Supplying to mom and pop: traditional retail channel selection in megacities

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Supplying to mom and pop:
Traditional retail channel selection in megacities

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Abstract

Problem definition: Nanostores are traditional, small and independent retailers that are
present in large numbers in the megacities of the developing world. CPG manufacturers can
choose to deliver to nanostores either directly - visiting thousands of stores per day - or via
wholesalers - saving on distribution cost but forfeiting the direct access to the store owners to
develop demand. We study a manufacturer’s channel strategy within a finite time horizon.

Academic / Practical Relevance: The channel strategy in emerging markets has both market-
ing and operational elements which lead to a newly formulated problem with novel characteris-
tics. High costs are involved in the nanostore distribution and the difference in wholesale price,
logistics cost, product availability and market growth leads to a multi-dimensional problem that
is not trivial to analyze.

Methodology: We develop an analytical model to derive the optimal channel policy. We
conduct a numerical study with parameters tuned by field data. We develop managerial insights
based on our formal results and our numerical analysis.

Results: The optimal channel policy structure depends mainly on two channel metrics: the
gross profitability, which is the gross margin at a particular moment in time and the growth-
adjusted profitability, which includes the growth potential of a particular channel strategy to de-
velop the market and realize future profits. With demand growth over time, we show that, in the optimal policy, there is at most one switch between the wholesale and direct channel strategies within the time horizon.

**Managerial Implications:** Depending on the two metrics, it may be optimal to first expand the market using the direct channel and then switch to the wholesale channel to exploit the expanded market. In other cases, it may be optimal to first expand the market slowly using the wholesale channel then switch to the direct channel to benefit from high demand growth. The optimal channel strategy is also dependent on the time horizon, with a longer time horizon leading to relatively longer use of the direct channel.

**Keywords:** traditional retail, emerging markets, nanostores, distribution channel strategy

### 1 Introduction

Emerging markets have been the major growth driver for the world’s Consumer Packaged Goods (CPG) industry in the past several decades. These markets, including China and India, will continue to see rapid growth in the next decade as new shoppers enter the market and the per capita spending increases (Severin et al. 2011). Economic growth is especially substantial in the megacities in China, South and Southeast Asia, Latin America, and some parts of Africa and the Middle East. Due to increasing levels of urbanization, the ability to sell and deliver in these large cities is of increasing importance to the competitiveness of CPG manufacturers (Fransoo et al. 2017).

Although large retailers such as Walmart and Carrefour have entered these emerging markets in the 1990s, the traditional retail channel remains a market force to be reckoned with. The traditional channel is mostly composed of small family-operated stores that Fransoo et al. (2017) refer to as nanostores. These small stores exist in large numbers in the megacities of the developing world. For instance, Beijing counts about 60,000 nanostores serving about 20 million consumers; similarly, Bogota counts about 100,000 nanostores serving about 8 million consumers. In India and sub-Saharan Africa, the market share of nanostores can amount to more than 90% of CPG sales (Nielsen 2015a). In further developed countries in East Asia and Latin America, this channel typically still serves more than 40% of total CPG sales (Nielsen 2015a, Nielsen 2015b). Recent studies also show that this channel has been growing faster in recent years than sales in hypermarkets (Nielsen 2015b). For CPG manufacturers, it is therefore of high importance to have their products on the shelves of nanostores in order to profit from the growth in consumer spending in the megacities of the developing world.

Large multinational manufacturers such as Unilever, AB Inbev, and Nestle, and strong local players such as Colombina in Colombia and Jarritos in Mexico, choose to serve nanostores by making direct deliveries. By serving each store up to a few times per week, these CPG manufacturers
ensure that their products remain available for sale in the store. Manufacturers have significant control over this channel. Price competition is virtually absent within a neighborhood due to nanostores not having any negotiating power over the price and manufacturers effectively controlling the price. In some cases, this even happens explicitly since manufacturers may print the consumer price on the products’ packaging. Sales and distribution agents help to drive sales by promoting their products to the nanostore owners and offering enhanced product merchandising and display on the nanostore shelves through promotional material. Obviously, these frequent store visits entail a high cost to the manufacturers.

To avoid these costs, other manufacturers choose to sell their products via wholesalers instead: they contract with local wholesalers to distribute their products to the nanostores. Wholesalers in these markets also tend to be small. For instance, in a city like Beijing, there are several thousands of small wholesalers, each serving three or four dozen nanostores. Using the wholesale channel strategy reduces the cost of logistics, but prevents the manufacturers from reaching the store owners directly to drive sales. Unlike in developed markets, the supplier is also the price setter in the wholesale channel, since the negotiating power of these small wholesalers is barely more than that of a nanostore: a small margin of a few percentage points is typically granted to such wholesalers. We refer to Huang et al. (2017) for more details on the role of wholesalers in Beijing.

