is equivalent to the summation of the estimator variance and squared bias, an equivalent MSE vector can be defined in the vector parameter case. Accordingly, an estimate of the QBE bias can be obtained through (25). Moreover, by defining \(\hat{V}(\epsilon, \sigma)\) as the vector that contains the diagonal elements in \(C_\hat{\theta}\), the estimated mean square error vector associated with the vector estimator \(\hat{\theta}\) is given by \(\tilde{M}(\epsilon, \sigma) = \hat{V}(\epsilon, \sigma) + \hat{B}(\epsilon, \sigma) \circ \hat{B}(\epsilon, \sigma)\), where “\(\circ\)” is the Hadamard product operator that provides the entrywise multiplication of the elements in \(\hat{B}(\epsilon, \sigma)\). Thus, the optimal value \(\epsilon_{\text{opt}}\) of \(\epsilon\) can be found by solving the following constrained minimization problem, where \(\sigma\) is given by

\[
\min_{\epsilon} \|\tilde{M}(\epsilon, \sigma)\| \\
\text{s.t. } 0 < \epsilon < \epsilon_B(P) = \frac{1}{2P + 1}. \tag{36}
\]

The solution to (36) can be found by a constrained numerical minimization procedure. To appreciate the characteristics of the minimization procedure, consider the behavior of \(\|\tilde{M}(\epsilon, \sigma)\|\) as a function of \(\epsilon\), shown in Fig. 4 for several values of \(P\) and \(N\), obtained through Monte Carlo simulations. It can be recognized that by increasing the number of samples, this function has a smooth behavior and a single minimum resulting in simple numerical processing aimed at the computation of the minimum value.

2) Selection Based on Same Variance Factors: The value of \(\epsilon\) influences the variances with which each component in the observation vector and each parameter is estimated. In fact, (19) shows that

\[
\text{var}(\sigma \Phi^{-1}(\hat{\theta}_n)) \propto \frac{1}{|z_n|} \simeq \frac{1}{N\epsilon}. \tag{37}
\]

At the same time, (21) and (32) show that each estimator component has a variance approximately proportional to \(1/H \simeq \epsilon\). By equating both factors, the information carried by the \(N\) samples is distributed between two components. This occurs when \(\epsilon = 1/\sqrt{N}\). When this value is larger than \(\epsilon_B(P)\), this latter value is selected.

B. Selecting \(\sigma\)

Although \(\sigma\) may not always be user selectable, its value affects the QBE performance. The Fisher information matrix associated with the estimation problem described in this paper was derived in [30]

\[
I = \sum_{n=0}^{N-1} J_n^T \frac{e^{-\Delta_n^2}}{2\pi \sigma^2 \Phi(\Delta_n)[1 - \Phi(\Delta_n)]} J_n \tag{38}
\]

where \(J_n\) denotes the Jacobian of \(x_n\) and \(\Delta_n = (T - x_n)/\sigma\). Note that the magnitude of \(\Delta_n\) must not exceed 3–4 for the corresponding sample \(x_n\) to add significant information about the searched parameter. When this does not occur, the probability of \(x_n + \eta\) to cross the threshold will vanish the information associated with \(x_n\). In order to compare the behavior of the QBE and the MLE with respect to \(\sigma\), Monte Carlo simulations were carried out under the assumption of \(P = 1\), \(N = 10^4\), \(T = 0\), \(\theta_1 = 0.03\), \(\theta_2 = 0.07\), and \(\theta_3 = 0.05\). The normalized Euclidean norm of the estimator variance was calculated using \(R = 10^3\) records and plotted in Fig. 5 as a function of \(\sigma\), in the case of both the QBE and the MLE. The behavior of the estimator variances largely coincides for the two estimation procedures. Fig. 5 also shows the graph of the normalized Euclidean norm of the CRLB, obtained using (38). The lack of superposition among the three curves for small values of \(\sigma\) is due to the contribution of the estimator bias.

