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Energy analysis of the Von Schlippe tyre model with application to shimmy

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Shimmy is an engineering example of self-excited vibrations. Much research on shimmy has considered the tyre as a positive feedback or negative damping to introduce instability of the entire system. In this context, we focus on the behaviour of the tyre under periodic excitations. The Von Schlippe tyre model is selected and the energy flow method is applied to illustrate the energy transfer by the tyre during shimmy. The energy flow method evaluates the tyre performance with a prescribed sinusoidal motion and provides a novel evaluation method for tyre models. With the help of straight contact line assumption in the Von Schlippe tyre model, the relative motion between the contact line and the wheel centre is studied to understand the path dependency of the energy transfer. It turns out that the tyre is extracting energy from the forward motion to induce unstable lateral and yaw vibrations when the motion or orientation of the contact line has a phase lead with respect to the wheel centre.

Keywords: shimmy; Von Schlippe tyre model; energy flow method; tyre motion

1. Introduction

In nature and engineering, sustained periodic motions exist in both mechanical systems and other dynamical systems, which are caused by the interaction among the elements in the system. The generation, maintaining and quenching of this periodic motion, both useful and destructive, has attracted lots of research and applications, with a diversity of terminology, such as self-excited vibration, self-oscillation and hunting. Self-excited vibration is manifested as the instability of a dynamical system’s static equilibrium. The amplitude of the oscillation of an unstable system grows with time, until nonlinearities become important and limit the amplitude. Finally, a steady and sustained oscillation is produced.

Jenkins carried out an excellent review on research of self-excited vibration with various fields of application in [1]. It emphasises the energetics of self-excited vibration and provides impressive examples from mechanical engineering, music, biology, electronics and medicine, showing that much technology ultimately depends on self-excited vibration, since it is the only way that can turn a steady source of power into a regular periodic motion. Ding [2] summarised the theory and research method of self-excited vibrations, and analysed several examples in engineering. Both [1,2] point out that a general explanation of self-excited

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vibrations is the negative damping, which is caused by a positive feedback between the oscillation and the external source of power. It introduces instability of the linearised system and keeps drawing energy from its surroundings, until the amplitude is limited by nonlinearities. In the sustained periodic motion, the energy fed into the system has to compensate the energy that is dissipated, indicating that the flow of energy can be considered as a mechanism of self-excited vibration.

As an example of self-excited vibrations in engineering, shimmy is an oscillatory, combined lateral–yaw motion of an aircraft landing gear or an automotive steering system, which is caused by the interaction of the dynamic behaviour of the structure and tyre. Pacejka [3] and Besselink [4] investigated the shimmy phenomenon in detail and lots of attention have been paid on the tyre dynamic behaviour and its impact on the stability. The study of shimmy is still challenging and appealing, while numerous publications appear in recent years [5–7]. Not surprisingly, methods considering an energy point of view are also applied in shimmy research. Katayama and Nishimi used the concept of energy flow from acoustic engineering to study the motorcycle wobble mode, which resembles the shimmy phenomenon [8]. The energy flowing into the steering system was considered coming from different so-called external torque channels. Each component can be determined by numerical simulations or eigenvector analysis. The stability estimation shows that the vibrational energy is dissipated by the total tyre torque which is one of the channels corresponding to the tyre behaviour. Kovacs generalised this method to analyse shimmy for a mid-size truck with the focus on a computational analysis tool that measures the power and corresponding work in dissipating the excitation energy [9]. Two designs were developed with components that effectively dissipate energy to suppress shimmy both in simulations and a test vehicle. It is of interest that the tyre energy is considered to be only associated with the self-aligning moment and the yaw vibration of a worn tyre adds energy to the system due to the relaxation behaviour. The author also claims that a new tyre generally dissipates energy.