Typically, manufacturers deploy a one-size-fits-all strategy in a city; they either serve all nanostores directly or make use of wholesalers across the board. Some manufacturers are however more sophisticated and make such trade-offs at a neighborhood level, choosing to serve one neighborhood directly, while wholesalers may be used to serve the nanostores in another neighborhood. In this paper, we study this channel selection decision at a neighborhood level. Neighborhoods in emerging market megacities show substantial variation between them, and it is this variety that we exploit. For instance, the nanostore density tends to be higher in poor communities than in rich neighborhoods. In areas of high store density, the direct distribution may outperform wholesale distribution since the logistics costs can be justified by a higher sales volume; in areas of low store density, the reverse may be true. Also on the commercial side, growth sensitivity to channel choice may be different. In some areas, particularly the more underdeveloped areas of a city, this difference in sensitivity between the two channels may be substantial. For instance, during a store visit made in Casablanca (Morocco), one of the authors of this paper noted that certain personal care products were nicely positioned in the front of the store, benefiting from daylight to be better visible, while the products from the direct competitor were positioned in a far back corner of the store, barely visible because of poor lighting. Unsurprisingly, the former group of products was supplied directly, while the competing manufacturer made use of wholesalers.

Especially in first and second-tier cities in China, and in some neighborhoods of cities in South-
east Asia and Latin America, the modern retail channel is also present. Especially convenience stores like 7-Eleven in Asia or Oxxo in Latin America have grown substantially. We do not consider this channel in our study for several reasons. First, it is important to realize that there is a fairly strong market segmentation between these channels, with the modern stores targeting the upper middle and upper class consumers. In neighborhoods where these consumers are dominant, such as for instance the (upmarket) Polanco district in Mexico City, few nanostores remain. However, in the very same cities, there are populous neighborhoods where nanostores thrive and dominate the market. In Mexico City, this is the case in lower middle class districts like Azcapotzalco or poor districts like Ecatepec. In most of Latin America, for instance, about 75% of the urban population lives in such neighborhoods, with little presence of the modern channel. Competition with the hypermarket channel may exist in some way from a consumer perspective, as some consumers may choose to occasionally travel to a hypermarket such as a Walmart or Carrefour to buy certain goods. This typically involves extensive travel on public transport to other neighborhoods. Furthermore, for a consumer like this, the quantities bought in this channel will be limited as immediate payment is due. In the nanostore channel, many consumers will make use of informal credit provided by the store (Mejía-Argueta et al. 2017). For these reasons, we do not study the impact of the modern channel in our paper as it is not directly related to our problem at hand.

Research papers on distribution channel selection typically study the incentives of a supplier to integrate or disintegrate the distribution function, or in essence, to reach the entire market directly or indirectly. To date, substantial work has been conducted from the perspective of marketing and economics, but little has been done that includes operations considerations. Given the significant influence of the cost of logistics to nanostores in megacities, it appears appropriate to model this explicitly.

Early conceptual and empirical work in the retail marketing literature on distribution channel selection categorizes the factors that impact distribution channel selection. Cited relevant factors include purchasing frequency (e.g., Aspinwall 1962, Miracle 1965), margin or value (e.g., Aspinwall 1962, Miracle 1965) and fixed investment costs or asset specificity (e.g., Anderson and Schmittelein 1984, Anderson and Coughlan 1987, John and Weitz 1988). Suppliers tend to use the direct channel strategy for products with lower purchasing frequency, higher margin or value, and larger fixed investment cost. For complete coverage, please refer to Coughlan (2006). Similarly, our paper incorporates parameters such as channel-building fixed investment costs. Differently, we explicitly model transportation cost parameters such as store density and distance (time) - based logistics costs. Fundamentally, using a modeling approach, our paper derives relevant terms such as the demand and margin as functions of the distribution channel policy.

Modeling work on distribution channel selection usually trades off disintegration versus inte-
gration in a competitive setting. For a monopoly supplier, the integration strategy dominates the disintegration strategy since integration avoids double marginalization (e.g., see Lin et al. 2014). For competing two-stage supply chains, the integration strategy loses the advantage of pricing control as it intensifies cross-chain price competition (e.g., see Coughlan 1985, Pun and Heese 2010). However, integration is more favorable if the competing products are complements (Moorthy 1988, Coughlan 1987), if the supplier is closer to the market or produces at a lower cost (Pun and Heese 2010), or if the suppliers also consider backward integration (Lin et al. 2014). In our paper, we study the distribution channel strategies of a monopoly manufacturer who decides when to sell through a wholesaler and when to eliminate the wholesaler through direct distribution where the wholesale prices are considered as exogenous parameters. Our work also differs from these papers in that we model the operations in more detail.