Observe that, when \(\sigma\) is a small value and \(\epsilon\) is given, the number of rows in \(A\) will be much lower than \(H = \lfloor 1/\epsilon \rfloor\). This will result in an overall increase in the estimator bias. On the contrary, by allowing the noise to extend overall input signal span, the triggering probability will never become negligible for any of the considered input signal samples. In fact, when \(\sigma\) increases, the number of rows in \(A\) will grow, for a given value of \(\sigma\) and the maximum value \(H\). This will occur when the magnitude of \(\Delta_n\) does not exceed 3–4 for every \(n\). Beyond this value, the number of rows in \(A\) will not grow, so that any increase in \(\sigma\) will increase the variance of the estimator. Fig. 6(a) shows the Euclidean norm of the bias vector as a function of \(\sigma\), when \(P = 1, 3, 4, 7, 15, 20\) and assuming that \(N = 10^4\), \(\epsilon = 0.01\), and \(\sigma = 0.1\). In Fig. 6(a), overall decrease in the bias is noticed when \(\sigma\) increases. Also,
observe that increasing \( N \) does not significantly modify the behavior of the graphs shown in Fig. 6(a). However, when \( N \) increases, the optimal value of \( \epsilon \) will decrease resulting in an overall decrease in the estimator bias.

The optimal value of \( \sigma \) can be determined by looking at the behavior of the Euclidean norm of the root-mean-square-error, graphed in Fig. 6(b) obtained using the same simulation parameters. This figure shows that the normalization of \( \sigma \) by the input signal span will result in a minimum MSE that is almost insensitive to the number \( P \) of harmonics. Thus, as a practical rule, \( 6/7\sigma \) should be comparable to the signal voltage span to exploit the information associated with every signal sample.

### VI. Estimating the Noise CDF After Binary Quantization

This section shows how to validate the assumption about the noise Gaussianity. The noise CDF can be estimated using a point estimator based on (11). In fact

\[
p_j = P(\eta_j \leq T - x_j) = F_\eta(T - x_j), \quad j = 0, \ldots, N - 1.
\]

A very rough estimate of \( p_j \) is \( y_j \), that is, the binary-valued quantizer output. Thus,

\[
\hat{p}_j = \tilde{F}_\eta(\tilde{T} - \hat{x}_j) = y_j
\]

where \( \hat{x}_j \) is obtained from (9), in which all occurrences of the unknown parameters are substituted by the corresponding estimates, obtained through (20). By sorting \( \tilde{F}_\eta(\cdot) \) according to increasing values of its argument \( \tilde{T} - \hat{x}_j \), a 1-bit coded version \( \tilde{F}_{1M}(x_j) \) of the noise CDF results.

In order to obtain a higher resolution estimate of the noise CDF, \( \tilde{F}_{1M}(x_j) \) needs to be interpolated. This can be done by:

1. a parametric approach based on the mathematical expression of the Gaussian CDF;
2. a nonparametric approach, based on zero-phase numerical filtering [49].

After interpolation, a higher resolution estimate \( \hat{F}_\eta(\hat{x}_j) \) of \( F_\eta(x_j) \) results. Observe that estimates are obtained at values of their argument that are not uniformly spaced. Thus, \( \tilde{F}_\eta(\cdot) \) is the 1-bit unsorted CDF estimator, \( \tilde{F}_{1M}(\cdot) \) is its 1-bit sorted version, and \( \hat{F}_\eta(\cdot) \) is the higher resolution estimator obtained after interpolation.

### VII. Simulations

Monte Carlo simulations were done to validate the BQBE and to compare its properties with those of the MLE, under several different parameters’ values. Solution in the MLE case is assured by the log-likelihood function being a convex
function, under the made assumptions [30]. Assuming \( m = 1, \ldots, P \), the dc component \( \theta_1 \) and the amplitudes of the \( P \) harmonics were chosen according to

\[
\begin{align*}
\theta_1 &= 0.05 + U_1 \\
\theta_{2m} &= A_m \cos(\phi_m) \\
\phi_{2m+1} &= A_m \sin(\phi_m) \\
A_m &= 0.25 + U_{2m} \\
\phi_m &= U_{3m}
\end{align*}
\]

(40)

where \( U_1 \) and \( U_{2m} \) are the outcomes of uniformly distributed random variables, respectively, in the intervals \((-0.025, 0.025), (0, 0.05)\), whereas \( U_{3m} \) is an outcome of a uniformly distributed random variable in the interval \([0, 2\pi)\). Simulations were performed using \( R = 30 \) records of data. Each time, amplitudes and phases were generated according to (40). All simulations assumed a normalized frequency \( \lambda = \pi/(1250\sqrt{2}) \) and \( T = 0 \). Results, displayed in Table I, were obtained by processing records of \( N \) samples, where \( N = 5 \cdot 10^4, 1 \cdot 10^5, 5 \cdot 10^5, 7 \cdot 10^5 \), with \( P = 10, 20, 50, 80 \) and \( \sigma = 0.7, 1.0, 2.0, 2.4 \). Also, Table I reports the mean of the normalized Euclidean norm of the error vectors when using both the BQBE with \( \epsilon \) chosen as described in subsection V-A2 and the MLE to estimate

\[
A = \begin{bmatrix} \theta_1 & A_1 & \cdots & A_P \\ A_1 \end{bmatrix}, \quad \varphi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_P \end{bmatrix}, \quad S = A \varphi
\]

(41)

and the mean processing times, as measured on a computer having a 2.5-GHz dual core i7 processor. Processing of phase errors included the wrapping operations to be applied when calculating differences in phase angles, as suggested in [50].

