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Competitive Searching for a Line on a Line Arrangement

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Abstract
We discuss the problem of searching for an unknown line on a known or unknown line arrangement by a searcher $S$, and show that a search strategy exists that finds the line competitively, that is, with detour factor at most a constant when compared to the situation where $S$ has all knowledge. In the case where $S$ knows all lines but not which one is sought, the strategy is $79$-competitive. We also show that it may be necessary to travel on $\Omega(n)$ lines to realize a constant competitive ratio. In the case where initially, $S$ does not know any line, but learns about the ones it encounters during the search, we give a $414.2$-competitive search strategy.

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1 Introduction

Given a set $L$ of $n$ lines $\ell_0, \ell_1, \ldots, \ell_{n-1}$ in the plane, consider the arrangement $A$ that they form as a geometric graph. Technically, $A$ is not a graph due to half-infinite edges, but in our problem we can end each line at its extreme intersection points, and hence we can use the term graph without complications. We consider paths on $A$. The cost of a path on $A$ is the Euclidean length of that path. The distance between two points on $A$ is the cost (or length) of the shortest path that stays on $A$ between those points.

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Assume that a searcher $S$ is located on some vertex or edge of the graph. Denote its initial position by $O$. The searcher $S$ can only travel on the arrangement and is hence restricted to paths on $A$. Searcher $S$ is looking for a target line $\ell_t \in L$, but does not know which of the lines in $L$ corresponds with $\ell_t$. The searcher $S$ will recognize $\ell_t$ when it reaches any point on $\ell_t$ (necessarily at an intersection point with another line). We call this special line the target line, and assume that $O$ does not lie on $\ell_t$. If it would, the problem would be solved immediately. We consider two versions of the problem: one where $S$ knows the lines in $L$ and therefore $A$ completely, and one where $S$ only knows about the existence and parameters of a line once it reaches some point on it.

We will show that a search strategy exists by which $S$ can reach the target line competitively in both versions. In other words, $S$ can reach the target line with a detour factor bounded by a constant, when compared to the shortest path on $A$ to the target line. Competitive analysis is commonly used to compare “the cost of not knowing” with “the cost of knowing”. The maximum detour factor of a search strategy is known as its competitive ratio. The competitive ratio of a search problem is the infimum of the competitive ratios of all search strategies that solve that search problem.

The best known search problem is perhaps the one-dimensional problem of finding a point on a line from a starting position. If we know the distance $d$, but not whether it is to the left or to the right, the optimal strategy is to go left over a distance $d$ and then right over a distance $2d$. We find the point with competitive ratio 3, which is optimal. If we don’t know the distance but we do know some (very) small lower bound $\epsilon$ on the distance, it is best to go $\epsilon$ to the left, then back and another $2\epsilon$ to the right, then back and another $4\epsilon$ to the left, and so on. This doubling strategy gives a competitive ratio of 9, which is known to be optimal as proved by Beck and Newman [4] in 1970, see also [2, 13].

The problem of searching for a line in the plane without obstacles was studied by Baeza-Yates et al. [2] in various settings. The settings refer to the knowledge we have of the line, which can be its slope, its distance, both, or neither. If the slope of the line is known, the problem reduces to the one-dimensional problem just discussed. If only the distance is known, the optimal competitive ratio is $6.39\ldots$. The problem of searching for a line a given distance away was posed by Bellman [5] in 1956 and solved by Isbell [20] in 1957. It is a classic in recreational mathematics and often posed as a swimmer in the fog, trying to reach the (straight) shore which is a known unit distance away, while swimming the least in the worst case. If the slope nor the distance of a line to be found is known, the best known competitive ratio is $13.81\ldots$, which is realized by a logarithmic spiral search strategy.

Competitive analysis of algorithms was introduced by Sleator and Tarjan for analyzing the list update problem [24]. Here the lack of knowledge is the next online requests. In geometric situations, the lack of knowledge is often the environment itself or the location of something to be found (by seeing or reaching it). The main motivation of such problems comes from the navigation of robots in unknown environments. More generally, searching for a target in environments where either the target or the environment is unknown is a basic problem, and competitive analysis is a fundamental way to understand what is in principle possible in such exploration problems. We list a few main results on searching and competitive analysis in geometric and geometric-graph environments; for an extensive overview see also [15]. We begin by noting that there is no $c$-competitive search strategy to find an unknown target node in a known graph, for example when the graph is a star.