Although the concept of energy flow in [8,9] is useful in shimmy analysis, the role of the tyre has been underestimated since relatively simple tyre models are used in the research and the idea of adding and dissipating energy through tyres is not clarified. In fact, even a new tyre may be responsible for introducing shimmy. Besselink confirmed this by the energy flow method, as discussed in [4]. Although the name and some concepts of this method are similar to that proposed in [8], Besselink developed this method independently by only focusing on the tyre behaviour. The tyre moves with prescribed sinusoidal lateral and yaw motion with different amplitudes and phases, and produces lateral force and self-aligning moment as a response. The work that is done by these force and moment in one time period can be used to analyse if the tyre is feeding energy into the lateral–yaw motion or vice versa. Various linear tyre models used in shimmy analysis are compared in [4], and the results show that the zero energy boundaries are circles in a polar plot with different centres and radii, depending on the tyre models, parameters and path wavelength. The positive energy area inside the cycle indicates that for this special input it is possible for the tyre to drive an unstable vibration. Pacejka presented this method to investigate shimmy also in his book [10]; he also generalised it to examine the transformation of forward energy into shimmy energy, with the assumption that the rate of the supplied driving energy is equal to the sum of the change of self-excitation energy, tyre potential energy and tyre dissipation energy. More details of the work of Besselink and Pacejka will also be discussed in this paper. Guo and Liu [11] applied a similar method based on an energy balance to study the influences of non-steady-state tyre cornering properties on automobile shimmy. The tyre model has been developed under the so-called entry/adhesive condition, and translation, bending and twisting of the carcass were considered. The energy was calculated using transfer functions, however, only the yaw motion of the wheel was considered in their research.
Although the shimmy phenomenon is of great concern and has been investigated for a long time, the theory and mechanisms behind it are still ambiguous. It is agreed that the tyre plays an essential role in shimmy and therefore different types of dynamic tyre models are developed and applied to study the shimmy phenomenon. Most of the previous research in this field focuses on stability and simulation methods of the entire wheel/suspension assemblies or landing gear systems. The tyre component, in this approach, is usually considered as a feedback loop of the entire system which introduces instability by pumping energy into the system. However, it is for sure that the tyre is a passive element that never generates energy. There remains a need for a theory that explains how the tyre interacts with other components in the system to introduce instability. Within this paper, we study the behaviour of an elastic tyre in the case of periodic motions and attempt to understand the contribution of tyre behaviour during shimmy from an energy and mechanical point of view.

This paper is organised as follows: First, the Von Schlippe tyre model is briefly described in Section 2. In Section 3, the energy flow method is evaluated in detail; the shimmy energy and its frequency response are presented to support the energy criterion. After that attention is paid to the motion of the contact line during a periodic vibration. The leading motion of the contact line moving along a path determined by the predefined lateral–yaw input explains how energy is extracted from the forward motion and its path dependency.

2. Von Schlippe tyre model

In the study of the shimmy phenomenon, probably the most popular tyre model is the stretched string tyre model with a finite contact length, which was introduced by Von Schlippe in [12]. Since then its derivatives with various approximations and levels of complexity of modelling the string have been introduced. Pacejka used the leading contact point to approximate the string deflection, which is referred as the straight tangent model [3]. An excellent approximation with good accuracy by Von Schlippe considers the contact line by connecting a straight line between the leading and the trailing contact points [12], causing a pure time delay in the model equations. Segel derived the frequency response characteristics without any restriction to the shape of the contact line, which leads to exact solutions of the stretched string model [13]. In more recent research, Takács et al. described the delay or memory effect of the string-like tyre with a delay differential equation [14], which corresponds to the exact solution of Segel; in their later study, the sliding zone of the contact line was also taken into account and resulted in micro-shimmy vibration [15], but an analytical solution becomes impossible. Detailed descriptions and comparisons of many variations of the stretched string model have been conducted by Besselink [4] and Pacejka [10]. In practical cases, shimmy usually occurs at a path wavelength which is at least 25 times larger than half of the contact length. The comparison of these various tyre models shows a reasonable agreement with each other, while the Von Schlippe approximation can hardly be distinguished from the exact solution in the practical shimmy frequency range.

Due to the fact that the Von Schlippe model excellently matches with the exact solution, being easy to use in vehicle simulation, and having a clear physical interpretation of the deformation at the contact line, it is the tyre model of choice in this study.