Solutions, in the MLE case, were obtained using standard numerical procedures for unconstrained optimization. The function \textit{fminunc} (MATLAB R2017b) was used to perform a quasi-Newton numerical maximization. Each time the MLE was initialized with a dc value equal to 0, with \( \theta_{2m} \) and \( \phi_{2m+1} \) in (40) as uniform random variables in the interval \((-0.5, 0.5)\). Derivatives of the likelihood function were supplied to the numerical algorithm and vectorized operators were used, whenever possible, to maximize its time performance. The same data used to obtained results shown in Table I were used to calculate data shown in Table II, where the performance of the MLE, when initialized by the BQBE, is shown.

### A. Results and Discussion

Data show that both estimators are characterized by similar accuracy performance. The BQBE outperforms the MLE in terms of mean processing time, especially when the problem complexity increases, and it is about 30 faster than the MLE in the most complex case (\( P = 80 \)) reported in Table I. Also, observe that normalized error norm related to the entire error vector \( S \) is always the lowest when using the MLE.

While careful optimization of the numerical maximization procedure might result in improvements in the MLE processing times, the iterative nature of this approach makes it less appealing in case of problems with a large number of parameters, for example, \( P = 80 \). The two right-most columns in Table I report the mean number of function evaluations and algorithm iterations. Moreover, if the MLE is used regardless of its time performance, the BQBE can still provide accurate initial estimates, very close to the optimal ones. In this case,
the MLE can be used as a postprocessing step starting from BQBE estimates. In fact, by comparing the corresponding data shown in Tables I and II, it can be appreciated that the initialization by the BQBE still reduces by a factor of about equal to 4–5 the processing time, in large-scale problems.

VIII. EXPERIMENTS

Using off-the-shelf components and measurement instruments, experiments were performed to further validate the BQBE, as described in the following.

A. Experimental Setup

The scheme of the realized measurement setup is shown in Fig. 7. The signal measured by a commercial 1 GSa/s 8-bit digital storage oscilloscope (DSO) was generated through the mixing of a two-tone signal generated by a Stanford Research Systems DS360 signal generator and by a commercial Gaussian noise source. The tone frequencies were set to 1.839.07 and 3.678.14 Hz. The nominal values of their maximum amplitudes were set to 0.4 V in both cases. The DSO recorded an 8-bit sequence of $N = 10^6$ samples at a rate of 100 MSa/s, in ac mode, to filter the contribution of dc values in the input signal. The recorded sequence was then requantized to provide the 1-bit stream of data and was processed by the BQBE. As a consequence, (1) results in $P = 2$, $\theta_1 = 0$ and

$$\lambda = \frac{D}{M} = \frac{183907}{10^8}$$

where the fraction $D/M$ is already in an irreducible form. Thus, the condition described in Section III-B2 applies with $M = 10^8 > N = 10^6$ and distinct time indices are generated by (5), when $j = 0, \ldots, N - 1$. A noise-only record of data was first collected by switching off the two-tone generator to allow the estimation of the noise standard deviation, as $\hat{\sigma} = 0.2359 \cdots$ V. Results presented in Section VIII-B were obtained by selecting $\epsilon$ through the solution of (36).

B. Experimental Results

Obtained experimental results are shown in Fig. 8. The 8-bit data sequence collected from the DSO was first requantized to provide the 1-bit sequence processed by the BQBE. Automatic selection of $\epsilon$ provided by the method described in Section V-A1, resulted in $\epsilon = 0.0031$. The four estimated signal components were used to reconstruct the input signal, which is shown in white in Fig. 8(a) and (b). In Fig. 8(a), the 8-bit noisy version of the recorded signal is also shown, whereas in Fig. 8(b), the same 8-bit signal was obtained by first turning off the noise generator. This sequence was collected for validation purposes. The graph in Fig. 8(c) shows the difference between the estimated signal, based on 1-bit data not shown in Fig. 8(a), and the measured sequence shown in Fig. 8(b). Finally, the estimated noise CDF is graphed in Fig. 8(d), along with the superimposed interpolated
Gaussian CDF. The error sequence in Fig. 8(c) highlights the presence of some residual bias when the estimated input signal is near the local maxima crossing the zero line. This might be explained by limitations both in the BQBE and in circuital changes when switching off the noise generator, to collect the data shown in Fig. 8(b).