When searching for an unknown target on a line, but additional information on the distance to the target is known, alternative results can be obtained [8, 17]. Demaine et al. [13] show that searching for an unknown target on a line with cost depending on both
search distance and turns can be done competitively with cost $9OPT + 2d$, where $d$ is the cost of one turn. Searching on multiple rays is studied in various papers [8, 13, 16, 23]; Kao et al. [22] give an optimal randomized algorithm. In yet other variants one can search with multiple searchers [3, 16].

Kalyanasundaram and Pruhs [21] consider visibility-based searching for a recognizable point in an unknown scene with convex obstacles. Their result on the competitive factor is not constant, but depends on the number of obstacles and their aspect ratio. Blum et al. [6] investigate similar problems for different classes of obstacles. Hoffmann et al. [18] show that an unknown simple polygon can be discovered completely with a competitive ratio of 26.5. There are various other visibility-based search problems addressed with competitive analysis (e.g., [14, 19]).

A different setting where competitive strategies are investigated is routing in geometric graphs. Here an unknown geometric graph is given along with a source and target with known coordinates. We route a package from source to target over the nodes, but learn about the existence and coordinates of a node when we are at a neighbor. For triangulations, no $c$-competitive strategy exists, but for special triangulations like Delaunay and certain other geometric graphs, a constant competitive strategy does exist [7, 9, 11, 12]. Searching for an unknown target on a planar straight line graph with discovery based on Pokémon Go was investigated with competitive analysis recently [25].

**Contributions.** In Section 2 we give a preliminary result where we use only two lines and obtain a competitive ratio depending on their angle. Moreover, we show that, if we want to obtain a constant competitive ratio that does not depend on parameters of the arrangement, then the search strategy must allow for traversing at least half the lines in an arrangement. In Section 3 we describe and analyze such a strategy and show that this leads to a 79-competitive strategy. This is an upper bound on the relative cost of not knowing which line is sought. (Note that for slightly more complex objects like half-lines, no constant-competitive strategy exists by mimicking a star graph, see Figure 1.) In Section 4 we generalize the problem to the situation where the searcher does not know all lines beforehand. They learn about the existence of a line and its parameters only when the line is reached. We show that in this case a search strategy exists with competitive ratio 414.2. This is an upper bound on the relative cost of not knowing the lines at all.

Although our search problems and competitive ratios are new, the existing literature implies lower bounds for our versions. When all lines are known, we have a lower bound of 9, because the problem is at least as hard as the one-dimensional problem of finding a point on a line. Moreover, it is essentially also at least as hard as finding a fully unknown line in the plane, because we could be given a very dense set of lines where all movement is approximately possible and every line could be the target. The best known competitive ratio is 13.81... to find an unknown line, but this is not known to be optimal so it does not provide a true lower bound. In case we do not know the lines of the arrangement at all, we inherit the lower bound of searching on four rays (half-lines) for a point, which is 19.96... [1, 13].
line arrangement consists of two perpendicular lines, we start on their intersection, and we must explore. If we do not follow the optimal strategy for four rays, the target line was just out of reach at the place where we went less far, and perpendicular to that ray. With more than four rays, lines will intersect more than one ray and the argument no longer works.

2 Competitive searching on an arrangement

As a warm-up, assume that $S$ starts at the intersection of two lines $\ell_1$ and $\ell_2$ whose smaller intersection angle is $\alpha \leq \pi/2$. Furthermore, $S$ only traverses $\ell_1$ and $\ell_2$, disregarding all other lines for traversal.

▶ **Theorem 1.** The target line can be found with competitive ratio at most $\frac{29}{\sin(\alpha/2)}$.

**Proof.** Denote the starting point by $O$ and the target line by $\ell_t$. As a lower bound for reaching $\ell_t$ we use the Euclidean distance between $O$ and $\ell_t$, denoted by $x$, because a line $\ell_3$ through $O$ and normal to $\ell_t$ could exist.