2.1. Model description

A top view of the stretched string and its Von Schlippe approximation is given in Figure 1. In this model, the tyre is considered as a massless string under a constant pre-tension force and
it is uniformly supported elastically in the lateral direction. The string contacts the road over a length of $2a$; for this contact region, the assumption is made that no sliding occurs with respect to the road. Furthermore, it is assumed that the angles under consideration remain small. The kinematic constraints of the rolling tyre imply that the string in the contact region follows the leading contact point, resulting in a retarding behaviour of points on the ground contact line. The lateral deflection of the string outside the contact line will gradually tend to zero. Since no bending stiffness is considered in the string, a kink in its lateral deformation will only occur at the aft contact point. In the Von Schlippe model, a straight line connecting the leading and aft contact points approximates the contact line. The lateral force and self-aligning moment applied to the rim can be determined by integrating the lateral tyre deflection of the string with respect to the rim.

The governing equations of the Von Schlippe model will be given in this section; for more details reference is made to [10]. The lateral position of the leading point with respect to the ground for the stretched string models is given as

$$\frac{\sigma}{V} \dot{y}_1 + y_1 = y_c + (\sigma + a)\psi. \quad (1)$$

In the Von Schlippe model, due to the delay effect and non-sliding assumption, the lateral position of the aft contact point will be identical to the leading contact point, once the tyre has moved a distance $2a$ forward. In the time domain, this equation becomes

$$y_2(t) = y_1 \left( t - \frac{2a}{V} \right), \quad (2)$$

where $y_1$ and $y_2$ are the lateral position of the leading and aft points of the contact line in the ground frame, $\sigma$ is the relaxation length of tyre, $V$ is the forward velocity, $y_c$ and $\psi$ are the lateral and yaw motion at the wheel centre.
The lateral force $F_y$ and self-aligning moment $M_z$ can be determined from the deflection of the contact patch with respect to the wheel plane:

\[
F_y = c_v \left( \frac{v_1 + v_2}{2} \right) = c_v \left( \frac{y_1 + y_2}{2} - y_c \right),
\]

\[
M_z = c_\beta \left( \frac{v_1 - v_2}{2a} \right) = c_\beta \left( \frac{y_1 - y_2}{2a} - \psi \right),
\]

In these equations $v_1$ and $v_2$ are the lateral deflections of leading and aft points of the contact line, $c_v$ and $c_\beta$ represent the lateral and yaw stiffness of the contact line, all with respect to the wheel plane. For a straight contact line, the relation with the distributed stiffness carcass stiffness per unit length $c_c$ of the stretched string is given by

\[
c_v = 2c_c(\sigma + a),
\]

\[
c_\beta = 2c_c a \left( \sigma (\sigma + a) + \frac{1}{3}a^2 \right).
\]

The lateral force and self-aligning moment can also be expressed as functions of the side slip angle $\alpha$ when considering steady-state side slip conditions. For small values of $y_c$ and $\psi$, we have

\[
\alpha = \psi - \frac{\dot{y}_c}{V},
\]

and

\[
F_y = C_{Fa} \alpha,
\]

\[
M_z = C_{Ma} \alpha,
\]

where $C_{Fa}$ is the cornering stiffness and $C_{Ma}$ is the self-aligning stiffness. In the case of the Von Schlippe model, they are

\[
C_{Fa} = c_v(\sigma + a),
\]

\[
C_{Ma} = -c_\beta.
\]

### 2.2. Transfer functions of the Von Schlippe model

From Equations (1) to (7), the transfer functions of the lateral force and self-aligning moment for the Von Schlippe tyre model with respect to the lateral and yaw motion at the wheel centre, can be determined:

\[
\begin{bmatrix}
F_y(s) \\
M_z(s)
\end{bmatrix} =
\begin{bmatrix}
H_{11}(s) & H_{12}(s) \\
H_{21}(s) & H_{22}(s)
\end{bmatrix}
\begin{bmatrix}
y_c(s) \\
\psi(s)
\end{bmatrix},
\]

where $s$ is the Laplace variable and the transfer function matrix reads

\[
H_{11}(s) = H_{F_y,y_c}(s) = C_{Fa} \left( \frac{2s(V/s) + 1 - e^{-2as/V}}{2(\sigma + a)(\sigma(s/V) + 1)} \right),
\]