In fact, when the range of the input signal is much larger than that shown in Fig. 8, the noise amplitude span might not be sufficiently large to toggle the quantizer output. Accordingly, these samples will contribute with a reduced amount of the Fisher information. As an example, consider the same signal as in Fig. 8, but having larger amplitudes of both harmonics. In this case, the measured noise standard deviation is \( \hat{\sigma} = 0.2169 \ldots \) V, and the BQBE provides the error sequence as shown in Fig. 9 with a selected value of \( \epsilon = 0.0111 \). The estimation error is shown in Fig. 9(b).

It highlights a larger bias in the estimation of the signal extreme values. This is not surprising because the CRLB in the estimation of noisy dc amplitudes using binary information increases significantly with the increasing distance between the dc value and the quantizer threshold, for a given amount of noise [19]. Thus, if the distance between the maximum signal amplitude in (9) and the quantizer threshold increases, we may expect worse results for a given amount of noise, as the probability that the noise will toggle the binary quantizer, will decrease accordingly. Improvements in the performance of the BQBE or in the performance of any other estimator in this case would require a larger amount of processed samples or a larger noise span. Finally, Fig. 8(d) shows the noise CDF estimated as described in Section VI, both by using the zero-phase filter based on 1000 taps (discontinuous line) and through the Gaussian parametric interpolation (smoothed line). All results confirm the validity of the proposed approach.

**IX. CONCLUSION**

We proved in this paper that it is possible to estimate the parameters of an asynchronously and incoherently sampled periodic signal with known frequency, based on its binary quantized version. This is possible because of the information encoding nature of the additive noise affecting the quantizer input. Being based on matrix inversion, the described algorithm is not computationally intensive and it is easy to code on simple microprocessor-based platforms. Presented simulations and experimental results were used to validate the properties of the BQBE.

Given the ease both in generating wideband Gaussian noise and in performing binary quantization using a comparator, the BQBE can easily be applied even at high frequencies, when accurately measuring the parameters of periodic signals becomes increasingly more difficult.

**APPENDIX A**

**APPROXIMATING THE PROBABILITY (14)**

At first, let us highlight the dependance of \( \overline{x}_h \) on the parameters in \( \Theta \)

\[
\overline{x}_h = \frac{1}{| I_h |} \sum_{i \in I_h} x_i
\]

\[
= \theta_1 + \frac{1}{| I_h |} \sum_{i \in I_h} \sum_{m=1}^{P} \left[ \theta_{2m} \sin(2\pi m \lambda_i) + \theta_{2m+1} \cos(2\pi m \lambda_i) \right]
\]

(A.1)

from which (16) results. Moreover, a first-order Taylor series expansion of \( \Phi \left( \frac{T-x_i}{\sigma} \right) \) about \( \overline{x}_h \) provides

\[
\Phi \left( \frac{T-x_i}{\sigma} \right) \approx \Phi \left( \frac{T-\overline{x}_h}{\sigma} \right) - \frac{1}{\sigma} f_\theta \left( \frac{T-\overline{x}_h}{\sigma} \right) \epsilon_i, \quad \epsilon_i \ll \sigma
\]

(A.2)

where \( f_\theta(\cdot) \) is the probability density function of a standard normal random variable and \( \epsilon_i = x_i - \overline{x}_h \). Thus,

\[
E(\hat{p}_h) = \frac{1}{| I_h |} \sum_{i \in I_h} \Phi \left( \frac{T-x_i}{\sigma} \right) \approx \frac{1}{| I_h |} \sum_{i \in I_h} \Phi \left( \frac{T-\overline{x}_h}{\sigma} \right)
\]

(A.3)

By observing that

\[
\sum_{i \in I_h} \epsilon_i = \sum_{i \in I_h} x_i - | I_h | \overline{x}_h = 0
\]

(A.4)

it follows that the error contribution by the first-order terms is zero and that the approximation holds up to the averaged summation of \( \epsilon_i^2 \). This approximation can be further refined by recalling that the \( n \)th derivative of \( f_\theta(\cdot) \) and \( f_\theta^{(n)}(\cdot) \) is given by [51]

\[
f_\theta^{(n)}(x) = (-1)^n \text{He}(n,x) f_\theta(x),
\]

