On a generalization of spikes

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ON A GENERALIZATION OF SPIKES

NICK BRETTELL†, RUTGER CAMPBELL‡, DEBORAH CHUN§, KEVIN GRACE¶, AND GEOFF WHITTLE∥

Abstract. We consider matroids with the property that every subset of the ground set of size \( t \) is contained in both an \( \ell \)-element circuit and an \( \ell \)-element cocircuit; we say that such a matroid has the \((t, \ell)\)-property. We show that for any positive integer \( t \), there is a finite number of matroids with the \((t, \ell)\)-property for \( \ell < 2t \); however, matroids with the \((t, 2t)\)-property form an infinite family. We say a matroid is a \( t \)-spike if there is a partition of the ground set into pairs such that the union of any \( t \) pairs is a circuit and a cocircuit. Our main result is that if a sufficiently large matroid has the \((t, 2t)\)-property, then it is a \( t \)-spike. Finally, we present some properties of \( t \)-spikes.

Key words. matroid, spike, circuit, cocircuit

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1. Introduction. For all \( r \geq 3 \), a rank-\( r \) spike is a matroid on \( 2r \) elements with a partition \( (X_1, X_2, \ldots, X_r) \) into pairs such that \( X_i \cup X_j \) is a circuit and a cocircuit for all distinct \( i, j \in \{1, 2, \ldots, r\} \). Spikes frequently arise in the matroid theory literature (see, for example, [2, 4, 8, 10]) as a seemingly benign, yet wild, class of matroids. Miller [5] proved that if \( M \) is a sufficiently large matroid having the property that every two elements share both a 4-element circuit and a 4-element cocircuit, then \( M \) is a spike.

We consider generalizations of this result. We say that a matroid \( M \) has the \((t, \ell)\)-property if every \( t \)-element subset of \( E(M) \) is contained in both an \( \ell \)-element circuit and an \( \ell \)-element cocircuit. It is well known that the only matroids with the \((1, 3)\)-property are wheels and whirls, and Miller’s result shows that if \( M \) is a sufficiently large matroid with the \((2, 4)\)-property, then \( M \) is a spike.

We first show that when \( \ell < 2t \), there are only finitely many matroids with the \((t, \ell)\)-property. However, for any positive integer \( t \), the matroids with the \((t, 2t)\)-property form an infinite class: when \( t = 1 \), this is the class of matroids obtained by taking direct sums of copies of \( U_{1,2} \); when \( t = 2 \), the class contains the infinite family of spikes. Our main result is the following theorem.

**Theorem 1.1.** There exists a function \( f \) such that if \( M \) is a matroid with the
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(t, 2t)-property, and \(|E(M)| \geq f(t)|, then \(E(M)\) has a partition into pairs such that
the union of any \(t\) pairs is both a circuit and a cocircuit.

We call a matroid with such a partition a \(t\)-spike. (A traditional spike is a 2-spike.
Note also that what we call a spike is sometimes referred to as a tipless spike.)

We also prove some properties of \(t\)-spikes, which demonstrate that \(t\)-spikes are
highly structured matroids. In particular, a \(t\)-spike has \(2t\) elements for some positive
integer \(t\), it has rank \(r\) (and corank \(r\)), any circuit that is not a union of \(t\) pairs avoids
at most \(t - 2\) of the pairs, and any sufficiently large \(t\)-spike is \((2t - 1)\)-connected.
We show that a \(t\)-spike’s partition into pairs describes crossing \((2t - 1)\)-separations
in the matroid; that is, an appropriate concatenation of this partition is a \((2t - 1)\)-
flower (more specifically, a \((2t - 1)\)-anemone), following the terminology of [1]. We
also describe a construction of a \((t + 1)\)-spike from a \(t\)-spike, and show that every
\((t + 1)\)-spike can be obtained from some \(t\)-spike in this way.

Our methods in this paper are extremal, so the lower bounds on \(|E(M)|\) that we
obtain, given by the function \(f\), are extremely large, and we make no attempts to
optimize these. For \(t = 2\), Miller [5] showed that \(f(2) = 13\) is best possible, and he
described the other matroids with the \((2, 4)\)-property when \(|E(M)| \leq 12\). We see no
reason why a similar analysis could not be undertaken for, say, \(t = 3\).

There are a number of interesting variants of the \((t, \ell)\)-property. In particular, we
say that a matroid has the \((t_1, \ell_1, t_2, \ell_2)\)-property if every \(t_1\)-element set is contained in
an \(\ell_1\)-element circuit, and every \(t_2\)-element set is contained in an \(\ell_2\)-element cocircuit.
Although we focus here on the case where \(t_1 = t_2\) and \(\ell_1 = \ell_2\), we show, in section 3,
that there are only finitely many matroids with the \((t_1, \ell_1, t_2, \ell_2)\)-property when \(\ell_1 < 2t_1\) or \(\ell_2 < 2t_2\).
Oxley et al. [7] recently considered the case where \((t_1, \ell_1, t_2, \ell_2) = (2, 4, 1, k)\) and \(k \in \{3, 4\}\). In particular, they proved, for \(k \in \{3, 4\}\), that a \(k\)-connected
matroid \(M\) with \(|E(M)| \geq k^4\) has the \((2, 4, 1, k)\)-property if and only if \(M \cong M(K_{n,n})\)
for some \(n \geq k\). This gives credence to the idea that sufficiently large matroids with
the \((t_1, \ell_1, t_2, \ell_2)\)-property, for appropriate values of \(t_1, \ell_1, t_2, \ell_2\), may form structured
classes. In particular, we conjecture the following generalization of Theorem 1.1.

**Conjecture 1.2.** There exists a function \(f(t_1, t_2)\) such that if \(M\) is a matroid
with the \((t_1, 2t_1, t_2, 2t_2)\)-property, for positive integers \(t_1\) and \(t_2\), and \(|E(M)| \geq f(t_1, t_2),
then \(E(M)\) has a partition into pairs such that the union of any \(t_1\) pairs is a circuit,
and the union of any \(t_2\) pairs is a cocircuit.

The study of matroids with the \((t, 2t)\)-property was motivated by problems in
matroid connectivity. Tutte proved that wheels and whirls (that is, matroids with
the \((1, 3)\)-property) are the only \(3\)-connected matroids with no element whose
deletion or contraction preserves \(3\)-connectivity [11]. Moreover, spikes (matroids with
the \((2, 4)\)-property) are the only \(3\)-connected matroids with \(|E(M)| \geq 13\) having no
triangles or triads, and no pair of elements whose deletion or contraction preserves
\(3\)-connectivity [12]. We envision that \(t\)-spikes could also play a role in a connectivity
“chain theorem”: they are \((2t - 1)\)-connected matroids, having no circuits or cocircuits
of size \((2t - 1)\), with the property that for every \(t\)-element subset \(X \subseteq E(M)\),
neither \(M/X\) nor \(M\setminus X\) is \((t + 1)\)-connected. We conjecture the following.

**Conjecture 1.3.** There exists a function \(f(t)\) such that if \(M\) is a \((2t - 1)\)-
connected matroid with no circuits or cocircuits of size \(2t - 1\), and \(|E(M)| \geq f(t),
then either

(i) there exists a \(t\)-element set \(X \subseteq E(M)\) such that either \(M/X\) or \(M\setminus X\) is
\((t + 1)\)-connected, or

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prove that there are only finitely many matroids with the (\(p\)-property, there is some 
Therefore, each 
\(S\) with 
we may assume that there is some maximal 
\(D \subseteq S\) for every subcollection 
\(J\) with 
\(s\). We denote the set of positive integers by \(\mathbb{N}\).

**Lemma 2.1.** There exists a function \(f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}\) such that, if \(S\) is a collection of distinct \(s\)-sets and \(|S| \geq f(s,n)\), then there is some \(S' \subseteq S\) with \(|S'| = n\), and a set \(J\) with \(0 \leq |J| < s\), such that \(S_1 \cap S_2 = J\) for all distinct \(S_1, S_2 \in S'\).

**Proof.** We define \(f(1,n) = n\) and \(f(s,n) = (s-1)f(s-1,n)\) for \(s > 1\). Note that \(f\) is increasing. We claim that this function satisfies the lemma. We proceed by induction on \(s\). If \(s = 1\), then the claim holds with \(J = \emptyset\).