\[
H_{12}(s) = H_{F_y,\psi}(s) = C_{Fa} \left( \frac{1 + e^{-2as/V}}{2(\sigma(s/V) + 1)} \right),
\]

\[
H_{21}(s) = H_{M_z,y_c}(s) = C_{Ma} \left( \frac{2s(V/s) + 1 - e^{-2as/V}}{2(\sigma + a)(\sigma(s/V) + 1)} \right),
\]

\[
H_{22}(s) = H_{M_z,\psi}(s) = C_{Ma} \left( \frac{1 + e^{-2as/V}}{2(\sigma(s/V) + 1)} \right).
\]
H_{21}(s) = H_{M_{z}, \psi}(s) = C_{Ma} \left( \frac{1 - e^{-2a/s/V}}{2a(\sigma(s/V) + 1)} \right),

H_{22}(s) = H_{M_{z}, \psi}(s) = C_{Ma} \left( \frac{2a\sigma(s/V) - (\sigma - a) + (\sigma + a)e^{-2a/s/V}}{2a(\sigma(s/V) + 1)} \right).

The frequency response functions for the force and moment constitute the response to sinusoidal motion of the wheel and can be easily obtained by replacing the Laplace variable $s$ by $j\omega$ in the transfer functions, where $\omega$ is the frequency of the input motion. As tyre behaviour is essentially path dependent when ignoring tyre inertia, the path wavelength $\lambda$ is introduced ($\lambda = 2\pi V/\omega$), such that the above frequency responses are functions of $\lambda$ and the tyre parameters. In Figures 2 and 3, these transfer functions are shown with respect to the normalised path wavelength. The parameters of the Von Schlippe tyre model are listed in Appendix, and the same set of parameters will be used in the remainders of this article.

3. Energy flow method

The energy flow method, initially proposed by Besselink in [4], was used to evaluate the performance of tyre models for shimmy analysis under sinusoidal input and was generalised by Pacejka in [10]. Similar to other energy methods used to analyse self-excited vibrations, the sustained periodic motion needs energy transmitted from the surroundings to the system. In the case of shimmy, the power to drive the lateral and yaw motion can only originate from the vehicle’s propulsion system, indicating a transformation of energy from forward motion into shimmy oscillations via the tyre.

The tyre is considered to roll freely, without driving or braking torques, and travel at a constant velocity $V$. The forces and movement of the Von Schlippe tyre model are presented in Figure 4, where $F_y$ and $M_z$ are the lateral force and self-aligning moment at the contact patch discussed in Section 2.1; at the wheel centre $F_d$ is the driving force from the propulsion system and further $F_a$ and $M_d$ are the resulting force and yaw moment exerted from axle to wheel.
3.1. Energy transfer in shimmy vibration

A basic assumption of the stretched string tyre model is that no sliding occurs between the string and the road, so no energy is lost at the contact line. The work that has been done by the force and moment at the wheel centre causes the change of energy inside the tyre in a small time step $\Delta t$:

$$F_d \dot{y}_c \Delta t + F_d \dot{x}_c \Delta t + M_d \dot{\psi} \Delta t = \Delta U + \Delta E_k. \quad (9)$$
Here, $\Delta U$ is the change of potential energy due to the elastic deformation of the carcass, and $\Delta E_k$ is the change of kinetic energy of the tyre.

Without considering the mass and inertia effects, the force and moment equilibrium reads

$$F_a = -F_y \cos \psi,$$
$$M_a = -M_z.$$  \hfill (10)

Substituting Equation (10) into Equation (9) leads to

$$F_d \dot{x}_c \Delta t = F_y \cos \psi \dot{y}_c \Delta t + M_z \dot{\psi} \Delta t + \Delta U + \Delta E_k.$$  \hfill (11)