(A.5)
where \( \text{He}(n, x) \) is the \( n \)th Hermite polynomial. Thus, we have
\[
E(\hat{p}_h) = \frac{1}{|I_h|} \sum_{i \in I_h} \Phi \left( \frac{T-x_i}{\sigma} \right) = \frac{1}{|I_h|} \sum_{i \in I_h} \Phi \left( \frac{T-x_{ih}}{\sigma} \right) - \epsilon
\]  
(A.6)
where the development of the error using a Taylor series expansion provides
\[
e = \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(n+1)!|I_h|!} \right] \text{He} \left( n, \frac{T-x_{ih}}{\sigma} \right) \times f_{\phi} \left( \frac{T-x_{ih}}{\sigma} \right) \sum_{i \in I_h} \epsilon_i^{n+1} \right].
\]  
(A.7)

**APPENDIX B**

**APPROXIMATING THE SUMMATIONS (30)**

For given values of \( \omega \) and \( \phi \), [52], [53], and [45] provide
\[
\sum_{h=1}^{H} \cos(\omega h + \phi) = \begin{cases} 
\frac{H \cos(\phi)}{\sin\left(\frac{H \omega}{2}\right)} \cos\left(\frac{H+1}{2} \omega \phi + \phi\right) & \omega = 0 \\
\frac{\sin\left(\frac{H \omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \cos\left(\frac{H}{2} \omega \phi + \phi\right) & \omega \neq 0.
\end{cases}
\]  
(B.8)
By using (B.8), the Werner formulas, and by rearranging terms, when \( \omega \neq 0 \) and \( \epsilon > 0 \), we have
\[
S_1(\epsilon, m, n) = \sum_{h=0}^{H-1} \cos(\pi m(2h+1)\epsilon) \sin(\pi n(2h+1)\epsilon)
\]  
\[
= \frac{\sin^2 \left( \pi \epsilon \left( \frac{1}{2} \right) (m+n) \right)}{2 \sin(\pi \epsilon (m+n))} - \frac{\sin^2 \left( \pi \epsilon \left( \frac{1}{2} \right) (m-n) \right)}{2 \sin(\pi \epsilon (m-n))}.
\]  
(B.9)
By recalling that \( x = \langle x \rangle + [x] \), where \( \langle \cdot \rangle \) is the fractional part operator, it follows that
\[
g(\epsilon) = \epsilon \left( \frac{1}{\epsilon} \right) = \epsilon M, \quad \frac{1}{M+1} < \epsilon \leq \frac{1}{M}
\]  
(B.10)
where \( M \) is a positive integer value. Thus, when \( 0 < \epsilon < 1 \), \( g(\epsilon) \) is a piecewise linear function such that \( 0.5 < g(\epsilon) < 1 \) and \( g(\epsilon) = 1 \) when \( \epsilon = 1/M \). Moreover
\[
\lim_{\epsilon \to 0^+} g(\epsilon) = \lim_{\epsilon \to 0^+} (1 - \epsilon \langle \epsilon \rangle) = 1
\]  
(B.11)
since the fractional part takes values in the interval \([0, 1)\). Moreover, by applying the definition of fractional part
\[
\sin(\pi \epsilon \left( \frac{1}{\epsilon} \right) (m \pm n)) = -\cos(\pi (m \pm n)) \sin(\pi \epsilon \left( \frac{1}{\epsilon} \right) (m \pm n))
\]  
(B.12)
so that because of (B.11), when \( \epsilon \to 0 \)
\[
\frac{\sin^2 \left( \pi \epsilon \left( \frac{1}{\epsilon} \right) (m \pm n) \right)}{2 \sin(\pi \epsilon (m \pm n))} = 0
\]  
(B.13)
and
\[
\lim_{\epsilon \to 0^+} S_1(\epsilon, m, n) = 0.
\]  
(B.14)
When \( m = n \)
\[
S_1(\epsilon, m, m) = \sum_{h=0}^{H-1} \sin(2\pi (2h+1)\epsilon m) = \frac{1}{2} \sum_{h=0}^{H-1} \sin(2\pi (2h+1)\epsilon m)
\]  
(B.15)
and again
\[
\lim_{\epsilon \to 0^+} S_1(\epsilon, m, m) = 0.