Our paper also relates to the new product diffusion literature which illustrates the evolving nature of demand growth using the common framework of innovation adoption, imitation, and word-of-mouth effect. Diffusion processes are normally modeled as the well-known logistic or S-shape curve. Among all the functional forms, the most representative variants have been constructed by Bass (1969) and Mansfield (1961). New product diffusion processes are usually modeled as functions of marketing and operations variables or parameters such as advertising (e.g., Mesak et al. 2011, Dockner and Jorgensen 1988, Swami and Dutta 2010), pricing (e.g., Bass and Bultez 1982, Liu et al. 2011), and most recent social network effect (e.g., Ho et al. 2012, Peres and Van den Bulte 2014). Among these, a few papers explicitly incorporate distribution in a product diffusion model. In order to study the role of retailers in the diffusion of manufacturers’ products to consumers, Jones and Ritz (1991) construct a two-stage diffusion process with retailers as the intermediary. Empirically, Bronnenberg et al. (2000) characterize a positive relation between retailer coverage and product diffusion. Alternatively, others study the influence of marketing variables and the channel switch timing for a sequential two-channel strategy for services like movies entering into a foreign market (e.g., Lehmann and Weinberg 2000, Elberse and Eliashberg 2003).

We model demand using a distribution channel dependent diffusion process. This generates an S-shape demand growth curve for any channel policy to switch between channel strategies without restricting any particular sequence. We model a one-stage product diffusion where nanostores as retailers are fully covered by the manufacturer at the beginning of the time horizon regardless of the channel strategy in use and the CPG manufacturer can use the direct channel to influence its product diffusion positively through nanostores. There are three key aspects of nanostores that play an important role in our model and that set it apart from organized retailers in a similar market. First, the channel choice for distribution directly affects sales, so the sales channel choice and the distribution channel choice are one and the same. In organized retail, this does not happen, even in
developing markets, as it is the retail chain that makes important decisions such as promotions and planogram design, irrespective of whether the manufacturer supplies directly to the stores or via a retail distribution center (DC). In organized retail, the decision of the retailer to negotiate direct delivery by the manufacturer is purely driven by logistics efficiency and cost. Second, the number of independent stores is considerably higher than the number of organized stores, allowing for the type of continuous approximation of the distribution costs that we use. For instance, the number of organized retail stores in Bogota (Colombia) is less than 1,000 (supermarkets and convenience stores). The number of nanostores is about 100,000 (Mejía-Argueta et al. 2017). Third, wholesalers still play an important role in the supply to nanostores, while their role is absent in the organized retail. In organized retail, typically manufacturers supply to the DC of the retailer, although in some countries such as the US, it is also common to see large manufacturers supply directly to (large) stores.

Note that, with the exception of the high density of stores that is important in this paper, the problem studied in our work also exists in developed markets. For instance, so-called Direct Store Delivery to independent stores is still common in the United States (see Grocery Manufacturers Association 2008 and Grocery Manufacturers Association 2011).

Tang et al. (2017) also consider the distribution channel selection of micro-retailers. Like ours, their paper considers a nanostore distribution setting, but the problem is modeled from the perspective of the (rural) micro-retailers. Instead of both buying independently from one manufacturer, one micro-retailer can buy in bulk and then act as a wholesaler by selling the product to the other micro-retailer. At the same time, the store can sell as a retailer, which represents a “hybrid channel”. With Cournot competition models, they show that the hybrid strategy is preferable with high fixed costs and quantity discounts.

In our paper, we study the channel selection decision of a CPG manufacturer serving nanostores in a densely populated megacity. We provide insights into the trade-offs that manufacturers face, depending on characteristics such as the nanostore density and the demand growth rate. We first analyze a static setting where the demand of each nanostore does not grow throughout the time horizon. Then, we consider a demand growth setting where the demand growth rate is dependent on the selected channel strategy. We model the growth of demand as an S-curve. We maximize the manufacturer’s cumulative profit over the length of the time horizon and present the optimal sequence of distribution channel strategies in each setting. We conduct numerical analyses with a range of realistic parameter settings to provide further insights.

Our contribution is threefold. First, our model captures the operations dimensions explicitly in the store density and the associated logistics costs; earlier channel choice models typically exclude these important operational costs, potentially because the fraction of logistics costs in operations
in North America or Europe is small. However, in our setting, the logistics costs are substantial because of deliveries in relatively high frequency and in relatively small drop size. We show that the operations costs difference between channels plays a key role in the channel selection trade-off. Second, the earlier literature assumes a single channel to be selected throughout while we model the possibility of channel switching. We believe this is an important contribution since this allows for strategies where new market entrants could decide between first going direct and then wholesale, or the reverse; these types of strategies are not discussed in the prior literature. Third, the prior literature has not studied the effect that the time horizon plays in the channel selection decision. Our model helps to explain one important reason why some companies would choose to use the wholesale channel throughout the horizon, i.e., because they are not prepared to take a longer project evaluation time horizon.