Let \(\mathcal{S}\) be a collection of \(s\)-sets with \(|\mathcal{S}| \geq f(s,n)\). Suppose there are \(n\) pairwise disjoint sets in \(\mathcal{S}\). Then the desired conditions are satisfied if we take \(J = \emptyset\). Thus, we may assume that there is some maximal \(\mathcal{D} \subseteq \mathcal{S}\) consisting of pairwise disjoint sets, with \(|\mathcal{D}| \leq n - 1\). Each \(S \in \mathcal{S} - \mathcal{D}\) meets some \(D \in \mathcal{D}\). Each such \(D\) has \(s\) elements. Therefore, each \(S \in \mathcal{S}\) contains at least one of \((n-1)s\) elements \(e \in \mathcal{D}\). By the pigeonhole principle, there is some \(e \in \cup \mathcal{D}\) such that

\[|\{S \in \mathcal{S} : e \in S\}| \geq \frac{f(s,n)}{(n-1)s} = f(s-1,n).\]

Let \(\mathcal{T} = \{S - \{e\} : e \in S \in \mathcal{S}\}\). Then, for every \(T \in \mathcal{T}\), we have \(|T| = s - 1\). Moreover, \(|\mathcal{T}| = |\{S \in \mathcal{S} : e \in S\}| \geq f(s-1,n)\). By the induction assumption, there is a subset \(T' \subseteq \mathcal{T}\), with \(|T'| = n\), and a set \(J'\), with \(|J'| < s - 1\), such that \(T_1 \cap T_2 = J'\) for all distinct \(T_1, T_2 \in T'\). Let \(S' = \{T \cup \{e\} : T \in T'\}\). Then, \(S' \subseteq S\) with \(|S'| = n\) such that \(S_1 \cap S_2 = J' \cup \{e\}\) for all distinct \(S_1, S_2 \in S'\) and \(|J' \cup \{e\}| < s\). \(\square\)

**3. Matroids with the \((t,\ell)\)-property for \(\ell < 2t\).** Recall that a matroid has the \((t_1, t_2, \ell_2)\)-property if every \(t_1\)-element set is contained in an \(\ell_2\)-element circuit, and every \(t_2\)-element set is contained in an \(\ell_2\)-element cocircuit. In this section, we prove that there are only finitely many matroids with the \((t_1, t_2, \ell_2)\)-property if \(\ell_2 < 2t_2\). By duality, the same is true if \(\ell_1 < 2t_1\). As a special case, we have that there are only finitely many matroids with the \((t,\ell)\)-property for \(\ell < 2t\).

**Lemma 3.1.** Let \(\mathcal{C}\) be a collection of circuits of a matroid \(M\) such that, for some \(J \subseteq E(M)\) with \(|J| \leq k\), we have \(C \cap C' = J\) for all distinct \(C, C' \in \mathcal{C}\). Then, for every subcollection \(\{C_1, \ldots, C_{2^k}\} \subseteq \mathcal{C}\) of size \(2^k\), there is a circuit contained in \(\bigcup_{i=1}^{2^k} C_i - J\).

**Proof.** We may assume \(|\mathcal{C}| \geq 2^k\); otherwise, the result holds vacuously. Also, we may assume \(k > 0\) as the result holds for any singleton subcollection of \(\mathcal{C}\) with \(J = \emptyset\). Therefore, \(\mathcal{C}\) has at least one subcollection \(\mathcal{C}' = \{C_1, \ldots, C_{2^k}\}\), with \(|\mathcal{C}'| = 2^k \geq 2\).

Let \(x_1, x_2, \ldots, x_{|J|}\) be the elements of \(J\). Define \(Z_{i,0} = C_i\), for \(i \in [2^k]\), and recursively define \(Z_{i,j} = Z_{2i-1,j-1} \cup Z_{2i,j-1}\) for \(j \in [k]\) and \(i \in [2^k-j]\). Note that...
each $Z_{i,j}$ is the union of $2^j$ members of $C$. We will show, by induction on $j$, that

$$Z_{i,j} - \{x_1, x_2, \ldots, x_j\}$$

contains a circuit. This is clear when $j = 0$. Now let $j \geq 1$. By the induction hypothesis, $Z_{2i-1,j-1}$ and $Z_{2i,j-1}$ each contain a circuit, $C'_1$ and $C''_2$, respectively, disjoint from $\{x_1, x_2, \ldots, x_{j-1}\}$, for each $i \in [2^{k-j}]$. (Moreover, $C'_1 \neq C''_2$ since $C'_1 \cap C''_2 \subseteq Z_{2i-1,j-1} \cap Z_{2i,j-1} \subseteq J$, which is independent since $J$ is the intersection of at least two circuits.) We may assume that neither $Z_{2i-1,j-1}$ nor $Z_{2i,j-1}$ contains a circuit disjoint from $\{x_1, x_2, \ldots, x_j\}$; otherwise, so does $Z_{i,j}$. Thus, $C'_1$ and $C''_2$ both contain $x_j$. By circuit elimination, there is a circuit $C'_2$ contained in $(C'_1 \cup C''_2) - \{x_j\} \subseteq Z_{i,j} - \{x_1, x_2, \ldots, x_j\}$. This completes the induction argument.

In particular, there is a circuit contained in $Z_{1,k} - \{x_1, x_2, \ldots, x_{|J|}\} = \bigcup_{i=1}^{2^k} C_i - J$, as required.

\[ \text{Lemma 3.2. There exists a function } g : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \text{ such that if } M \text{ is a matroid having at least } g(\ell, d) \text{-many } \ell\text{-element circuits, then } M \text{ has a collection of } d \text{ pairwise disjoint circuits.} \]

\[ \text{Proof. Let } C \text{ be the collection of } \ell\text{-element circuits of } M, \text{ let } f \text{ be the function of Lemma 2.1, and let } g(\ell, d) = f(\ell, 2^{\ell-1}d). \text{ Then, by Lemma 2.1, there is a subset } C' \subseteq C, \text{ with } |C'| = 2^{\ell-1}d, \text{ and a set } J, \text{ with } 0 \leq |J| \leq \ell - 1, \text{ such that } C \cap C' = J \text{ for every pair } C, C' \in C'. \text{ Say } C' = \{C_1, C_2, \ldots, C_{2^{\ell-1}d}\}. \]

If $J = \emptyset$, then $M$ contains a circuit disjoint from $\{x_1, x_2, \ldots, x_{|J|}\}$, as required. Thus, we may assume that $J \neq \emptyset$. For each $C_i \in C'$, let $D_i = C_i - J$, and observe that the $D_i$’s are pairwise disjoint. For $j \in [d]$, let

$$D'_j = \bigcup_{i=1}^{2^{\ell-1}} D_{(j-1)(2^{\ell-1})+i}.$$ 

By Lemma 3.1, each $D'_j$ contains a circuit $C'_j$, and the $C'_j$’s are pairwise disjoint.

\[ \text{Theorem 3.3. Let } t_1, \ell_1, t_2, \text{ and } \ell_2 \text{ be positive integers. If } \ell_1 < 2t_1 \text{ or } \ell_2 < 2t_2, \text{ then there is a finite number of matroids with the } (t_1, \ell_1, t_2, \ell_2)\text{-property.} \]

\[ \text{Proof. By duality, it suffices to prove the result when } \ell_2 < 2t_2. \text{ So let } \ell_2 < 2t_2, \text{ and let } g \text{ be the function given in Lemma 3.2.} \]

Suppose $M$ has at least $g(\ell_1, t_2)$-many $\ell_1$-element circuits. By Lemma 3.2, $M$ has a collection of $t_2$ pairwise disjoint circuits. Call this collection $C = \{C_1, \ldots, C_{t_2}\}$. Let $b_i$ be an element of $C_i$, for each $i \in [t_2]$. By the $(t_1, \ell_1, t_2, \ell_2)$-property, there is an $\ell_2$-element cocircuit $C^*$ containing $\{b_1, \ldots, b_{t_2}\}$. By orthogonality, for each $i \in [t_2]$ there is an element $b'_i \neq b_i$ such that $b'_i \in C_i \cap C^*$. This implies that $\ell_2 = |C^*| \geq 2t_2$; a contradiction. Thus, $M$ has fewer than $g(\ell_1, t_2)$-many $\ell_1$-element circuits.

Suppose $|E(M)| \geq \ell_1 \cdot g(\ell_1, t_2)$. Partition a subset of $E(M)$ into $[\ell_1/t_1] \cdot g(\ell_1, t_2)$ pairwise disjoint $t_1$-sets. By the $(t_1, \ell_1, t_2, \ell_2)$-property, each of these $t_1$-sets is contained in an $\ell_1$-element circuit. The collection consisting of these $\ell_1$-element circuits contains at least $g(\ell_1, t_2)$ distinct circuits. This contradicts the fact that $M$ has fewer than $g(\ell_1, t_2)$-many $\ell_1$-element circuits. Therefore, $|E(M)| < \ell_1 \cdot g(\ell_1, t_2)$. The result follows.