Note that $\ell_t$ must intersect at least one of $\ell_1$ and $\ell_2$. Let $y$ be the distance on $\ell_1$ or $\ell_2$ to the closest intersection point $u$ of $\ell_t$ with $\ell_1$ and/or $\ell_2$. Since $\alpha$ is the smaller angle, the worst case occurs when the target line $\ell_t$ spans a triangle with the two initial lines $\ell_1$ and $\ell_2$ with an angle of $\pi - \alpha$; the worst ratio between $x$ and $y$ occurs when this triangle is equilateral with apex $O$. This is illustrated in Figure 2. By elementary geometry, we then have $y \leq \frac{x}{\sin(\alpha/2)}$.

The strategy to find $\ell_t$ is as follows. Let $d$ be the distance between $O$ and the vertex $v$ on $\ell_1$ or $\ell_2$ closest to it. First, $S$ travels to $v$ and back to $O$. Then $S$ travels the same distance $d$ in each of the other three directions on $\ell_1$ and $\ell_2$, and back to $O$ each time. After that we double $d$ and repeat. $S$ has achieved its goal when it reaches $u$, and therefore $\ell_t$.

We can view the traversal of $S$ on $\ell_1$ and $\ell_2$ as the traversal on four half-lines induced by $O$. One of these half-lines crosses $\ell_t$ at distance $y$. This is, by definition, where $u$ is. By the doubling strategy, $S$ will have traversed a total distance less than $5y$ on the half-line with $u$. On each of the other half-lines, $S$ has traversed at most a distance of $8y$. Summing up yields that the searcher travelled at most a distance of $29y$; using $y \leq \frac{x}{\sin(\alpha/2)}$, we find that the competitive ratio, bounded by $\frac{29y}{x}$, gives the claimed bound of $\frac{29}{\sin(\alpha/2)}$. ◀

We note that a tighter analysis of the same strategy will give a slightly better competitive ratio, and a different strategy where we traverse the half-lines over different distances will also give a better competitive ratio. However the strategy is not $c$-competitive for any constant $c$, since $\alpha$ can be arbitrarily small. Moreover, since this is a special case of the problem, we explore this strategy no further.

Below, we show that for any constant $c$, any $c$-competitive strategy must traverse $\Omega(n)$ lines. So the strategy of the proof of Theorem 1 cannot work, not even with the usage of some carefully chosen additional lines besides $\ell_1$ and $\ell_2$.
Theorem 2. For any constant $c \geq 1$, there is an arrangement $A$ of $n$ lines such that any $c$-competitive strategy must traverse at least $\frac{n}{2} = \Omega(n)$ lines of $A$ in the worst case.

Proof. We construct an arrangement $A$ of $n = 2m + 1$ lines. The line $h_0$ is the $x$-axis, and searcher $S$ starts on $h_0$ at the origin $O$. Let $h_1, \ldots, h_m$ be horizontal lines that together with $h_0$ have a bottom-to-top order $h_0, h_1, \ldots, h_m$. Let $\ell_1, \ldots, \ell_m$ be $m$ lines with positive slope $\leq 1$, such that the upper envelope of $\ell_1, \ldots, \ell_m$ is a convex increasing function that contains all these lines in the same order. We ensure that these lines intersect $h_0$ on the positive side and in the order $\ell_1, \ldots, \ell_m$. The construction will be such that the part of $\ell_i$ between its intersection with $h_i$ and its intersection with $\ell_{i-1}$ must be used by $S$ to reach $h_i$ with detour no more than $c$, because even the intersection of $\ell_{i-1}$ with $h_i$ has an $x$-coordinate that is too high.

In more detail, we construct the lines incrementally from $m$ down to 1, in pairs $\ell_i$ and then $h_i$, see Figure 3. We start with $\ell_m : y = x - 2$ and $h_m : y = 1$. Assume $\ell_{i+1}$ and $h_{i+1}$ are placed, and their intersection point $p_{i+1}$ is such that $d_{i+1} = \text{dist}(O, h_0 \cap \ell_{i+1}) > \text{dist}(h_0 \cap \ell_{i+1}, p_{i+1})$ (for $\ell_m$ and $h_m$ we made sure of this condition). Then we define $\ell_i$ by constructing two points on it. One is the point $(d_{i+1}/(2c), 0)$ on $h_0$; the other is the point on $h_{i+1}$ with $x$-coordinate $2cd_{i+1}$. This defines $\ell_i$. The line $h_i$ is chosen horizontal and low enough so that $\text{dist}(h_0 \cap \ell_i, p_i) < \text{dist}(O, h_0 \cap \ell_i) = d_{i+1}/(2c)$. Note that $d_m = 2$ and $d_i = 2/(2c)^{m-i}$.