Integration over one cycle of the periodic oscillation with constant amplitude will eliminate both $\Delta U$ and $\Delta E_k$ due to the periodicity of the displacement and velocity. Consequently, the expression reduces to

$$\int_0^T F_d \dot{x}_c \, dt = \int_0^T (F_y \cos \psi \dot{y}_c + M_z \dot{\psi}) \, dt,$$  \hfill (12)

where $T$ is the period of one shimmy vibration. Since the yaw angle $\psi$ is assumed to be small in the Von Schlippe model, we set $\cos \psi \approx 1$ and define the shimmy energy as

$$W = \int_0^T (F_y \dot{y}_c + M_z \dot{\psi}) \, dt.$$  \hfill (13)

It should be mentioned that the definition of the shimmy energy $W$ is only valid for periodic oscillations with a constant amplitude and can be calculated over one period of vibration $T$ starting at an arbitrary point in time. Equation (12) implies that when the shimmy energy is positive, energy is extracted from the forward motion to drive the sustained lateral and yaw vibrations; or energy goes into the forward motion through the tyre when $W$ is negative.

### 3.2. Energy criterion of shimmy vibration

In the previous section, we have introduced the definition of shimmy energy, which is crucial to understand the energy flow in shimmy vibrations. In order to compare and evaluate the performance of tyres using the shimmy energy, the motion of the wheel centre has to be specified further: not only periodic, but also sinusoidal, which is common in the shimmy phenomenon. Without examining the whole system with structure or suspension, the tyre is presumed to undertake a prescribed sinusoidal lateral and yaw input with the same radial frequency $\omega$ but different amplitudes and phases

$$y_c(t) = A\eta \sin(\omega t + \xi),$$
$$\psi(t) = A \sin(\omega t).$$  \hfill (14)

In these equations, $A$ is the amplitude of the motion, $\eta$ is the amplitude ratio. If $\eta$ is zero, only a yaw input is considered; with increasing $\eta$ the relative magnitude of the lateral motion increases. The relative phase angle $\xi$ indicates the phase lead of the lateral motion with respect to the yaw motion.

The calculation of shimmy energy with a sinusoidal input will be determined using the frequency response functions derived in Section 2.2. First, we designate the absolute value
and phase angle of the frequency response functions as

\[ H_{mn} = |H_{mn}(j\omega)|, \theta_{mn} = \angle H_{mn}(j\omega); \quad m, n = 1, 2. \]

The steady-state responses of force and moment with the sinusoidal input in Equation (14) are also harmonic waves with the same frequency but different amplitudes and phases. As a result, the shimmy energy can be calculated as follows:

\[ W = W_{11} + W_{12} + W_{21} + W_{22}, \quad (15) \]

where

\[ W_{11} = \int_0^T (A\eta H_{11} \sin(\omega t + \xi + \theta_{11}) \cdot A\omega \eta \cos(\omega t + \xi)) \, dt = \pi \eta^2 A^2 H_{11} \sin \theta_{11}, \]

\[ W_{12} = \int_0^T (AH_{12} \sin(\omega t + \theta_{12}) \cdot A\omega \eta \cos(\omega t + \xi)) \, dt = -\pi \eta A^2 H_{12} \sin(\xi - \theta_{12}), \]

\[ W_{21} = \int_0^T (A\eta H_{21} \sin(\omega t + \xi + \theta_{21}) \cdot A\omega \cos(\omega t)) \, dt = \pi \eta A^2 H_{21} \sin(\xi + \theta_{21}), \]

\[ W_{22} = \int_0^T (AH_{22} \sin(\omega t + \theta_{22}) \cdot A\omega \cos(\omega t)) \, dt = \pi A^2 H_{22} \sin \theta_{22} \]

are the components of the shimmy energy determined from the integration over one time period of the product of the force or moment due to lateral or yaw input motion and the corresponding velocity.