Note that there may still be infinitely many matroids where every $\ell_1$-element set is in an $\ell_1$-element circuit for fixed $\ell_1 < 2t_1$; it is necessary that the matroids in Theorem 3.3 have the property that every $t_2$-element set is in an $\ell_2$-element cocircuit, for fixed $t_2$ and $\ell_2$. To see this, observe that projective geometries on at least three elements form an infinite family of matroids with the property that every pair of elements is in a 3-element circuit.
COROLLARY 3.4. Let $t$ and $\ell$ be positive integers. When $\ell < 2t$, there is a finite number of matroids with the $(t, \ell)$-property.

4. Echidnas and $t$-spikes. We now focus on matroids with the $(t, 2t)$-property. In section 5, we will show that every sufficiently large matroid with the $(t, 2t)$-property has a partition into pairs such that the union of any $t$ of these pairs is both a circuit and a cocircuit. We call such a matroid a $t$-spike. We first define a related structure:

A $t$-echidna of order $n$ is a partition $(S_1, \ldots, S_n)$ of a subset of $E(M)$ such that

(i) $|S_i| = 2$ for all $i \in [n]$ and

(ii) $\bigcup_{i \in I} S_i$ is a circuit for all $I \subseteq [n]$ with $|I| = t$.

For $i \in [n]$, we say $S_i$ is a spine. We say $(S_1, \ldots, S_n)$ is a $t$-echidna of $M$ if $(S_1, \ldots, S_n)$ is a $t$-echidna of $M^*$.

DEFINITION 4.1. Let $M$ be a matroid. A $t$-echidna of order $n$ is a partition $(S_1, \ldots, S_n)$ of a subset of $E(M)$ such that

(i) $|S_i| = 2$ for all $i \in [n]$ and

(ii) $\bigcup_{i \in I} S_i$ is a circuit for all $I \subseteq [n]$ with $|I| = t$.

For $i \in [n]$, we say $S_i$ is a spine. We say $(S_1, \ldots, S_n)$ is a $t$-echidna of $M$ if $(S_1, \ldots, S_n)$ is a $t$-echidna of $M^*$.

DEFINITION 4.2. A matroid $M$ is a $t$-spike of order $r$ if there exists a partition $\pi = (A_1, \ldots, A_r)$ of $E(M)$ such that $\pi$ is a $t$-echidna and a $t$-coechidna, for some $r \geq t$. We say $\pi$ is the associated partition of the $t$-spike $M$, and $A_i$ is an arm of the $t$-spike for each $i \in [r]$.

Note that if $M$ is a $t$-spike, then $M^*$ is a $t$-spike.

In this section, we prove, as Lemma 4.5, that if $M$ is a matroid with the $(t, 2t)$-property, and $M$ has a $t$-echidna of order $4t - 3$, then $M$ is a $t$-spike.

LEMMA 4.3. Let $M$ be a matroid with the $(t, 2t)$-property. If $M$ has a $t$-echidna $(S_1, \ldots, S_n)$, where $n \geq 3t - 1$, then $(S_1, \ldots, S_n)$ is also a $t$-echidna of $M$.

Proof. Let $S_i = \{x_i, y_i\}$ for each $i \in [n]$. By definition, if $J$ is a $t$-element subset of $[n]$, then $\bigcup_{i \in J} S_i$ is a circuit. Consider such a circuit $C$; without loss of generality, we let $C = \{x_1, y_1, \ldots, x_t, y_t\}$. By the $(t, 2t)$-property, there is a $2t$-element cocircuit $C^*$ that contains $\{x_1, \ldots, x_t\}$.

Suppose that $C^* \neq C$. Then there is some $i \in [t]$ such that $y_i \notin C^*$. Without loss of generality, say $y_1 \notin C^*$. Let $I$ be a $(t-1)$-element subset of $[t+1, n]$. For any such $I$, the set $S_i \cup \bigcup_{i \in I} S_i$ is a circuit that meets $C^*$. By orthogonality, $\bigcup_{i \in I} S_i$ meets $C^*$ for every $(t-1)$-element subset $I$ of $[t+1, n]$. Thus, $C^*$ avoids at most $t-2$ of the $S_i$’s for $i \in [t+1, n]$. In fact, as $C^*$ meets each $S_i$ with $i \in [t]$, the cocircuit $C^*$ avoids at most $t-2$ of the $S_i$’s with $i \in [n]$. Thus $|C^*| \geq n - (t-2) \geq (3t-1) - (t-2) = 2t+1 > 2t$; a contradiction. Therefore, we conclude that $C^* = C$, and the result follows.

LEMMA 4.4. Let $M$ be a matroid with the $(t, 2t)$-property, and let $(S_1, \ldots, S_n)$ be a $t$-echidna of $M$ with $n \geq 3t - 1$. Let $I$ be a $(t-1)$-element subset of $[n]$. For $z \in E(M) - \bigcup_{i \in I} S_i$, there is a $2t$-element circuit each containing $\{z\} \cup (\bigcup_{i \in I} S_i)$.

Proof. By duality, it suffices to show that there is a $2t$-element cocircuit containing $\{z\} \cup (\bigcup_{i \in I} S_i)$. For $i \in [n]$, let $S_i = \{x_i, y_i\}$. By the $(t, 2t)$-property, there is a $2t$-element circuit $C$ containing $\{z\} \cup \{x_i : i \in I\}$. Let $J$ be a $(t-1)$-element subset of $[n]$ such that $C$ and $\bigcup_{i \in J} S_i$ are disjoint (such a set exists since $|C| = 2t$ and $n \geq 3t-1$). For $i \in I$, let $C_i^* = S_i \cup (\bigcup_{j \in J} S_j)$, and observe that $x_i \in C_i^* \cap C$, and $C_i^* \cap C \subseteq S_i$. By Lemma 4.3, $(S_1, \ldots, S_n)$ is a $t$-echidna as well as a $t$-echidna; therefore, $C_i^*$ is a cocircuit. Now, for each $i \in I$, orthogonality implies that $|C_i^* \cap C| \geq 2$, and hence $y_i \in C$. So $C$ contains $\{z\} \cup (\bigcup_{i \in I} S_i)$, as required.

Let $(S_1, \ldots, S_n)$ be a $t$-echidna of a matroid $M$. If $(S_1, \ldots, S_m)$ is a $t$-echidna of
Lemma 4.4 implies that there is a 2-echidna, and let \( J \) be a 2-echidna. Thus, \( z \in E(M) - X \). By Lemma 4.4, there is a 2t-element circuit \( C = \{z, z'\} \cup (\bigcup_{i \in \llbracket t-1 \rrbracket} S_i) \), for some \( z' \in E(M) - (\{z\} \cup (\bigcup_{i \in \llbracket t-1 \rrbracket} S_i)) \).

We claim that \( z', \pi S \) is a 2-echidna. By orthogonality, \( z \in X \); a contradiction. Thus, \( z', \pi S \notin X \). Towards a contradiction, suppose that \( z' \notin S_k \) for some \( k \in [t, m] \). Let \( J \) be a t-element subset of \([t, m] \) containing \( k \). Then, since \((S_1, \ldots, S_m)\) is a t-echidna, \( \bigcup_{i \in J} S_j \) is a cocircuit that contains \( z \). Now, by orthogonality, \( z \in X \); a contradiction. Thus, \( z' \notin X \), as claimed.

We next show that \( \{z, z', S_1, S_2, \ldots, S_m\} \) is a 2-echidna. It suffices to show that \( \{z, z'\} \cup (\bigcup_{i \in \llbracket t \rrbracket} S_i) \) is a cocircuit for each \((t - 1)\)-element subset \( I \) of \([t, m] \). Let \( I \) be such a set. Lemma 4.4 implies that there is a 2t-element cocircuit \( C^* \) of \( M \) containing \( \{z\} \cup (\bigcup_{i \in I} S_i) \). By orthogonality, \(|C \cap C^*| > 1\). Therefore, \( z' \in C^* \).

Thus, \( \{z, z', S_1, S_2, \ldots, S_m\} \) is a t-echidna. Since this t-echidna has order \( 1 + m - (t - 1) \geq 3t - 1 \), the dual of Lemma 4.3 implies that \( \{z, z', S_1, S_2, \ldots, S_m\} \) is also a t-echidna.

Now, we claim that \( \{z, z', S_1, S_2, \ldots, S_m\} \) is a t-echidna. It suffices to show that \( \{z, z'\} \cup (\bigcup_{i \in \llbracket t \rrbracket} S_i) \) is a cocircuit for any \((t - 1)\)-element subset \( I \) of \([m] \). Let \( I \) be such a set, and let \( J \) be a \((t - 1)\)-element subset of \([t, m] \) containing \( I \). By Lemma 4.4, there is a 2t-element cocircuit \( C^* \) containing \( \{z\} \cup (\bigcup_{i \in J} S_i) \). Moreover, \( C = \{z, z'\} \cup (\bigcup_{i \in J} S_i) \) is a circuit since \( \{z, z', S_1, S_2, \ldots, S_m\} \) is a t-echidna. By orthogonality, \( z' \in C^* \).