The rest of the paper is structured as follows. We present the model in Section 2. We then analyze both the static and the growth model in Section 3 and present the numerical study in Section 4. We conclude in Section 5. All proofs can be found in Online Appendix C.

2 Model

A CPG manufacturer produces a non-perishable product at a cost of $c$ per unit and has the option to sell it to a cluster of identical nanostores located in a rectangular selling region over a well-defined time horizon. In practice, the length of the time horizon, denoted by $T$, may correspond to the decision making horizon of the company, i.e., the period for which the company normally evaluates projects. If the manufacturer decides to sell to the region, she can use two channel strategies: she can either use the direct channel strategy, i.e., she sells and delivers the product to the nanostores using her own sales force and transportation fleet, or she can use the wholesale channel strategy, i.e., she sells to a wholesaler located inside the region who then sells and delivers the product to the nanostores. Let $p_D$ be the price at which the manufacturer sells its product to the nanostores when using the direct channel strategy and let $p_W$ denote the price at which she sells her product to the wholesaler when using the wholesale channel strategy. We assume that $p_D > p_W$, i.e., the (wholesale) price is higher in the direct channel strategy, as the retail price is identical regardless of the chosen channel strategy and the difference corresponds to the wholesaler’s cut.

When using the direct channel strategy, the manufacturer faces a demand rate of $\omega_D(t) = \lambda(t)\beta_D$ per nanostore at time $t$ where $\lambda(t)$ corresponds to a nanostore’s market size for the manufacturer’s product at time $t$ and $\beta_D \in (0,1]$ represents the availability multiplier when selling via the direct channel strategy. This $\beta_D$ parameter captures the consumer access to the product at the nanostores and it is a function of the amount of space allocated to the product on the nanostore shelves as well as the actual product on-shelf availability which is high if the product is actively
replenished at the nanostores. Similarly, when using the wholesale channel strategy, the manufacturer faces a demand rate per nanostore at time $t$ of $\omega_W(t) = \lambda(t)\beta_W$ where $\beta_W \in (0, 1]$ is the availability multiplier when selling via the wholesale channel strategy. We assume that $\beta_D \geq \beta_W$, that is, the manufacturer achieves higher availability and captures more consumer demand when dealing directly with the nanostores.

We model demand growth using models commonly used in the literature on product and innovation diffusion (e.g., Bass 1969, Mansfield 1961). In those models, the growth curve is determined based on the interaction between adopters and non-adopters of a new product and is represented by an S-curve. In our case, once a consumer has adopted the product, he or she repeatedly purchases the product as commonly observed with Consumer Packaged Goods. Following this modeling perspective, we model the instantaneous adoption rate of the product at time $t$ as:

$$\frac{\partial \lambda(t)}{\partial t} = \alpha_i \lambda(t)(1 - \lambda(t))$$

(1)

which is the multiplication of the interaction intensity between adopters and non-adopters, i.e., $\lambda(t)(1 - \lambda(t))$, and the average conversion rate $\alpha_i$ for $i = D, W$ of each interaction. For exposition purpose, we term $\alpha_D$ and $\alpha_W$ as the market growth rates of using the direct and the wholesale channel respectively. We assume that $\alpha_D \geq \alpha_W$, i.e., the market grows at a faster pace when using the direct channel strategy because of more specific and localized commercial activities at the nanostores. For instance, the manufacturer’s sales representative who visits the store directly may ensure a better quality shelf space, and use special product displays and other promotional material and activities which promote the brand and lead to faster growth than if the product was replenished by the wholesaler. Also, we assume that the market size does not grow if the manufacturer chooses not to sell the product in the region over a certain period of time.

The manufacturer’s objective is to maximize total profits over a $[0, T]$ time horizon by deciding when to sell the product and, when doing so, whether to use the direct or the wholesale channel strategy. More specifically, we define a channel policy as a sequence of $N$ channel strategies (i.e., wholesale, direct or not-selling) and time thresholds denoted $(S, t)$ where $S = (S_1, ..., S_N)$ and $t = (t_1, ..., t_N)$ with $S_1 \in \{W, D, \emptyset\}$ for $i = 1, ..., N$, $S_i \neq S_{i+1}$ for $i = 1, ..., N - 1$ and $0 \leq t_1 \leq \ldots \leq t_N = T$. Here, $S_i = \emptyset$ means that the manufacturer does not sell to the region in time interval $[t_{i-1}, t_i)$, $S_i = W$ means that the manufacturer uses the wholesale channel strategy and $S_i = D$ means that the manufacturer uses the direct channel strategy. In essence, the manufacturer divides the time horizon $[0, T]$ into $N$ disjoint intervals so that channel strategy $S_i$ is used in time interval $[t_{i-1}, t_i)$ and a different channel strategy is used in adjacent intervals. Note that, in theory, the same channel strategy can be used in non-adjacent intervals over the time horizon, that is, we do not exclude situations where, for example, the manufacturer could enter the market using the
direct channel strategy, then switch to the wholesale strategy only to switch back later to the direct channel strategy (however we show in Section 3 that such a case is never optimal).