3.2.1. Case I: pure yaw input

First, let us consider the simple case where only yaw input of the wheel is present \((\eta = 0)\). Then, the sign of the shimmy energy

\[ W = W_{22} = \pi A^2 H_{22} \sin \theta_{22} \quad (16) \]

only depends on the angle \(\theta_{22}\) of the corresponding frequency response function \(H_{22}\), which is a function of the wavelength \(\lambda\). Therefore, the shimmy energy can also be expressed by the dimensionless wavelength \(a/\lambda\) as shown in Figure 5. The critical path wavelength \(\lambda_0\) is also marked, at which the shimmy energy passes through zero. It only exits for the pure yaw input and the shimmy energy is positive if the wavelength is larger than it. This indicates that the tyre is transferring energy from the forward motion into the yaw motion. The shimmy energy changes quadratically with the amplitude of the input motion (see Equation 16); the larger the shimmy energy is, the more the tendency of shimmy will occur. The shimmy energy is negative for short wavelengths, and the tyre is extracting energy from yaw motion. Comparing to the frequency response function in Figure 3, where the critical path wavelength is also displayed, apparently \(\lambda_0\) is the wavelength that distinguishes whether the phase of self-aligning moment has a leading or lag with respect to the yaw motion. It is concluded that for the Von Schlippe tyre model, shimmy can only occur at relatively large wavelength, which also agrees with the knowledge of shimmy frequency in practice.
3.2.2. Case II: pure lateral input

In case of only lateral input, all the energy components vanish except $W_{11}$. The shimmy energy with respect to the wavelength is also depicted in Figure 5. It persists to be negative over the practical wavelength range. This consists with the frequency response in Figure 2 that the lateral force always delays against lateral motion in phase. In this way, we confirm that the tyre will never provoke shimmy when only a lateral degree of freedom is present.

3.2.3. Zero energy boundary

In the above case studies, the frequency responses of the shimmy energy for pure yaw and pure lateral input are discussed. It is found that in order to maintain the prescribed sinusoidal motion, the tyre acts as an energy transmitter between the forward motion and the lateral–yaw motion. However when shimmy in a landing gear or an automotive steering system arises, the motion of tyre consists of lateral and yaw movements, which are coupled by the dynamics of the suspension structure. Accordingly, the coupling terms of shimmy energy $W_{12}$ and $W_{21}$ start to influence the total energy balance and the combination of lateral and yaw motion becomes more important for determining the tyre behaviour with energy transfer. Instead of examining the individual energy components with respect to the wavelength, it is more efficient to focus on the zero energy boundary with different amplitude ratio $\eta$ and phase difference $\xi$. If the sinusoidal inputs in Equation (14) are presumed, Besselink proved the zero energy dissipation per cycle of the straight tangent tyre model displays as a circle in a polar plot, where the distance to the origin is $\eta$ and the angle to the positive $x$-axis is represented as $\xi$ [4].

With the shimmy energy expression of the Von Schlippe model in Equation (15), the zero energy boundary in the polar plot is solved by setting $W$ equals to zero which gives

$$\eta = \frac{H_{12} \sin(\xi - \theta_{12}) - H_{21} \sin(\xi + \theta_{21}) \pm \sqrt{\Delta}}{2H_{11} \sin \theta_{11}}, \quad (17)$$

where

$$\Delta = (H_{12} \sin(\xi - \theta_{12}) - H_{21} \sin(\xi + \theta_{21}))^2 - 4H_{11}H_{22} \sin \theta_{11} \sin \theta_{22}.$$ 

These boundaries for different wavelengths are presented in Figure 6. Inside these circles $W$ is greater than zero and energy is fed into the lateral–yaw motion: shimmy is possible to occur. The origin of the plot where $\eta = 0$ corresponds to the case of a pure yaw input. For
the large wavelength \((\lambda = 40a)\), the origin is inside the circle, and it moves outside when the wavelength becomes short \((\lambda = 10a)\). The critical path wavelength \(\lambda_0\) is approximately 20a and its circle almost pass the origin. It is consistent with the observation of the critical wavelength in Figure 5 as the transition point of stability.

The case of pure lateral input can be approximated by letting \(\eta\) go to infinity which is always outside the boundary circle. This also agrees with the previous case study showing that the tyre is extracting energy from the lateral motion and making the system less tendency to shimmy when only the lateral input is considered.