To argue that this arrangement forces a searcher $S$ to walk on every line $\ell_i$ (and also $h_0$ where $S$ starts), we observe that we can reach the line $h_i$ in distance at most $d_{i+1}/c$ simply by following $h_0$ and $\ell_j$ only (we can do a little bit better but for the proof this is not needed). To reach $h_i$ $c$-competitively we must thus travel less than $d_{i+1}$ along $A$.

We cannot use line $\ell_{i+1}$ or any higher-indexed line, because all their vertices have $x$-coordinates at least $d_{i+1}$ so it must take $d_{i+1}$ or more to even reach $\ell_{i+1}$ or a later line. Thus if we do not use $\ell_i$, we must reach line $h_i$ on line $\ell_{i-1}$ or a lower-indexed line. By construction the intersection of $\ell_{i-1}$ and $h_i$ has $x$-coordinate $d_{i+1}$. Thus reaching $h_i$ from $\ell_{i-1}$ is not $c$-competitive. Furthermore, any line $\ell_j$ with $1 \leq j < i - 1$ must intersect $h_i$ right of the intersection with $\ell_{i-1}$ and thus for the same reason reaching $h_i$ via $\ell_j$ cannot be $c$-competitive.

In other words, we must use $\ell_i$ to get $c$-competitively to $h_i$, and any of the $m$ horizontal lines $h_1, \ldots, h_m$ can be the target line. Hence, a $c$-competitive strategy must visit and walk on each of $\ell_1, \ldots, \ell_m$. As $S$ starts on $h_0$, it thus walks on at least $m + 1 \geq n/2$ lines.
3 \textbf{A} \textit{c}-\textit{competitive search strategy on a known arrangement}

We continue with the general case where \( S \) may start anywhere on any line and we make no assumptions on the angles between intersecting lines. For convenience we will assume the starting point to be at the origin \( O \) and the line crossing through \( O \) to be \( \ell_0 \). If multiple lines cross the origin, we pick \( \ell_0 \) to be the line that intersects any other line closest to \( O \). We will assume \( \ell_0 \) is horizontal. As the problem is rotation and translation invariant these assumptions do not change the problem. As before let \( d \) be the distance to the closest intersection point on \( \ell_0 \).

Consider the following search strategy for \( S \). Searcher \( S \) iteratively explores the arrangement starting from the origin. In iteration \( i \) four paths of length \( 2^i \cdot d \) are explored starting at \( O \). These paths are picked such that they maximize (minimize) the \( x \)-respectively \( y \)-coordinate that \( S \) can achieve on the arrangement within distance \( 2^i \cdot d \) from \( O \) (see Figure 4(a)). Specifically this results in the following strategy. First, \( S \) traverses \( \ell_0 \) over a distance \( 2d \) in the direction \(+x\) and then returns back to \( O \). Second, \( S \) traverses \( \ell_0 \) for a distance \( 2d \) in the direction \(-x\) and back. Third, \( S \) determines the point on \( A \) with maximum \( y \)-coordinate it can reach when traversing over a distance \( 2d \); \( S \) goes there and back. Symmetrically, \( S \) also visits the point with lowest \( y \)-coordinate reachable within distance \( 2d \) from \( O \). Upon returning to the origin the allowed distance is doubled and the process is repeated until \( S \) finds \( \ell_t \).