We adapt the well-known logistic curve, which is a solution to Equation (1) and also has been widely used to model new product diffusion, to illustrate the market dynamics over the time horizon. Specifically, the market size in time interval \( t \in [t_{i-1}, t_i) \) is:

\[
\lambda(t) = \frac{1}{1 + e^{-a_S(t-t_{i-1})+b-\sum_{j=1}^{i-1}a_S(t_j-t_{j-1})}}
\]

where the market potential is normalized to be 1 and \( a_S \) is the market growth rate of using the channel strategy \( S_i \) for interval \( i = 1, ..., N \). Herein, the term \( b \) is a positive constant which reflects the difficulty of growing the market and \( \frac{1}{1+b} \) is the initial market size per nanostore at the beginning of the time horizon. A product with a higher value of \( b \) diffuses at a lower pace and takes a longer time to saturate the market. Given Equation (2), initially, the sales volume grows at an accelerating rate. Later, after an inflection point, the sales volume grows in a decreasing rate until the entire market potential is reached. This characterizes a \( S \) curve which is a common market growth pattern. We also have conducted our analysis for a general exponential growth demand model; this leads to similar results and insights.

Suppose, for example, that the manufacturer decides to sell her product via the direct channel strategy from time 0 to time \( t_1 \), then switches to the wholesale channel strategy from time \( t_1 \) until the end of the time horizon \( T \). The demand rate per nanostore at time \( t \) is equal to \( \omega_D(t) = \frac{1}{1+e^{-a_D(t-t_{1})+b-a_Dt_1}}\beta_D \) for \( t \in [0, t_1] \) and equal to \( \omega_W(t) = \frac{1}{1+e^{-a_W(t-t_{1})+b-a_Wt_1}}\beta_W \) for \( t \in (t_1, T] \). Figure 1 illustrates such a policy.

![Figure 1: Market size growth curve of a channel policy switching from the direct to wholesale](image)

The manufacturer uses vans to deliver to nanostores when using the direct channel distribution, which is common for the CPG sector in emerging markets. In this sales model, the sales representative is also the vehicle driver who delivers products to nanostores, optimizes shelf display and conducts store-level brand promotion in a single trip. Let \( C_W \) (\( C_D \)) denote the capacity of
the vehicles used for deliveries, $\mu_W (\mu_D)$ be the vehicle operating cost per unit of distance, $f_W (f_D)$ denote the fixed logistics cost per vehicle trip and $h_W (h_D)$ be the unloading time (plus the time for the sales representative to optimize shelf display and conduct store-level brand promotion) spent by the manufacturer at the wholesaler (each nanostore) for deliveries using the wholesale (direct) channel strategy. Because she can use major roads to deliver to the wholesaler outside of peak traffic hours, the manufacturer is likely to use large but expensive-to-operate trucks. In contrast, she is likely to use small, cheaper-to-operate vehicles (such as electric delivery bikes) to reach the nanostores which are often located in highly congested parts of the city where parking options are very scarce. Hence, we assume that $C_D < C_W$ and $\mu_D < \mu_W$. We also assume that $h_D < h_W$ since the manufacturer delivers a higher quantity to the wholesaler, resulting in a larger unloading time. We assume that both types of vehicles travel at an average speed of $v$ and denote the vehicle driver wage per unit of time as $s$.

We use a continuous approximation scheme to estimate the logistics cost of selling the product, following Daganzo (1987). We model the nanostores to be uniformly distributed over a rectangular selling region with horizontal length $x$, vertical length $y$ and density $\delta$ per area unit, such that the number of nanostores in the region is $xy\delta$. The depot from which the manufacturer transports her products is located outside of the selling region and the wholesaler is located in the center of it. More precisely, we assume that the manufacturer’s depot is located south of the selling region, at a vertical distance $\phi$ from the bottom edge and horizontally at the half way point (see Figure 2). Therefore, the distance between the manufacturer’s depot and the wholesaler’s location is $\phi + \frac{y}{2}$.