Therefore, \( \{z, z', S_1, S_2, \ldots, S_m\} \) is a t-echidna. By the dual of Lemma 4.3, it is also a t-echidna, contradicting the maximality of \((S_1, \ldots, S_m)\).

5. Matroids with the \((t, 2t)\)-property. In this section, we prove that every sufficiently large matroid with the \((t, 2t)\)-property is a t-spikes. Our primary goal is to show that a sufficiently large matroid with the \((t, 2t)\)-property has a large t-echidna or t-coechidna; it then follows, by Lemma 4.5, that the matroid is a t-spikes.

Lemma 5.1. Let \( M \) be a matroid with the \((t, 2t)\)-property, and let \( X \subseteq E(M) \).

(i) If \( r(X) < t \), then \( X \) is independent.

(ii) If \( r(X) = t \), then \( M|X \cong U_{t, |X|} \) and \( |X| < 3t \).

Proof. Clearly, \( M \) has the \((t, 2t)\)-property. \( M \) has no circuits of size at most \( t \). Thus, if \( r(X) < t \), then \( X \) contains no circuits and is therefore independent. If \( r(X) = t \), then a subset of \( X \) is a circuit if and only if it has size \( t + 1 \). Therefore, \( M|X \cong U_{t, |X|} \).

Suppose towards a contradiction that \( M|X \cong U_{t, 3t} \). Let \( x \in X \), and let \( C^* \) be a cocircuit of \( M \) containing \( x \). Then \( E(M) - C^* \) is closed, so \( \text{cl}(X - C^*) \subseteq \text{cl}(E(M) - C^*) = E(M) - C^* \). Therefore, \( r(X - C^*) < r(X) = t \), implying that \(|C^*| > 2t \). But then every cocircuit containing \( x \) has size greater than \( 2t \), contradicting the \((t, 2t)\)-property.

Lemma 5.2. Let \( M \) be a matroid with the \((t, 2t)\)-property. Let \( C_1^*, C_2^*, \ldots, C_{t-1}^* \) be a collection of \( t - 1 \) pairwise disjoint cocircuits of \( M \), and let \( Y = E(M) - \bigcup_{i \in [t-1]} C_i^* \). For all \( y \in Y \), there is a 2t-element circuit \( C_y \) containing \( y \) such that either
(i) \(|C_y \cap C^*_t| = 2\) for all \(i \in [t-1]\) or
(ii) \(|C_y \cap C^*_t| = 3\) for some \(j \in [t-1]\), and \(|C_y \cap C^*_t| = 2\) for all \(i \in [t-1] - \{j\}\).
Moreover, if \(C_y = S \cup \{y\}\) satisfies (ii), then there are at most \(3t-1\) elements \(w \in Y\) such that \(S \cup \{w\}\) is a circuit.

Proof. Choose an element \(c_i \in C^*_t\) for each \(i \in [t-1]\). By the \((t,2t)\)-property, there is a \(2t\)-element circuit \(C_y\) containing \(\{c_1,c_2,\ldots,c_{t-1},y\}\), for each \(y \in Y\). By orthogonality, \(C_y\) satisfies (i) or (ii).

Suppose \(C_y\) satisfies (ii), and let \(S = C_y - Y = C_y - \{y\}\). Let \(W = \{w \in Y : S \cup \{w\}\) is a circuit\}. It remains to prove that \(|W| < 3t\). Observe that \(W \subseteq \text{cl}(S) \cap Y\), and, since \(S\) contains \(t-1\) elements in pairwise disjoint cocircuits that avoid \(Y\), we have \(r(\text{cl}(S) \cup Y) \geq r(Y) + (t-1)\). Thus,

\[
\begin{align*}
r(W) &\leq r(\text{cl}(S) \cap Y) \\
&\leq r(\text{cl}(S)) + r(Y) - r(\text{cl}(S) \cup Y) \\
&\leq (2t - 1) + r(Y) - (r(Y) + (t-1)) \\
&= t,
\end{align*}
\]

using submodularity of the rank function at the second line.

Now, by Lemma 5.1(i), if \(r(W) < t\), then \(W\) is independent, so \(|W| = r(W) < t\). On the other hand, by Lemma 5.1(ii), if \(r(W) = t\), then \(M|W| = U_{t,|W|}\) and \(|W| < 3t\), as required.

Lemma 5.3. There exists a function \(h\) such that if \(M\) is a matroid with the \((t,2t)\)-property and having at least \(h(\ell,d,t)\) \(\ell\)-element circuits, then \(M\) has a collection of \(d\) pairwise disjoint \(2t\)-element cocircuits.

Proof. By Lemma 3.2, there is a function \(g\) such that if \(M\) has at least \(g(\ell,d)\) \(\ell\)-element circuits, then \(M\) has a collection of \(d\) pairwise disjoint circuits. We define \(h(\ell,d,t) = g(\ell,td)\), and claim that a matroid with the \((t,2t)\)-property and having at least \(h(\ell,d,t)\) \(\ell\)-element circuits has a collection of \(d\) pairwise disjoint \(2t\)-element cocircuits.

Let \(M\) be such a matroid. By Lemma 3.2, \(M\) has a collection of \(td\) pairwise disjoint circuits. We partition these into \(d\) groups of size \(t\): call this partition \((C_1,\ldots,C_d)\). Since the \(t\) circuits in any cell of this partition are pairwise disjoint, it now suffices to show that, for each \(i \in [d]\), there is a \(2t\)-element cocircuit contained in the union of the members of \(C_i\). Let \(C_i = \{C_1,\ldots,C_t\}\) for some \(i \in [d]\). Pick some \(c_j \in C_j\) for each \(j \in [t]\). Then, by the \((t,2t)\)-property, \(\{c_1,c_2,\ldots,c_t\}\) is contained in a \(2t\)-element cocircuit, which, by orthogonality, is contained in \(\bigcup_{j \in [t]} C_j\).

Lemma 5.4. There exists a function \(q\) such that if \(M\) is a matroid with the \((t,2t)\)-property and \(|E(M)| \geq g(t,q)\), then, for some \(M' \in \{M,M^*\}\), the matroid \(M'\) has \(t-1\) pairwise disjoint cocircuits \(C^*_1,\ldots,C^*_t\), and there is some \(Z \subseteq E(M') - \bigcup_{i \in [t-1]} C^*_i\) such that

(i) \(r_{M'}(Z) \geq q\) and
(ii) for each \(z \in Z\), there exists an element \(z' \in Z - \{z\}\) such that \(z,z'\) is contained in a \(2t\)-element circuit \(C\) of \(M'\) with \(|C \cap C^*_i| = 2\) for each \(i \in [t-1]\).

Proof. By Lemma 5.3, there is a function \(h\) such that if \(M'\) has at least \(h(\ell,d,t)\) \(\ell\)-element circuits, for \(M' \in \{M,M^*\}\), then \(M'\) has a collection of \(d\) pairwise disjoint \(2t\)-element cocircuits.

Suppose \(|E(M)| \geq 2t \cdot h(2t, t-1, t)\). Then, by the \((t,2t)\)-property, \(M'\) has at least \(h(2t, t-1, t)\) distinct \(2t\)-element circuits. Hence, by Lemma 5.3, \(M'\) has a collection
of \( t - 1 \) pairwise disjoint \( 2t \)-element cocircuits \( C_1^*, C_2^*, \ldots, C_{t-1}^* \).

Let \( X = \bigcup_{i \in [t-1]} C_i^* \) and \( Y = E(M) - X \). By Lemma 5.2, for each \( y \in Y \) there is a \( 2t \)-element circuit \( C_y \) containing \( y \) such that \(|C_y \cap C_j^*| = 3\) for at most one \( j \in [t-1] \) and \(|C_y \cap C_i^*| = 2\) otherwise. Let \( W \) be the set of all \( w \in Y \) such that \( w \) is in a \( 2t \)-element circuit \( C \) with \(|C \cap C_j^*| = 3\) for some \( j \in [t-1] \), and \(|C \cap C_i^*| = 2\) for all \( i \in [t-1] - \{j\} \). Now, letting \( Z = Y - W \), we see that (ii) is satisfied for both \( M' = M \) and \( M' = M^* \).