Let \( D_i \) be the distance travelled in iteration \( i \). Let the points where \( S \) ends when searching over a distance \( D_i \) with minimum and maximum \( x \)- and \( y \)-coordinate be denoted \( p_i^{-x}, p_i^{+x}, p_i^{-y}, \) and \( p_i^{+y} \), respectively. Figure 4(b) shows the four paths to \( p_i^{+y}, \ldots, p_4^{+y} \). Notice that the path for \( D_{i+1} \) does not necessarily follow the path for \( D_i \) as a less steep line may be followed to reach a steeper line sooner.

\begin{lemma}
\textbf{Lemma 3.} The \( y \)-coordinate of \( p_i^{+y} \) is at least twice that of \( p_{i-1}^{+y} \). The symmetric statement holds for \( p_i^{-y} \) and \( p_{i-1}^{+y} \).
\end{lemma}

\textbf{Proof.} Observe that for any \( p_{i-1}^{+y} \), the last line traversed on the path to \( p_{i-1}^{+y} \) must have the steepest absolute slope. If not, we could get higher by staying on the line with steepest slope. When we traverse a distance \( D_i \) instead of \( D_{i-1} \), we have the option of staying on this steepest absolute slope line, and since \( D_i = 2D_{i-1} \), we get at least twice as high just by staying on the line that contains \( p_{i-1}^{+y} \). \hfill \blacksquare

Let \( Q_i \) be the convex quadrilateral with the points \( p_i^{-x}, p_i^{+x}, p_i^{-y}, \) and \( p_i^{+y} \) as vertices and let \( R_i \) be the axis-parallel rectangle with these four points on its boundary (see Figure 5).
Trivially $Q_i \subset R_i$ and from Lemma 3 it immediately follows that $R_1 \subset R_2 \subset \cdots \subset R_k$.

Lemma 4. $R_i \subset Q_{i+2}$.

Proof. Without loss of generality only consider the half-plane above $\ell_0$. We show that the triangle $p_{i+2}^{-x}p_{i+2}^{x+y}p_{i+2}^{x}$ contains the rectangle with bottom vertices $p_{i+2}^{x}$ and $p_{i+2}^{x+y}$ and top side through $p_{i+2}^{y}$. We know that $p_{i+2}^{-x}p_{i+2}^{x+y}$ is exactly four times the length of $p_{i+2}^{x+y}$ as $\ell_0$ is horizontal. By Lemma 3 the $y$-coordinate of $p_{i+2}^{x+y}$ is at least four times that of $p_{i+2}^{y}$ (see Figure 6). By triangle inequality the $x$-coordinate of $p_{i+2}^{x+y}$ must be between $p_{i+2}^{-x}$ and $p_{i+2}^{x}$.

Let $x$ be the $x$-coordinate of $p_{i+2}^{x+y}$, and $r = (-x, y)$ the vertex at the top-left corner of $R_i$. Consider the side $p_{i+2}^{-x}p_{i+2}^{x+y}$ of the triangle and the line $p_{i+2}^{x+y}r$. The slope of $p_{i+2}^{x+y}r$ is $y/(3x)$. The slope of $p_{i+2}^{x+y}r$ depends on the exact location of $p_{i+2}^{x+y}$. In the (impossible) worst case $p_{i+2}^{x+y}$ is located at $(4x, 4y)$. Thus the slope of $p_{i+2}^{x+y}r$ is at least $y/(2x)$ and $r$ is below $p_{i+2}^{x+y}p_{i+2}^{x}$. Containment of $R_i$ in $Q_{i+2}$ trivially follows.

We observe that if the target line $\ell_i$ intersects $Q_i$ then $\ell_i$ will be found in iteration $i$ or before. Hence the distance travelled by the searcher is upper-bounded by the distance travelled up to and including iteration $i$. Suppose the searcher $S$ finds the target line $\ell_i$ in iteration $k$. We will use the rectangle $R_{k-3}$ as a lower bound on the length of the shortest path to $\ell_i$ to prove an upper bound on the competitive ratio.

Lemma 5. The target line $\ell_i$ intersects $Q_k$ and does not intersect $R_{k-3}$.

Proof. If $\ell_i$ intersects $Q_{k-1}$, then $\ell_i$ would have been found in phase $k-1$. Since $R_{k-3} \subset Q_{k-1}$, the lemma follows.

As $\ell_i$ does not intersect $R_{k-3}$ the closest point of $\ell_i$ to $O$ must be outside of $R_{k-3}$. But then the shortest path to $\ell_i$ must have length larger than $D_{k-3}$. Assume for contradiction that the closest point $p_i$ on $\ell_i$ has distance less than $D_{k-3}$. As in iteration $k-3$ we followed the paths that maximize (minimize) the $x$- and $y$-coordinate, $p_i$ could be reached and must thus be contained in $R_{k-3}$. Contradiction. Thus $D_{k-3}$ is a lower bound on the distance from $O$ to $\ell_i$, and $D_{k-3} = D_k/8$.