We now provide expressions for the manufacturer’s profit when using either strategy. First, consider the wholesale channel strategy. The aggregate demand rate from all the nanostores in the selling region at time $t$ is $\Omega_W(t) = xy\delta \omega_W(t)$. Given the vehicle capacity $C_W$, the manufacturer has to make $\frac{\Omega_W(t)}{C_W}$ delivery trips to the wholesaler, which results in a total distance traveled of $\frac{\Omega_W(t)}{C_W} (2\phi + y)$ and a total delivery (transit + on-site operations) time of $\frac{\Omega_W(t)}{C_W} v (2\phi + y) + \frac{\Omega_W(t)}{C_W} h_W$. $10$
Note that we allow for fractional number of trips so as to avoid discreteness issues. As a result, the instant profit (sales revenues minus logistics costs) at time $t$ when using the wholesale channel strategy is given by:

$$\pi_W(t) = \Omega_W(t)(p_W - c) - \mu_W \left( \Omega_W(t) \left( 2\varphi + y \right) - s \left( \Omega_W(t) \left( 2\varphi + y + \frac{\Omega_W(t)}{C_W} \right) - \frac{\Omega_W(t)}{C_W} f_W \right) \right)$$

$$= \left[ \frac{(p_W - c) - \frac{1}{C_W} \left( f_W + sh_W + \left( \mu_W + \frac{s}{v} \right) \left( 2\varphi + y \right) \right)}{M_W} \right] \Omega_W(t).$$

In the first row, the first term is the sales revenue, the second term is the distance-based operating cost, the third term is the labor cost and the last term is the fixed logistics cost. On the second row, $M_W$ is the marginal return from selling an extra product unit using the wholesale channel strategy. Note that we may have $M_W < 0$.

Let us now consider the direct channel strategy. The aggregate demand rate from all the nanostores at time $t$ is $\Omega_D(t) = xy\delta \omega_D(t)$. Given the vehicle capacity $C_D$, the manufacturer has to make $\frac{\Omega_D(t)}{C_D}$ delivery trips to serve the $xy\delta$ nanostores which are uniformly distributed over the selling region (the average number of nanostores visited in one trip is $\frac{xy\delta C_D}{\Omega_D(t)} = \frac{C_D}{\omega_D(t)}$). Using the half-width routing continuous approximation proposed by Daganzo (1987), we calculate that the total distance traveled by all the delivery vehicles within the region at time $t$ is equal to $\frac{\Omega_D(t)}{C_D} \left( 2\varphi + y \right) + y \left( 1 + \frac{x^2\delta}{6} \right)$ and that the corresponding total delivery time is $\frac{\Omega_D(t)}{\omega_D(t)} \left( 2\varphi + y \right) + \frac{y}{v} \left( 1 + \frac{x^2\delta}{6} \right) + xy\delta h_D$ (see derivations in Online Appendix A). As a result, the instant profit at time $t$ using the direct channel strategy is given by:

$$\pi_D(t) = \Omega_D(t)(p_D - c) - \mu_D \left[ \frac{\Omega_D(t)}{C_D} \left( 2\varphi + y \right) + y \left( 1 + \frac{x^2\delta}{6} \right) \right]$$

$$- s \left[ \frac{\Omega_D(t)}{C_D} \left( 2\varphi + y \right) + \frac{y}{v} \left( 1 + \frac{x^2\delta}{6} \right) + xy\delta h_D \right] - \frac{\Omega_D(t)}{C_D} f_D$$

$$= \left[ \frac{(p_D - c) - \frac{1}{C_D} \left( f_D + \left( \mu_D + \frac{s}{v} \right) \left( 2\varphi + y \right) \right)}{M_D} \right] \Omega_D(t) - \left[ \left( \mu_D + \frac{s}{v} \right) y \left( 1 + \frac{x^2\delta}{6} \right) + sxy\delta h_D \right].$$

In the first row, the first term is the sales revenue, the second term is the distance-based operating cost, the third term is the labor cost and the last term is the fixed logistics cost for all vehicle trips. On the second row, $M_D$ is the marginal return from selling an extra product unit using the direct channel strategy and $k_D$ is the fixed logistics cost. Note that we may have $M_D < 0$. Also note that, unlike with the wholesale channel, the instant profit at time $t$ using the direct channel includes a term which does not increase with the aggregate demand size. This is due to the fixed nature of part of the logistics costs from serving a preset number of stores over the selling area.

On top of the costs already mentioned, we assume that the manufacturer incurs a fixed in-
vestment cost when she starts selling her product and every time she switches from one channel strategy to the other channel strategy. Let $F_D$ and $F_W$ denote the channel-building fixed investment costs when entering a period where the manufacturer uses the direct or wholesale channel strategy respectively. We assume that $F_D \geq F_W$ since using the direct channel strategy is likely to require larger initial investments than selling via the wholesaler as it entails hiring a larger sales force and establishing business contacts with more clients.

Finally, for a given channel policy $(S, t)$, the cumulative profit in the entire $[0, T]$ horizon is:

$$\Pi(S, t) = \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \left( \pi_W(t) I_{\{S_i=W\}} + \pi_D(t) I_{\{S_i=D\}} \right) dt - \sum_{i=1}^{N} I_{\{S_i=W\}} F_W - \sum_{i=1}^{N} I_{\{S_i=D\}} F_D.$$  

Next, we study the optimal channel policy under the static demand model where $\alpha_D = \alpha_W = 0$ and the demand growth model where $\alpha_D > \alpha_W > 0$.