Since the \( C_i^* \)'s have size 2, there are \((t - 1)\binom{2t}{3} \binom{t - 2}{2}\) sets \( X' \subseteq X \) with \(|X' \cap C_i^*| = 3\) for some \( j \in [t-1] \) and \(|X' \cap C_i^*| = 2\) for all \( i \in [t-1] - \{j\} \). It follows, by Lemma 5.2, that \(|W| \leq s(t)\) where

\[
s(t) = (3t - 1) \left[ (t - 1) \binom{2t}{3} \binom{2t}{2} \bigg]^{t - 2}\right.
\]

We define

\[
g(t, q) = \max \left\{ 2t \cdot h(2t, t - 1, t), 2q + s(t) + 2t(t - 1) \right\}
\]

Suppose that \(|E(M)| \geq g(t, q)\). Recall that (ii) holds for both \( M' = M \) and \( M' = M^* \). Moreover, we can choose \( M' \in \{M, M^*\} \) such that \( r(M') \geq q + s(t) + 2t(t - 1) \). Then,

\[
r_{M'}(Z) \geq r_{M'}(Y) - |W|
\]
\[
\geq (r(M') - 2t(t - 1) - s(t)
\]
\[
\geq g,
\]

so (i) holds as well, as required.

**Lemma 5.5.** Let \( M \) be a matroid with the \((t, 2t)\)-property. Suppose \( M \) has \( t - 1 \) pairwise disjoint cocircuits \( C_1^*, C_2^*, \ldots, C_{t-1}^* \), and, for some positive integer \( p \), there is some \( Z \subseteq E(M) - \bigcup_{i \in [t-1]} C_i^* \) such that

(a) \( r_M(Z) \geq (\binom{2t}{3} - 1) \) \( (p + 2(t - 1)) \) and

(b) for each \( z \in Z \), there exists an element \( z' \in Z - \{z\} \) such that \( \{z, z'\} \) is contained in a \( 2t \)-element circuit \( C \) of \( M \) with \(|C \cap C_i^*| = 2\) for each \( i \in [t-1] \).

Then there exist a subset \( Z' \subseteq Z \) and a partition \( Z' = (Z'_1, \ldots, Z'_p) \) of \( Z' \) into pairs such that

(i) each circuit of \( M[Z'] \) is a union of pairs in \( Z' \) and

(ii) the union of any \( t \) pairs of \( Z' \) contains a circuit.

**Proof.** We first prove the following claim.

**Claim 5.5.1.** There exist a \((2t - 2)\)-element set \( X \), with \(|X \cap C_i^*| = 2\) for each \( i \in [t - 1] \), and a set \( Z' \subseteq Z \), with a partition \( Z' = (Z'_1, \ldots, Z'_p) \) into \( p \) pairs, such that

(I) \( X \cup Z_i' \) is a circuit for each \( i \in [p] \) and

(II) \( Z' \) partitions the ground set of \((M/X)|Z'\) into parallel classes, and we have that \( r_{M/X \bigcup_{i \in [p]} Z'_i} = p \).

**Proof.** For each \( z \in Z \), there exist an element \( z' \in Z - \{z\} \) and a set \( X' \) such that \( \{z, z'\} \cup X' \) is a circuit of \( M \), and \( X' \) is the union of pairs \( Y_i \) for \( i \in [t - 1] \), with \( Y_i \subseteq C_i^* \). There are \((\binom{2t}{3} - 1) \) choices of such pairs \( Y_i \subseteq C_i^* \). Thus, for some \( m \leq (\binom{2t}{3} - 1) \), there are \((2t - 2)\)-element sets \( X_1, \ldots, X_m \), each of which intersects \( C_i^* \) in two elements for each \( i \in [t - 1] \), and sets \( Z_1, \ldots, Z_m \) whose union is \( Z \), such that

\[
 \sum_{i=1}^{m} (|X_i| + |Z_i|) = 2t - 1
\]

\[
 |X| + \sum_{i=1}^{m} Z_i = 2t - 1
\]
for each \(j \in [m]\) and each \(z_j \in Z_j\), there is an element \(z'_j \in Z_j\) such that \(X_j \cup \{z_j, z'_j\}\) is a circuit. Moreover, \(r(Z_1) + \cdots + r(Z_m) \geq r(Z)\). Thus, by the pigeonhole principle, there exists some \(j \in [m]\) with

\[
r(Z_j) \geq \frac{r(Z)}{2^{m-1}} \geq p + 2(t - 1).
\]

Let \(Z' = Z_j\) and \(X = X_j\). Now, observe that \(X \cup \{z, z'\}\) is a circuit, for some pair \(\{z, z'\} \subseteq Z'\), if and only if \(\{z, z'\}\) is a parallel pair in \(M/X\). So the ground set of \((M/X)|Z'\) has a partition into parallel classes, where each parallel class has size at least two. Let \(Z' = \{\{z_1, z'_1\}, \ldots, \{z_n, z'_n\}\}\) be a collection of pairs from each parallel class such that \(\{z_1, z_2, \ldots, z_n\}\) is independent in \((M/X)|Z'\). Since \(r_{M/X}(Z') = r(Z' \cup X) - r(X) \geq r(Z') - 2(t - 1) \geq p\), there exists such a collection \(Z'\) of size \(p\), and this collection satisfies Claim 5.5.1.

Let \(X\) and \(Z' = \{Z'_1, \ldots, Z'_p\}\) be as described in Claim 5.5.1, let \(Z' = \bigcup_{i \in [p]} Z'_i\), and let \(X' = \{X_1, \ldots, X_{t-1}\}\), where \(X_i = \{x_i, x'_i\} = X \cap C_i^*\).

\textbf{Claim 5.5.2.} Each circuit of \(M|(X \cup Z')\) is a union of pairs in \(X' \cup Z'\).

\textbf{Proof.} Let \(C\) be a circuit of \(M|(X \cup Z')\). If \(x_i \in C\), for some \(\{x_i, x'_i\} \subseteq X', \) then, by orthogonality with \(C_i^*\), we have \(x'_i \in C\). Towards a contradiction, say \(\{z, z'\} \subseteq Z'\) and \(C \cap \{z, z'\} = \{z\}\). Choose \(W\) to be the union of the pairs of \(Z'\) that contain elements of \((C - \{z\}) \cap Z'\). Then \(z \in cl(X \cup W)\). Hence \(z \in cl_{M/X}(W)\), contradicting Claim 5.5.1(I).

\textbf{Claim 5.5.3.} The union of any \(t\) pairs of \(X' \cup Z'\) contains a circuit.

\textbf{Proof.} Let \(W\) be a subcollection of \(X' \cup Z'\) of size \(t\). We proceed by induction on the number of pairs in \(W \cap Z'\). If there is only one pair in \(W \cap Z'\), then the union of the pairs in \(W\) contains a circuit (indeed, is a circuit) by Claim 5.5.1(I). Suppose the result holds for any subcollection containing \(k\) pairs in \(Z'\), and let \(W\) be a subcollection containing \(k + 1\) pairs in \(Z'\). Let \(\{x, x'\}\) be a pair in \(X' - W\), and let \(W = \bigcup_{W' \subseteq W} W'\). By the induction hypothesis, \(W \cup \{x, x'\}\) contains a circuit \(C_1\). If \(\{x, x'\} \subseteq \bigcup M - C_1\), then \(C_1 \subseteq W\), in which case the union of the pairs in \(W\) contains a circuit, as desired. Therefore, we may assume, by Claim 5.5.2, that \(\{x, x'\} \subseteq C_1\). Since \(X\) is independent, there is a pair \(\{z, z'\} \subseteq Z' \cap C_1\). By the induction hypothesis, there is a circuit \(C_2\) contained in \((W - \{z, z'\}) \cup \{x, x'\}\). Observe that \(C_1 \cap C_2\) are distinct, and \(\{x, x'\} \subseteq C_1 \cap C_2\). By circuit elimination on \(C_1\) and \(C_2\), and Claim 5.5.2, there is a circuit \(C_3 \subseteq (C_1 \cup C_2) - \{x, x'\} \subseteq W\), as desired. The result now follows by induction.

Now, Claim 5.5.3 implies that the union of any \(t\) pairs of \(Z'\) contains a circuit, and the result follows.

In order to prove Theorem 1.1, we use some hypergraph Ramsey theory [9].

\textbf{Theorem 5.6 (Ramsey’s theorem for \(k\)-uniform hypergraphs).} For positive integers \(k\) and \(n\), there exists an integer \(r_k(n)\) such that if \(H\) is a \(k\)-uniform hypergraph on \(r_k(n)\) vertices, then \(H\) has either a clique on \(n\) vertices, or a stable set on \(n\) vertices.

We now prove Theorem 1.1, restated below as Theorem 5.7.