### 3.3. Summary of energy flow method

By means of evaluating the performance of the tyre under periodic motion, the energy flow method demonstrates that the energy can be transferred between the forward and the lateral-yaw motion due to the tyre dynamic response. In the case of pure input, the energy flow only depends on the phase difference between the force/moment response and input. Shimmy can only occur for yaw input at large path wavelength since the moment has a leading phase response with respect to the yaw motion. When the wheel moves with combined lateral and yaw motion, the shimmy energy then is determined by the both inputs and path wavelength. The zero energy boundary is represented as a circle in the polar plot. Instability occurs inside the circles where \(W > 0\).

Although only the performance of the Von Schlippe tyre model with different path wavelengths is presented here, the comparison and evaluation of different tyre models can be conducted using the energy flow method in the polar plot. The difference between the zero energy boundaries, such as the change of position and shrink or expand in area, will indirectly distinguish their characteristics regarding shimmy stability.
4. Motion of the contact line

From the previous energy analysis, it can be observed that the phase response at different wavelengths is a crucial property of the tyre regarding stability: when the lateral force or aligning moment responses have a phase lead about the respective input motion, energy is extracted from the forward motion to maintain the periodic vibration of shimmy. With the straight contact line approximation, the Von Schlippe tyre model gives a clear physical representation: a two-point follower that represents the contact line moves along a sinusoidal path, and the stiffness of the carcass is replaced by a lateral and a yaw spring. In this section, we will focus on the motion of the contact line and study the mechanism of the energy transfer from a physical point of view.

4.1. Equivalent wheel motion

When the lateral and yaw motion of the wheel centre is prescribed, the path of the contact line is uniquely governed by Equation (1). For both pure lateral and yaw input, the path can be considered as being identical with a scaling factor that equals to $\sigma + a$, as illustrated in Figure 7. The solid lines present the lateral and yaw motion of the wheel, which are scaled within one period such that they produce the same path of the contact line. Obviously, the response of the path has a first-order delay with respect to the input, and the phase delay increases while reducing the path wavelength, that is, the path shifts horizontally to the right, as illustrated by the dashed lines.

Although both pure inputs generate the same path with a lag, it is only possible for the aligning moment response to realise a phase lead with respect to the yaw input. To understand this, we can select the motion of the wheel in the middle of one period (when $x = 0.5\lambda$) in Figure 7 to illustrate the force and moment generation. The paths of the contact line at three different path wavelengths are displayed in Figure 8. The wheel moves with only lateral and only yaw motion on the left and right of the figure, respectively. With the equivalent scaling between the lateral and the yaw motion, the paths for the leading point of the contact line are identical at the same path wavelength, and therefore so are the positions and orientations of the contact line.

Due to the delay effect of the leading point, the contact line also moves behind the wheel centre in the case of pure lateral input, which makes the force opposite against to the motion. The negative work has been done by the lateral force and the energy is being dissipated in the lateral direction.

On the other hand, the aligning moment is determined by the difference of orientation between the wheel and the contact line. As evident from Figure 8, the different positions of
the trailing point also cause a different slope of the contact line. For a larger wavelength, such as $\lambda = 40a$, the orientation of the contact line is ahead of the yaw motion of the wheel centre, and the aligning moment will try to accelerate the yaw motion. This is the point where the energy starts to be pumped into the yaw direction. For a shorter wavelength, such as $\lambda = 10a$ in Figure 8, the delaying movement of the contact line tries to slow down the yaw motion of the wheel. In the case of $\lambda = 20a$ which is almost the critical wavelength $\lambda_0$, the contact line moves in parallel with the wheel centre and no moment will be generated, hence no energy changes in the yaw motion at such a particular moment.

The motion of the wheel centre and contact line reported in Figure 8 is illustrative but the entire vibration cycle has to be studied to access the shimmy energy. For example when the wheel moves to the extremum position, that is, the maximum or the minimum, then there will be a short period that is contrary to the situation in Figure 8 since the movement of the wheel centre changes direction. However, contribution of energy will be minor during the whole period. The other situation is caused by the extremely short path wavelength. The curvature of the path takes the trailing point of the contact line to the different side of the wheel centre. But usually such short wavelength is not interesting in the research of shimmy. Therefore, the mechanism of energy transfer for the pure input still can be explained as Figure 8 and the energy flow method.