For an upper bound, we consider the distance we have travelled. Except for the last iteration, we traversed four paths of length $D_i$ in two directions in each iteration. Thus in previous iterations we traversed $8 \sum_{i=1}^{k-1} D_i$. In the last iteration in the worst-case we discover $\ell_i$ while traversing the fourth path all the way to its end. Hence we traverse three paths of length $D_k$ twice, and the last path of length $D_k$ once. The total travel is thus at most:

$$8 \sum_{i=1}^{k-1} D_i + 7D_k$$
Using the summation $\sum_{i=0}^{k-1} z^i = \frac{z^k-1}{z-1}$ and $D_i = 2^i d$ we can rewrite this to $15 \cdot 2^k d - 16d < 15D_k$. We thus upper-bound the competitive ratio by 120.

A more careful analysis shows that Lemma 4 is true even if we do not double $D_i$ but enlarge by only a factor $\sqrt{3}$. Let $D_1 = \sqrt{3}d$ and $D_i = \sqrt{3}D_{i-1}$ for $i \geq 2$, so $D_i = \sqrt{3}^i d$, and suppose $S$ finds $\ell_i$ in iteration $k$. Then $D_{k-3} = \sqrt{3}^{k-3} d$ is a lower bound for reaching $\ell_i$. With the described strategy $S$ travels at most

$$8 \sum_{i=1}^{k-1} \sqrt{3}^i d + 7\sqrt{3}^k d < 8 \cdot \frac{\sqrt{3}^k d}{\sqrt{3} - 1} + 7\sqrt{3}^k d$$

The competitive ratio becomes

$$\frac{8 \cdot \frac{\sqrt{3}^k d}{\sqrt{3} - 1} + 7\sqrt{3}^k d}{\sqrt{3}^{k-1} d} = \left(\frac{8}{\sqrt{3} - 1} + 7\right)\sqrt{3} < 94$$

Another improvement comes from organizing the four traversals in a phase conveniently so that we do not have to go back to $O$ at the end. In every even phase $i$ we start with going to $p_i^x$, then we do $p_i^y$ and $p_{i+1}^y$ in any order, and end with going to $p_{i+2}^x$. In every odd phase $j$ we go to $p_j^x$ first and to $p_{j+1}^x$ last. It is easy to see that we do not have to go back at the end of any phase, because we go out over the exact same stretch in the next phase anyway. Instead of traversing $8D_i$ in a phase $i$, we now traverse $(7 - 1/\sqrt{3}) \cdot D_i$. This also holds for the last phase $D_k$. With some basic calculation we obtain:

$\blacktriangleright$ Theorem 6. A 79-competitive search strategy exists to find an unknown target line in an arrangement of lines.

Alternatively, we may also triple $D_i$ because then $R_i \subset Q_{i+1}$; a lower constant factor than 3 will not ensure that $R_i \subset Q_{i+1}$ so that will not give improvements. The competitive ratio we get is worse, however, than when using $\sqrt{3}$ and $R_i \subset Q_{i+2}$.

We note that if we know the exact distance to the line, we can use some of the ideas just given. By the observations above, we can find the unknown line by going three times as far in each direction. For the last direction $S$ does not need to go back, so in total we will find the line with competitive ratio 21.

## 4 A $c$-competitive search strategy on an unknown arrangement

In this section we consider the situation where the searcher $S$ does not know the arrangement beforehand. In particular, we assume $S$ learns the slope and intercept of a line, only when $S$ reaches it. The question arises whether we can adapt our competitive strategy to still realize a constant competitive ratio. The exact same strategy cannot be used, because we can no longer determine the points $p^x$ and $p^y$ before we start walking.

First of all, this problem suffers from a technicality that has been observed in similar problems: as soon as we decide to walk any distance from the starting location in some direction on the starting line, the target line could have been arbitrarily much closer in the other direction [2]. So a constant competitive ratio cannot exist. This technicality is commonly circumvented by assuming that the target line is at least some known – possibly extremely small – distance away from the start. We will assume this as well.