\textbf{Theorem 5.7.} There exists a function \(f : \mathbb{N} \to \mathbb{N}\) such that if \(M\) is a matroid with the \((t, 2t)\)-property, and \(|E(M)| \geq f(t)\), then \(M\) is a \(t\)-spike.
Proof. We first consider the case where \( t = 1 \). Let \( M \) be a nonempty matroid with the \((1,2)\)-property. Then, for every \( e \in E(M) \), the element \( e \) is in a parallel pair \( P \) and a series pair \( S \). By orthogonality, \( P = S \), and \( P \) is a connected component of \( M \). Then \( M \cong U_{1,2} \oplus M \setminus P \), and the result easily follows.

We may now assume that \( t \geq 2 \). We define the function \( h_k : \mathbb{N} \to \mathbb{N} \), for each \( k \in [t] \), as follows:

\[
h_k(t) = \begin{cases} 
4t - 3 & \text{if } k = t, \\
r_k(h_{k+1}(t)) & \text{if } k \in [t-1],
\end{cases}
\]

where \( r_k(n) \) is the Ramsey number described in Theorem 5.6. Note that \( h_k(t) \geq h_{k+1}(t) \geq 4t - 3 \), for each \( k \in [t-1] \). Let \( p(t) = h_1(t) \), and let \( q(t) = (\frac{2t}{2})^{t-1}(p(t) + 2(t-1)) \).

By Lemma 5.4, there exists a function \( g \) such that if \( |E(M)| \geq g(t,q(t)) \), then, for some \( M' \in \{M,M*\} \), the matroid \( M' \) has \( t - 1 \) pairwise disjoint cocircuits \( C_1^*, C_2^*, \ldots, C_{t-1}^* \), and there is some \( Z' \subseteq E(M') - \bigcup_{i \in [t-1]} C_i^* \) such that \( r_M(Z') \geq q(t) \), and, for each \( z \in Z' \), there exists an element \( z' \in Z' - \{z\} \) such that \( \{z,z'\} \cup (\bigcup_{i \in [t-1]} \{x_i, x_i'\}) \) is a circuit of \( M' \), where \( \{x_i, x_i'\} \subseteq C_i^* \).

Let \( f(t) = q(t,q(t)) \), and suppose that \( |E(M)| \geq f(t) \). For ease of notation, we assume that \( M' = M \). Then, by Lemma 5.5, there exist a subset \( Z \subseteq Z' \) and a partition \( Z = (Z_1,\ldots,Z_{p(t)}) \) of \( Z \) into \( p(t) \) pairs such that

(I) each circuit of \( M|Z \) is a union of pairs in \( Z \) and

(II) the union of any \( t \) pairs of \( Z \) contains a circuit.

By Lemma 4.5, and since \( t \geq 2 \), it suffices to show that \( M \) has a \( t \)-echidna or a \( t \)-coechidna of order \( 4t-3 \). If the smallest circuit in \( M|Z \) has size \( 2t \), then, by (II), \( Z \) is a \( t \)-echidna of order \( p(t) \geq 4t - 3 \). So we may assume that the smallest circuit in \( M|Z \) has size \( 2j \) for some \( j \in [t-1] \).

Claim 5.7.1. If the smallest circuit in \( M|Z \) has size \( 2j \), for \( j \in [t-1] \), and \( |Z| \geq h_j(t) \), then either

(i) \( M \) has a \( t \)-coechidna of order \( 4t-3 \) or

(ii) there exists some \( Z'' \subseteq Z \) that is the union of \( h_{j+1}(t) \) pairs of \( Z \) for which the smallest circuit in \( M|Z'' \) has size at least \( 2(j+1) \).

Proof. Let \( 2j \) be the size of the smallest circuit in \( M|Z \). We define \( H \) to be the \( j \)-uniform hypergraph with vertex set \( Z \) whose hyperedges are the \( j \)-subsets of \( Z \) that are partitions of circuits in \( M|Z \). By Theorem 5.6 and the definition of \( h_k \), as \( H \) has at least \( h_j(t) \) vertices, it has either a clique or a stable set, on \( h_{j+1}(t) \) vertices. If \( H \) has a stable set \( Z' \) on \( h_{j+1}(t) \) vertices, then clearly (ii) holds, with \( Z' = \bigcup_{P \in Z} P \).

So we may assume that there is \( h_{j+1}(t) \) pairs in \( Z \) such that the union of any \( j \) of these pairs is a circuit. Let \( Z'' \) be the union of these \( h_{j+1}(t) \) pairs. We claim that the union of any set of \( t \) pairs contained in \( Z'' \) is a cocircuit. Let \( T \) be a transversal of \( t \) pairs of \( Z \) contained in \( Z'' \), and let \( C^* \) be the \( 2t \)-element cocircuit containing \( T \). Towards a contradiction, suppose that there exists some pair \( P \in Z \) with \( P \subseteq Z'' \) such that \( |C^* \cap P| = 1 \). Select \( j-1 \) pairs \( Z''_1, \ldots, Z''_{j-1} \) of \( Z \) that are each contained in \( Z'' - C^* \) (these exist since \( h_{j+1}(t) \geq 3t-1 \geq 2t + j - 1 \). Then \( P \cup (\bigcup_{i \in [j-1]} Z''_i) \) is a circuit that intersects the cocircuit \( C^* \) in a single element, contradicting orthogonality. We deduce that the union of any \( j \) pairs of \( Z \) that are contained in \( Z'' \) is a cocircuit. So \( M \) has a \( t \)-coechidna of order \( h_{j+1}(t) \geq 4t - 3 \), satisfying (i).

We now apply Claim 5.7.1 iteratively, for a maximum of \( t - j \) iterations. If (i) holds, at any iteration, then \( M \) has a \( t \)-coechidna of order \( 4t - 3 \), as required.
Otherwise, we let $Z'$ be the partition of $Z$ induced by $Z$; then, at the next iteration, we relabel $Z = Z'$ and $Z = Z'$. If (ii) holds for each of $t - j$ iterations, then we obtain a subset $Z'$ of $Z$ such that the smallest circuit in $M|Z'$ has size $2t$. Then, by (II), $M$ has a $t$-echidna of order $h_t(t) = 4t - 3$. This completes the proof.

6. Properties of $t$-spikes. In this section, we prove some properties of $t$-spikes, which demonstrate that $t$-spikes form a class of highly structured matroids. In particular, we show that a $t$-spike has order at least $2t - 1$; a $t$-spike of order $r$ has $2r$ elements and rank $r$; the circuits of a $t$-spike that are not a union of $t$ arms meet all but at most $t - 2$ of the arms; and a $t$-spike of order at least $4t - 4$ is $(2t - 1)$-connected. We also show that an appropriate concatenation of the associated partition of a $t$-spike is a $(2t - 1)$-anemone, following the terminology of [1].

It is straightforward to see that the family of $1$-spikes consists of matroids obtained by taking direct sums of copies of $U_{1,2}$. We also describe a construction that can be used to obtain a $(t + 1)$-spike from a $t$-spike, and show that every $(t + 1)$-spike can be constructed from some $t$-spike in this way.

**Basic properties.**

**Lemma 6.1.** Let $M$ be a $t$-spike of order $r$. Then $r \geq 2t - 1$.

**Proof.** Let $(A_1, \ldots, A_r)$ be the associated partition of $M$. By definition, $r \geq t$. Let $J$ be a $t$-element subset of $[r]$, and let $Y = \bigcup_{j \in J} A_j$. Pick some $y \in Y$. Since $Y$ is a cocircuit and a circuit, $Z = (E(M) - Y) \cup \{y\}$ spans and cospans $M$. Since $|Z| = 2(r - t) + 1$,

$$2r = |E(M)| = r(M) + r^*(M) \leq (2(r - t) + 1) + (2(r - t) + 1).$$

It follows that $r \geq 2t - 1$.

**Lemma 6.2.** Let $M$ be a $t$-spike of order $r$. Then $r(M) = r^*(M) = r$.

**Proof.** Let $(A_1, \ldots, A_r)$ be the associated partition of $M$, and label $A_i = \{x_i, y_i\}$ for each $i \in [r]$. Pick $I \subseteq J \subseteq [r]$ such that $|I| = t - 1$ and $|J| = r - t$. Let $X = \bigcup_{i \in I} A_i \cup \{x_j : j \in J\}$, and observe that $|X| = |I| + |J| = r - 1$. Now, $(A_1, \ldots, A_r)$ is a $t$-echidna, $\bigcup_{j \in J} A_j \subseteq cl(X)$. As $E(M) - \bigcup_{j \in J} A_j$ is a cocircuit, we deduce that $r(M) - 1 \leq r(X) \leq |X| = r - 1$, so $r(M) \leq r$. Similarly, as $(A_1, \ldots, A_r)$ is a $t$-cospine, we deduce that $r^*(M) \leq r$. Since $r(M) + r^*(M) = |E(M)| = 2r$, the lemma follows.