3 Results

3.1 Optimal Channel Policy under the Static Demand Model

In this section, we consider the special case where the nanostore market size does not grow over time, that is, $\alpha_D = \alpha_W = 0$. Let $\lambda = \frac{1}{1+e^{b \beta \delta}}$. As a result, the per-nanostore demand rate is constant over the time horizon and equal to $\omega_D = \lambda \beta_D$ and $\omega_W = \lambda \beta_W$ respectively for the direct and wholesale channel strategy. Let $\pi_D$ and $\pi_W$ denote the instant profit rates generated from using the direct and wholesale channel strategy at a given point in time, where $\pi_D = M_D \beta_D \lambda y \delta - k_D$ and $\pi_W = M_W \beta_W \lambda y \delta$. Note that we may have $\pi_D < 0$ and $\pi_W < 0$ (if $M_W < 0$). The cumulative profit within the entire $[0, T]$ time horizon from using channel policy $(S, t)$ can be written as:

$$\Pi(S, t) = \sum_{i=1}^{N} (t_i - t_{i-1}) I_{\{S_i=W\}} \pi_W + \sum_{i=1}^{N} (t_i - t_{i-1}) I_{\{S_i=D\}} \pi_D - \sum_{i=1}^{N} I_{\{S_i=W\}} F_W - \sum_{i=1}^{N} I_{\{S_i=D\}} F_D. \quad (3)$$

Let $(S^*, t^*)$ denote the optimal channel policy in this static demand model. We first show that, as expected, it is never optimal to switch channel strategies in this case.

**Lemma 1.** In the static demand model, i.e., $\alpha_D = \alpha_W = 0$, it is optimal to use the same channel strategy over the entire horizon, that is, either sell via the direct channel strategy, wholesale channel strategy or not at all.

From Lemma 1, optimally we always have $N^* = 1$, $t^*_1 = T$ and $S^* \in \{ \emptyset, W, D \}$. In what follows, we drop the $t$ notation from $\Pi(S, t)$ for simplicity. Then, the profit maximization problem of the static model reduces to a choice between the wholesale-only, direct-only or no-selling channel
policy, as follows:

$$\Pi^* = \max \{\Pi(W), \Pi(D), 0\}$$

where

$$\Pi(W) = \frac{M_W \beta_W \lambda \delta T - F_W}{\pi_W}$$ \hspace{1cm} (4)

$$\Pi(D) = \left(\frac{M_D \beta_D \lambda \delta - k_D}{\pi_D}\right) T - F_D = M_D \beta_D \lambda \delta T - F_D$$ \hspace{1cm} (5)

with $k_D = (\mu_D + \xi) y \left(1 + \frac{s^2 \delta}{6}\right) + s \delta \eta D$ and $F_D' = k_D T + F_D$.

From Equations (4) and (5), we see that the manufacturer needs a minimum market size $\lambda$ in order to recover fixed investments, which include the initial channel-building investment costs $F_D$ and $F_W$, plus, in the case of the direct-only channel policy, an extra fixed logistics cost, i.e., $k_D$. Since we assume $F_D \geq F_W$ and $k_D \geq 0$, the direct channel strategy must provide a higher or equal return per nanostore customer in order to be optimal, i.e., we must have $M_D \beta_D \geq M_W \beta_W$. We refer to $M_D \beta_D$ and $M_W \beta_W$ as the gross profitability of the direct and wholesale channel respectively, which measures the marginal return generated from one extra unit of base market demand, after accounting for the availability multiplier.

**Lemma 2.** The direct-only channel policy can only be optimal if $M_D \beta_D \geq M_W \beta_W$.

Next, we study how the optimal channel policy varies with the store density $\delta$ and the nanostore market size $\lambda$. Note that $\delta$ impacts not only total demand but also the logistics cost $k_D$ when using the direct channel.

**Theorem 1.** Let $\Lambda_W(\delta) = \frac{F_W}{M_W \beta_W \lambda \delta T}$, $\Lambda_D(\delta) = \frac{F_D(\delta)}{M_D \beta_D \lambda \delta T}$, $\lambda'(\delta) = \frac{F_D'(\delta) - F_W}{M_D \beta_D - M_W \beta_W |xy| \delta T}$ and $\bar{\delta}_\lambda = \frac{F_W M_D \beta_D - F_D M_W \beta_W - (\mu_D + \frac{s}{2}) M_W \beta_W \lambda T}{xy T M_W \beta_W \left(s h_D + \frac{(\mu_D + \frac{s}{2}) M_W \beta_W \lambda T}{6}\right)}$. The optimal channel policy varies with $\lambda$ and $\delta$ when $M_D \beta_D > M_W \beta_W > 0$ in accordance with Figure 3.