\]  
(B.16)
Similarly, when \( m \neq n \) and \( \epsilon > 0 \)
\[
S_2(\epsilon, m, n) = \sum_{h=0}^{H-1} \cos(\pi m(2h+1)\epsilon) \cos(\pi n(2h+1)\epsilon)
\]  
\[
= \frac{\cos(\pi \epsilon \left( \frac{1}{2} \right) (m+n)) \cos(\pi \epsilon \left( \frac{1}{2} \right) (m+n))}{2 \sin(\pi \epsilon (m+n))} + \frac{\cos(\pi \epsilon \left( \frac{1}{2} \right) (m-n)) \cos(\pi \epsilon \left( \frac{1}{2} \right) (m-n))}{2 \sin(\pi \epsilon (m-n))}.
\]  
(B.17)
Because of (B.12)
\[
\sin(\pi \epsilon \left( \frac{1}{2} \right) (m \pm n)) = \frac{-\cos(\pi (m \pm n)) \cos(\pi \epsilon \left( \frac{1}{2} \right) (m \pm n))}{2 \sin(\pi \epsilon (m \pm n))}
\]  
(B.18)
so that when \( \epsilon \to 0 \), \( S_2(\epsilon, m, n) \) becomes about equal to \(-1/2/(1/\epsilon)\). Thus, the limit of \( S_2(\epsilon, m, n) \) when \( \epsilon \to 0^+ \) does not exist, but when \( \epsilon = 1/M \), with \( M \) as a large positive integer, \( S_2(\epsilon, m, n) = 0 \). Moreover, when \( n = m \)
\[
S_2(\epsilon, m, m) = \sum_{h=0}^{H-1} \cos^2(\pi m(2h+1)\epsilon)
\]  
\[
= \sum_{h=1}^{H} \frac{1}{2} + \frac{1}{2} \cos(2\pi m(2h-1)\epsilon)
\]  
(B.19)
Thus, when \( \epsilon > 0 \)
\[
S_2(\epsilon, m, m) = \frac{H}{2} + \frac{\sin(2\pi m \epsilon \left( \frac{1}{2} \right)) \cos(2\pi m \epsilon \left( \frac{1}{2} \right))}{2 \sin(2\pi m \epsilon)}
\]  
(B.20)
and because of (B.12), when \( \epsilon \to 0^+ \)
\[
S_2(\epsilon, m, n) \simeq \frac{H}{2} - \frac{1}{2} \left( \frac{1}{\epsilon} \right) = \frac{1}{2} \frac{1}{\epsilon} \quad m = n
\]  
(B.21)
Finally, when \( m \neq n \) and \( \epsilon > 0 \)
\[
S_3(\epsilon, m, n) = \sum_{h=0}^{H} \sin(\pi m(2h+1)\epsilon) \sin(\pi n(2h+1)\epsilon)
\]  
\[
= \frac{\sin(\pi \epsilon \left( \frac{1}{\epsilon} \right) (m-n)) \cos(\pi \epsilon \left( \frac{1}{\epsilon} \right) (m-n))}{2 \sin(\pi \epsilon (m-n))} - \frac{\sin(\pi \epsilon \left( \frac{1}{\epsilon} \right) (m+n)) \cos(\pi \epsilon \left( \frac{1}{\epsilon} \right) (m+n))}{2 \sin(\pi \epsilon (m+n))}.
\]  
(B.22)
Moreover, when \( n = m \) and \( \epsilon > 0 \)
\[
S_3(\epsilon, m, m) = \frac{H}{2} - \frac{\sin(2\pi m \epsilon \left( \frac{1}{\epsilon} \right)) \cos(2\pi m \epsilon \left( \frac{1}{\epsilon} \right))}{2 \sin(2\pi m \epsilon)}
\]  
(B.23)
so that, because of (B.12), when \( \epsilon \to 0^+ \)
\[
S_3(\epsilon, m, n) \simeq \frac{H}{2} + \frac{1}{2} \left( \frac{1}{\epsilon} \right) = \frac{1}{2} \frac{1}{\epsilon} + \frac{1}{\epsilon} \quad m = n
\]  
(B.24)
When $\epsilon = 1/M$, with $M$ as a large positive integer, and $m \neq n$, $S_2(\epsilon, m, n) = S_3(\epsilon, m, n) = 0$, while $S_2(\epsilon, m, m) = S_3(\epsilon, m, m) = H/2$ when $n = m$. When $\epsilon \to 0^+$, these results approximately hold true by neglecting the fractional parts in (B.21) and (B.24).

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