The next lemma shows that a circuit $C$ of a $t$-spike is either a union of $t$ arms, or else $C$ meets all but at most $t - 2$ of the arms.

**Lemma 6.3.** Let $M$ be a $t$-spike of order $r$ with associated partition $(A_1, \ldots, A_r)$, and let $C$ be a circuit of $M$. Then either

(i) $C = \bigcup_{j \in J} A_j$ for some $t$-element set $J \subseteq [r]$ or
(ii) $|\{i \in [r] : A_i \cap C \neq \emptyset\}| \geq r - (t - 2)$ and $|\{i \in [r] : A_i \subseteq C\}| < t$.

**Proof.** Let $S = \{i \in [r] : A_i \cap C \neq \emptyset\}$, so $S$ is the minimal subset of $[r]$ such that $C \subseteq \bigcup_{i \in S} A_i$. If $C$ is properly contained in $\bigcup_{i \in S} A_i$, then $C$ is independent: a contradiction. So $|S| \geq t$. If $|S| = t$, then $C = \bigcup_{i \in S} A_i$, implying $C$ is a circuit, which satisfies (i). So we may assume that $|S| > t$. Now $|\{i \in [r] : A_i \subseteq C\}| < t$; otherwise $C$ properly contains a circuit. Thus, there exists some $j \in S$ such that $A_j - C \neq \emptyset$. If $|S| \geq r - (t - 2)$, then (ii) holds; thus we assume that $|S| \leq r - (t - 1)$. Let $T = (|S| - S) \cup \{j\}$. Then $|T| \geq t$, so $\bigcup_{i \in T} A_i$ contains a cocircuit that intersects $C$ in one element, contradicting orthogonality.
Connectivity. Let \( M \) be a matroid with ground set \( E \). Recall that the connectivity function of \( M \), denoted by \( \lambda \), is defined as
\[
\lambda(X) = r(X) + r(E - X) - r(M)
\]
for all subsets \( X \) of \( E \). It is easily verified that
\[
(6.1) \quad \lambda(X) = r(X) + r^*(X) - |X|.
\]

A subset \( X \) or a partition \((X, E - X)\) of \( E \) is \( k \)-separating if \( \lambda(X) < k \). A \( k \)-separating partition \((X, E - X)\) is a \( k \)-separation if \( |X| \geq k \) and \( |E - X| \geq k \). The matroid \( M \) is \( n \)-connected if, for all \( k < n \), it has no \( k \)-separations.

Lemma 6.4. Suppose \( M \) is a \( t \)-spike with associated partition \((A_1, \ldots, A_r)\). Then, for all partitions \((J, K)\) of \([r]\) with \(|J| \leq |K|\),
\[
\lambda \left( \bigcup_{j \in J} A_j \right) = \begin{cases} 2|J| & \text{if } |J| < t, \\ 2t - 2 & \text{if } |J| \geq t. \end{cases}
\]

Proof. Let \((J, K)\) be a partition of \([r]\) with \(|J| \leq |K|\).

Claim 6.4.1. The lemma holds when \(|J| \leq t\).

Proof. Suppose \(|J| < t\). Since \((A_1, \ldots, A_r)\) is a \( t \)-echidna (respectively, \( t \)-coechidna), \( \bigcup_{j \in J} A_j \) is independent (respectively, coindependent). So, by (6.1), \( \lambda(\bigcup_{j \in J} A_j) = 2|J| + 2|J| - 2|J| = 2|J| \).

Now suppose \(|J| = t\). Then, by definition, \( \bigcup_{j \in J} A_j \) is a circuit and a cocircuit. So \( \lambda(\bigcup_{j \in J} A_j) = (2t - 1) + (2t - 1) - 2t = 2t - 2 \), by (6.1). \( \square \)

Claim 6.4.2. Let \( X \subseteq Y \subseteq [r] \) such that \(|X| \geq t - 1\). Then
\[
\lambda \left( \bigcup_{x \in X} A_x \right) \geq \lambda \left( \bigcup_{y \in Y} A_y \right).
\]

Proof. Let \( X' \) be a \((t - 1)\)-element subset of \( X \), and let \( y \in Y - X \). Then \( \lambda(\bigcup_{x \in X'} A_x) = 2(t - 1) \), and \( \lambda(A_y \cup (\bigcup_{x \in X'} A_x)) = 2t - 2 \), by Claim 6.4.1. By submodularity of the connectivity function,
\[
\lambda \left( A_y \cup \bigcup_{x \in X} A_x \right) \leq \lambda \left( A_y \cup \bigcup_{x \in X'} A_x \right) + \lambda \left( \bigcup_{x \in X} A_x \right) - \lambda \left( \bigcup_{x \in X'} A_x \right)
\]
\[
= (2t - 2) + \lambda \left( \bigcup_{x \in X} A_x \right) - (2t - 2)
\]
\[
= \lambda \left( \bigcup_{x \in X} A_x \right).
\]

Claim 6.4.2 now follows by induction. \( \square \)

Now suppose \(|J| > t\). By Claims 6.4.1 and 6.4.2, \( \lambda(\bigcup_{j \in J} A_j) \leq 2t - 2 \). Recall that \(|K| \geq |J| > t\). Let \( K' \) be a \( t \)-element subset of \( K \). Let \( J' = [r] - K' \), and note that \( J \subseteq J' \). So, by Claim 6.4.2,
\[
\lambda \left( \bigcup_{j \in J} A_j \right) \geq \lambda \left( \bigcup_{j \in J'} A_j \right) = \lambda \left( \bigcup_{k \in K'} A_k \right) = 2t - 2.
\]
We deduce that \( \lambda(\bigcup_{j \in I} A_j) = 2t - 2 \), as required.

Given a \( t \)-spike \( M \) with associated partition \((A_1, \ldots, A_r)\), suppose that \((P_1, \ldots, P_m)\) is a partition of \( E(M) \) such that, for each \( i \in [m] \), \( P_i = \bigcup_{j \in I} A_i \) for some subset \( I \) of \([r] \), with \( |P_i| \geq 2t - 2 \). Using the terminology of \([1]\), it follows immediately from Lemma 6.4 that \((P_1, \ldots, P_m)\) is a \((2t - 1)\)-anemone. (Note that a partition whose concatenations give rise to a flower in this way has previously appeared in the literature \([3]\) under the name of “quasi-flowers.”)

**Lemma 6.5.** Let \( M \) be a \( t \)-spike of order at least \( 4t - 4 \), for \( t \geq 2 \). Then \( M \) is \((2t - 1)\)-connected.

**Proof.** Let \( r \) be the order of the \( t \)-spike \( M \), and let \((A_1, \ldots, A_r)\) be the associated partition of \( M \). Towards a contradiction, suppose \( M \) is not \((2t - 1)\)-connected, and let \((P, Q)\) be a \( k \)-separation for some \( k < 2t - 1 \). Without loss of generality, we may assume that \( |P| \geq |Q| \). Note, in particular, that \( \lambda(P) < k \leq |Q| \) and \( \lambda(P) < 2t - 2 \).

Suppose \( |P \cap A_j| \neq 1 \) for all \( j \in [r] \). Then, by Lemma 6.4, \( \lambda(P) = |Q| \) if \( |Q| < 2t \), otherwise \( \lambda(P) = 2t - 2 \); either case is contradictory. So \( |P \cap A_j| = 1 \) for some \( j \in [r] \).

Suppose \( |Q| \leq 2t - 2 \). Then, by Lemma 6.3 and its dual, \( Q \) is independent and co-independent, so \( \lambda(P) = |Q| \) by (6.1); a contradiction.

Now we may assume that \( |Q| > 2t - 2 \). Suppose \( \bigcup_{j \in I} A_j \subseteq P \), for some \((t - 1)\)-element set \( I \subseteq [r] \). Then \( A_j \subseteq \text{cl}(P) \) for each \( j \in [r] \) such that \( |P \cap A_j| = 1 \). For such a \( j \), it follows, by the definition of \( \lambda \), that \( \lambda(P \cup A_j) \leq \lambda(P) \); we use this repeatedly in what follows. Let \( U = \{u \in [r] : |P \cap A_u| = 1\} \). For any subset \( U' \subseteq U \), we have \( \lambda(P \cup \bigcup_{u \in U'} A_u) \leq \lambda(P) < 2t - 2 \). Let \( P' = P \cup \bigcup_{u \in U} A_u \), and let \( Q' = E(M) - P' \). If \( |Q'| > 2t - 2 \), then \( \lambda(P') = 2t - 2 \) by Lemma 6.4, contradicting that \( \lambda(P') \leq \lambda(P) < 2t - 2 \). So \( |Q'| \leq 2t - 2 \). Now, let \( d = |Q' - (2t - 2) \), and let \( U' \) be a \( d \)-element subset of \( U \). Then \( \lambda(P) \geq \lambda(P \cup \bigcup_{u \in U'} A_u) = \lambda(Q - \bigcup_{u \in U'} A_u) \). Since \( |Q - \bigcup_{u \in U'} A_u| = 2t - 2 \), we have that \( \lambda(Q - \bigcup_{u \in U'} A_u) = 2t - 2 \), so \( \lambda(P) \geq 2t - 2 \); a contradiction. We deduce that \( |\{i \in [r] : A_i \subseteq P\}| < t - 1 \). Since \( |Q| \leq |P| \), it follows that \( |\{i \in [r] : A_i \subseteq Q\}| \leq |\{i \in [r] : A_i \subseteq P\}| < t - 1 \).