Assume the starting location is at the origin $O$ and lies on a horizontal line $\ell_0$. We start by finding the closest intersection to $O$. If it is at distance $d$, then we let $D_1 = 2d$. Similar to the strategy for known arrangements in iteration $i$ we aim to find the leftmost, rightmost,
Thus the largest (absolute) slope of any line traversed to get to a height lies in the slab $[-D, D]$ for the full length of the path to height $h_2$ and a path on $L_1$ of length $2tD + 2D$ (red) reaching height $h_1$. We show $h_1 \geq h_2$.

lowest, and highest point we can reach with distance $D$. We, however, choose our movement as to also discover a suitable set of “nearby” lines to which we must necessarily restrict our movement as we do not know about the existence of other lines. We show that with this smaller set of lines we can still achieve the height that we could have reached with knowledge of all lines; however, we traverse a constant factor further to ensure this.

We start by walking left and right from $O$ over a distance $tD$ for some constant $t \geq 1$ to be specified later. In doing so, we discover a subset $L_1$ of the lines. Let $L_2 = L \setminus L_1$, see Figure 7. Let $h_2$ be the height we could achieve within distance $D$ if we had full knowledge of the arrangement. Let the sequence of lines used to reach $h_2$ be $\ell_0, \ell_1, \ell_2, \ldots, \ell_j$. We know that $\ell_j$ is the steepest line among these, by the proof of Lemma 3.

We want to reach the highest point in the vertical slab $[-D, D]$ using lines from $L_1$ only. Clearly within a distance $D$ we can get at most as high as $h_2$. Instead we allow a traversal of distance $2tD + 2D$ along the lines of $L_1$. Let $h_1$ be the maximum height we can achieve while ending in the vertical slab $[-D, D]$ and when travelling over distance at most $2tD + 2D$ along only lines of $L_1$.

**Lemma 7.** $h_2 \leq h_1$ if $t \geq 2$.

**Proof.** Assume for contradiction that $h_2 > h_1$. Let $\ell_0, \ell_1, \ldots, \ell_j$ be the lines on a path of length $D$ to height $h_2$ on $L = L_1 \cup L_2$. Either $\ell_j \in L_1$ or $\ell_j \in L_2$.

Assume first that $\ell_j \in L_1$. Specifically then there is a point $p$ we can reach along $\ell_j$ that lies in the slab $[-D, D]$ at height $h_2$. However, $\ell_j$ intersects $\ell_0$ at most $tD$ from the origin. Thus we can follow $\ell_0$ to the intersection with $\ell_j$, and then follow $\ell_j$ to height $h_2$. As $h_2 \leq D$ this takes at most $tD + (t + 1)D$ horizontal movement and $D$ vertical movement (see Figure 8). The total distance traversed along lines from $L_1$ is upper bounded by $2tD + 2D$, therefore $h_1 \geq h_2$. Contradiction.

Next, assume that $\ell_j \in L_2$. The line $\ell_j$ must intersect the rectangle $[-D, D] \times [0, h_2]$ since the path of length $D$ reaching $h_2$ cannot leave this rectangle. The maximum slope of a line $\ell_j \in L_2$ that intersects this rectangle is $\frac{h_2}{(t-1)D}$ as such a line must intersect $\ell_0$ at least $tD$ from the origin.

We must have that $\ell_j$ has the steepest absolute slope. If a previously traversed line had a steeper absolute slope we could follow it to get higher while staying in the slab $[-D, D]$. Thus the largest (absolute) slope of any line traversed to get to $h_2$ is $\frac{h_2}{(t-1)D}$. Take $t \geq 2$, then the largest slope is at most $\frac{h_2}{D}$. In the (unachievable) best case we traverse this slope for the full length of the path to height $h_2$, however then we still reach a height less than $h_2$. Contradiction.

Our constant competitive strategy, using $t = 2$, is therefore as follows: Go left over $2D$, then right over $4D$, then back to the starting point over $2D$, and form the set $L_1$. 