4.2. Combined lateral and yaw input

When the wheel undergoes a lateral and yaw motion at the same time, the response of the contact line will not only delay against wheel centre as in the pure input case. However, the energy transfer is still determined by the motion of the contact line with respect to the wheel centre. If the contact line moves ahead of the lateral input, energy is pumped into the lateral motion; similarly for the yaw motion, but the condition becomes that the slope of the contact line leads the yaw input of the wheel centre.

To confirm this observation, a combined input with $\eta = 1.5a$ and $\xi = 150^\circ$ is chosen with two path wavelengths, that is, $\lambda = 40a$ and $10a$. The corresponding position in the polar point is indicated as a red $\otimes$ in Figure 6. The responses of the contact line are given in Figures 9 and 10 within one cycle of vibration.

In these two figures, the shimmy powers of lateral and yaw motion are also presented, which are defined as $P_y = F_y \dot{y}_c$ and $P_z = M_z \dot{\psi}$. In the figures of motion, the force and moment exerted on the wheel centre are also depicted as arrows pointing to the contact line. In the power plots, the positive and negative areas are filled with red and green respectively. The shimmy energy requires the integration over one period, namely the net area enclosed by the power curves.
Figure 9. Motion and energy power for combined inputs. $\lambda = 40a$. Solid line: wheel centre motion; dashed line: path of contact line; arrow: force or moment exerted on the wheel centre.

Figure 10. Motion and energy power for combined inputs. $\lambda = 10a$. Solid line: wheel centre motion; dashed line: path of contact line; arrow: force or moment exerted on the wheel centre.
Obviously for the large wavelength in Figure 9, both the energy in lateral and yaw motion is positive, which means that the leading motion of the contact line predominates in such a case during one period. On the contrary, the total shimmy energy when \( \lambda = 10a \) is negative, even though the lateral contribution is positive. It is also interesting to notice that unlike the pure input, the lateral motion can also be responsible for the instability due to the coupling with the yaw motion.

As evidence from the generation of force and moment, the positive shimmy power occurs when the contact line moves ahead of the wheel centre such that the force or the moment is in the same direction with the prescribed motion.

The study of motion and energy power suggests the consistency with the energy boundary in the polar plot: it is inside the circle when the path wavelength is large and moves outside for a shorter wavelength. The results for other combinations of wheel motion confirm the same relation between the motion of contact line and the energy transfer.

5. Conclusions

The tyre behaviour is studied from an energy and mechanical point of view, with an application to shimmy analysis. After reviewing the Von Schlippe tyre model, the energy flow method is applied to manifest that, in the shimmy vibration, the tyre can be responsible for the energy transfer from the forward to lateral and yaw motion. In the case of a pure input, shimmy can only occur at large path wavelength for a yaw input, since the self-aligning moment response has a phase lead. When the tyre experiences a combined lateral and yaw input, the zero energy boundary turns out to be a circle in a polar plot. The circle moves as the path wavelength changes, and the conditions for instability consistently exist inside the circle where the shimmy energy is positive.

The mechanical representation of the contact line in the Von Schlippe tyre model shows that the energy transfer takes place due to the relative motion between the wheel centre and the contact line. If the movement or orientation of the contact line response is ahead of the wheel centre, the energy is extracted from the forward motion and fed into the lateral or yaw motion. It explains the mechanism of energy transfer from the forward motion and its path dependency.

Our study provides not only a novel criterion of tyre modelling with energy consideration, it also gives the insight into the mechanism of shimmy through a relatively simple physical interpretation based on the common accepted Von Schlippe tyre model. The performance of tyre models in the polar plot reveals their differences regarding shimmy stability. It can also be employed to evaluate tyre models with different characteristics and therefore guide the modelling of tyres for shimmy analysis. The discussion in this paper is limited to a linear tyre model. Future research could include nonlinear tyre behaviour with the energy flow method.

Disclosure statement

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References

Appendix. Tyre model parameters

\[ a = 0.0448 \text{ m}, \sigma = 0.483 \text{ m}, c_v = 1.27e5 \text{ N/m}, c_\theta = 2.75e3 \text{ Nm/rad}. \]