Now \( |\{i \in [r] : A_i \cap Q \neq \emptyset\}| \geq r - (t - 2) \), so \( r(Q) \geq r - (t - 1) \) by Lemma 6.3. Similarly, \( r(P) \geq r - (t - 1) \). So

\[
\lambda(P) = r(P) + r(Q) - r(M) \\
\geq (r - (t - 1)) + (r - (t - 1)) - r \\
\geq (4t - 4) - 2(t - 1) = 2t - 2;
\]

a contradiction. This completes the proof.

**Constructions.** We first describe a construction that can be used to obtain a \((t + 1)\)-spike of order \( r \) from a \( t \)-spike of order \( r \), when \( r \geq 2t + 1 \). We then show that every \((t + 1)\)-spike can be constructed from some \( t \)-spike in this way.

Recall that \( M_1 \) is an elementary quotient of \( M_0 \) if there is a single-element extension \( M_0^+ \) of \( M_0 \) by an element \( e \) such that \( M_1 = M_0^+/e \). A matroid \( M_1 \) is an elementary lift of \( M_0 \) if \( M_0^+ \) is an elementary quotient of \( M_0 \). Note also that if \( M_1 \) is an elementary quotient of \( M_0 \), then \( M_0 \) is an elementary lift of \( M_1 \).

Let \( M_0 \) be a \( t \)-spike of order \( r \geq 2t + 1 \) with associated partition \( \pi \). Let \( M_0^\dagger \) be an elementary quotient of \( M_0 \) such that none of the \( 2t \)-element cocircuits are preserved (that is, extend \( M_0 \) by an element \( e \) that blocks all of the \( 2t \)-element cocircuits, and then contract \( e \)). Now, in \( M_0^\dagger \), the union of any \( t \) cells of \( \pi \) is still a \( 2t \)-element circuit, but, as \( r(M_0^\dagger) = r(M_0) - 1 \), the union of any \( t + 1 \) cells of \( \pi \) is a \((2t + 1)\)-element
circuit. We then repeat this in the dual; that is, let $M_1$ be an elementary lift of $M_0$ such that none of the 2$t$-element circuits are preserved. Then $M_1$ is a $(t + 1)$-spike. Note that $M_1$ is not unique; more than one $(t + 1)$-spike can be constructed from a given $t$-spike $M_0$ in this way.

Given a $(t + 1)$-spike $M_1$, for some positive integer $t$, we now describe how to obtain a $t$-spike $M_0$ from $M_1$ by a specific elementary quotient, followed by a specific elementary lift. This process reverses the construction from the previous paragraph. The next lemma describes the single-element extension (or coextension, in the dual) that gives rise to the elementary quotient (or lift) we desire. Intuitively, the extension adds a “tip” to a $t$-echidna. In the proof of this lemma, we assume knowledge of the theory of modular cuts (see [6, section 7.2]).

**Lemma 6.6.** Let $M$ be a matroid with a $t$-echidna $\pi = (S_1, \ldots, S_n)$. Then there is a single-element extension $M^+$ of $M$ by an element $e$ such that $e \in \text{cl}_{M^+}(X)$ if and only if $X$ contains at least $t - 1$ spines of $\pi$ for all $X \subseteq E(M)$.

**Proof.** Let

$$F = \left\{ \bigcup_{i \in I} S_i : I \subseteq [n] \text{ and } |I| = t - 1 \right\}.$$  

By the definition of a $t$-echidna, $F$ is a collection of flats of $M$. Let $\mathcal{M}$ be the set of all flats of $M$ containing some flat $F \in F$. We claim that $\mathcal{M}$ is a modular cut. Recall that, for distinct $F_1, F_2 \in \mathcal{M}$, the pair $(F_1, F_2)$ is modular if $r(F_1) + r(F_2) = r(F_1 \cup F_2) + r(F_1 \cap F_2)$. It suffices to prove that for any $F_1, F_2 \in M$ such that $(F_1, F_2)$ is a modular pair, $F_1 \cap F_2 \in \mathcal{M}$.

For any $F \in \mathcal{M}$, since $F$ contains at least $t - 1$ spines of $\pi$, and the union of any $t$ spines is a circuit (by the definition of a $t$-echidna), it follows that $F$ is a union of spines of $\pi$. So let $F_1, F_2 \in \mathcal{M}$ such that $F_1 = \bigcup_{i \in I_1} S_i$ and $F_2 = \bigcup_{i \in I_2} S_i$, where $I_1$ and $I_2$ are distinct subsets of $[n]$ with $u_1 = |I_1| \geq t - 1$ and $u_2 = |I_2| \geq t - 1$. Then

$$r(F_1) + r(F_2) = (t - 1 + u_1) + (t - 1 + u_2) = 2(t - 1) + u_1 + u_2.$$  

Suppose that $|I_1 \cap I_2| < t - 1$. Let $s = |I_1 \cap I_2|$. Then $F_1 \cup F_2$ is the union of $u_1 + u_2 - s \geq t - 1$ spines of $\pi$. So

$$r(F_1 \cup F_2) + r(F_1 \cap F_2) = (t - 1 + (u_1 + u_2 - s)) + 2s = (t - 1) + s + u_1 + u_2.$$  

Since $s < t - 1$, it follows that $r(F_1 \cup F_2) + r(F_1 \cap F_2) < r(F_1) + r(F_2)$. So, for every modular pair $(F_1, F_2)$ with $F_1, F_2 \in \mathcal{M}$, we have $|I_1 \cap I_2| \geq t - 1$, in which case $F_1 \cap F_2$ is a flat containing the union of $t - 1$ spines of $\pi$, and hence $F_1 \cap F_2 \in \mathcal{M}$ as required.

Now, there is a single-element extension corresponding to the modular cut $\mathcal{M}$, and this extension satisfies the requirements of the lemma (see, for example, [6, Theorem 7.2.3]).

Let $M$ be a $t$-spike with associated partition $\pi = (A_1, \ldots, A_r)$, for some integer $t \geq 2$, where $r \geq 2t - 1$ by Lemma 6.1. Let $M^+$ be the single-element extension of $M$ by an element $e$ described in Lemma 6.6.

Consider $M^+/e$. We claim that $\pi$ is a $(t - 1)$-echidna and a $t$-coechidna of $M^+/e$. Let $X$ be the union of any $(t - 1)$ spines of $\pi$. Then $X$ is independent in $M$, and $X \cup \{e\}$ is a circuit in $M^+$, so $X$ is a circuit in $M^+/e$. So $\pi$ is a $(t - 1)$-echidna of $M^+/e$.  

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Now let $C^*$ be the union of any $t$ spines of $\pi$, and let $H = E(M) - C^*$. Then $H$ is the union of at least $t - 1$ spines, so $e \in cl_{M^+}(H)$. Now $H \cup \{e\}$ is a hyperplane in $M^+$, so $C^*$ is a cocircuit in $M^+$. Hence $\pi$ is a $t$-coechidna of $M^+/e$.

We now repeat this process on $N = (M^+/e)^*$. In $N$, the partition $\pi$ is a $t$-echidna and $(t - 1)$-coechidna. By Lemma 6.6, there is a single-element extension $N^+$ of $N$ (a single-element coextension of $M^+/e$) by an element $e'$. By the same argument as in the previous paragraph, $\pi$ is a $(t - 1)$-echidna and $(t - 1)$-coechidna of $N^+/e$, so $N^+/e$ is a $(t - 1)$-spike. Let $M' = (N^+/e)^*$.

Note that $M^+/e$ is an elementary quotient of $M$, so $M$ is an elementary lift of $M^+/e$ where none of the $2(t - 1)$-element circuits of $M^+/e$ are preserved in $M$. Similarly, $M^+/e$ is an elementary quotient of $M'$ where none of the $2(t - 1)$-element cocircuits are preserved. So the $t$-spike $M$ can be obtained from the $(t - 1)$-spike $M'$ using the earlier construction.

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