![Figure 7](image-url)
Figure 8 Assume for contradiction that \( h_2 > h_1 \). The last line traversed to get to height \( h_2 \) within distance \( D \) on \( L_1 \cup L_2 \) must then be from \( L_2 \). If \( \ell_j \in L_1 \) then \( h_1 \geq h_2 \) as we can traverse only \( \ell_0 \) and \( \ell_j \) to reach the same height within distance \( 2tD + 2D \).

Figure 9 Even with the worst-case placement of \( r_{k+2}^{+y} \), \( R_k \) is still contained in \( U_{k+2} \).

Use these lines, using distance \( 6D \) to get as high as possible in the vertical slab \([-D, D]\), and the same distance to get as low as possible, and back. In total we traverse a distance \( 8D + 12D + 12D = 32D \) in one phase. Then double \( D \) and repeat.

We once again argue that the true minimum and maximum \( x \) and \( y \) coordinates reachable in some phase \( i \) are covered completely by a quadrilateral on the discovered minima and maxima in a later phase. Let \( U_k \) be the quadrilateral created by our exploration of four paths on \( L_1 \) in phase \( k \).

Lemma 8. \( R_k \subseteq U_{k+2} \)

Proof. The proof of the lemma is identical to the proof of Lemma 4, with the following minor changes. See Figure 9 for an illustration of the proof.

Let \( r_i^{+y} \) be the highest point reachable in the slab \([-D_i, D_i]\) during phase \( i \). Once again let \( p_i^{+y} \) be the highest point achievable in distance \( D_i \) on the complete arrangement. From Lemma 7 we conclude that the \( y \)-coordinate of \( p_k^{+y} \) is less or equal than that of \( r_k^{+y} \). We also know that the \( x \)-coordinate of \( r_{k+2}^{+y} \) lies in the slab \([-D_{k+2}, D_{k+2}]\) so we do not need the triangle inequality of the proof. The proof follows directly.

We can now use the same method of analysis as for the case of a fully known line arrangement, except that we have to take into account that the searcher must move more in
every phase. Once again we can scale the distance walked in an iteration by a factor of $\sqrt{3}$ instead of 2 to improve the bound. For a line found in iteration $i$ we traverse at most:

$$32 \sum_{i=1}^{k-1} D_i + 36 D_k < 32 \frac{\sqrt{3}^k d}{\sqrt{3} - 1} + 36 \sqrt{3}^k d$$

A line found in iteration $i$ is at least at a distance of $D_k = 3 \sqrt{3}^k - 3$. Thus we obtain the following result.

\textbf{Theorem 9.} A $414.2$-competitive search strategy exists to find an unknown target line in an unknown arrangement of lines, where a line becomes known once we reach it.

\section{Conclusions}

We have developed and analyzed search strategies for reaching an unknown target line in an arrangements of lines. We did so by considering the competitive ratio: the worst-case ratio between the distance travelled by the searcher and the length of the shortest path from the searcher’s start location to the target line. We gave a search strategy for the case of known arrangements that achieves a competitive ratio of 79. Then we generalized our strategy so that it is competitive on line arrangements that are not known beforehand. The parameters of a line become known only when the line is reached. In this case we gave a $414.2$-competitive search strategy. There is a considerable gap between the known lower bounds and upper bounds.

\textbf{Future work.} In our work we assumed that the speed on every line is the same. When we drop this assumption we do not know whether searching for a line can be done competitively even if we know all lines and all speeds. Certain properties still hold, for example, if we search for the largest $y$-coordinate, then we can get twice as far if we double the travel time. However, a diagonal with high speed may cause the furthest reachable point in both horizontal and vertical direction to be along this diagonal, essentially preventing growth of the explored region into other directions. When we search with a cost $T$ from $O$, the relevant points to visit seem to be the vertices of a convex polygon that is the convex hull of all points reachable at cost $T$. This polygon can have more than constantly many vertices so we cannot visit all in a phase. It is unclear how to choose a constant-size subset so that the resulting, smaller convex hull at least contains the full convex hull from a previous iteration.

We note that searching (connected) arrangements of simple geometric objects like line segments, circles, and half-lines cannot be done with a constant competitive strategy. But it is possible that if we impose restrictions on the arrangement, constant-competitive search strategies can be developed.

\section*{References}

Competitive Searching for a Line on a Line Arrangement