We see that all else equal, larger values for the store density $\delta$ and nanostore market size $\lambda$ tend to make the direct-only channel policy more profitable as it allows the manufacturer to recover the higher investment costs faster. Also, we see that when $M_D \beta_D$ is very large, as in the graph on the right, there exist values of the market size $\lambda$ such that the wholesale-only channel policy is never optimal.

The manufacturer needs to sell for a certain period of time before recovering the initial fixed channel-building cost investment associated with running a distribution channel. Theorem 2 shows how the optimal channel policy varies with the length of the time horizon.
We see that all else equal, each channel strategy becomes more attractive when the sales force in the direct channel should guarantee sufficient shelf space allocation and higher on-shelf product availability. We assume that it is optimal not to sell when \( \beta_D \leq \frac{F_D + (\mu_D + \bar{\rho})\lambda T}{f_W} \) otherwise, it is optimal not to sell.

The optimal channel policy varies with the availability multipliers \( \beta_D \) and \( \beta_W \).

**Theorem 2.** Let \( T_W = \frac{f_W}{\pi_W}, T_D = \frac{f_D}{\pi_D} \) and \( T^e = \frac{F_D - f_W}{\pi_D - \pi_W} \).

When \( \pi_W \leq 0 \) and \( \pi_D \leq 0 \), it is optimal not to sell to the region.

When \( \pi_D > 0 \) and \( \pi_W \leq 0 \), it is optimal to use the direct-only channel policy if \( T \geq T_D \); otherwise, it is optimal not to sell.

When \( \pi_W > 0 \) and \( \pi_W \geq \pi_D \), it is optimal to use the wholesale-only channel policy if \( T \geq T_W \); otherwise, it is optimal not to sell.

When \( \pi_D > \pi_W > 0 \) and \( T_W \leq T_D \), it is optimal not to sell for \( T < T_W \), to use the wholesale-only channel policy for \( T \in [T_W, T^e] \), and the direct-only channel policy for \( T \geq T^e \).

When \( \pi_D > \pi_W > 0 \) and \( T_W > T_D \), it is optimal not to sell for \( T < T_D \), and to use the direct-only channel policy for \( T \geq T_D \).

We see that, for relatively short time horizons, it may be optimal to use the wholesale-only policy even in situations where the instant profit generated from the direct channel is greater than the instant profit from the wholesale channel, i.e., \( \pi_D > \pi_W > 0 \). This is because it may take longer to recoup the higher fixed cost investments which are necessary to serve the nanostores directly. However, for long enough time horizons, it is optimal to use the direct-only policy when its instant rate of profit is higher.

Next, we study how the optimal channel policy varies with the availability multipliers \( \beta_D \) and \( \beta_W \).

**Theorem 3.** Let \( \beta_W = \frac{f_W}{M_W \chi \lambda \tau} \), \( \beta_D = \frac{f_D}{M_D \chi \lambda \tau} \), \( \beta_D' = \beta_D + \frac{M_W}{M_D} (\beta_W - \beta_D) \) and \( \beta_D'' = \frac{F_D - f_W}{(M_D - M_W) \chi \lambda \tau} \).

The optimal channel policy varies with \( \beta_D \) and \( \beta_D' \) when \( M_D \geq 0 \) and \( M_W \geq 0 \) in accordance with Figure 4.

Note that, in all four graphs of Figure 4, the grey region is not to be considered since we have assumed that \( \beta_D \geq \beta_W \), which is further because the careful re-shelving efforts of the manufacturer’s sales force in the direct channel should guarantee sufficient shelf space allocation and higher on-shelf product availability. We see that all else equal, each channel strategy becomes more attractive
as its availability multiplier increases. Interestingly, we see on the bottom right graph that there are cases in which the wholesale-only channel policy is never optimal.

In summary, we find that, when the market does not grow over time, the manufacturer should use a single channel strategy, provided the market is profitable. The choice between the direct and wholesale strategies depends mainly on the gross profitability of each strategy, which measures the extra return per unit of base market demand, after accounting for the availability multiplier, that is, the impact of shelf space allocation and product on-shelf availability of the chosen strategy. Also, because selling via the direct channel usually comes with higher logistics and structural investments, it generally requires longer time horizons or a greater market size to be the profit-maximizing strategy in cases where it generates higher gross profitability than the wholesale channel.

3.2 Optimal Channel Policy under the Demand Growth Model

In this section, we consider the most general version of our model, where the nanostore market size grows over time, with the growth being faster under the direct channel strategy, i.e., \( \alpha_D > \alpha_W > 0 \). Our first result is to establish that it is never optimal to delay entry to the market or exit the market early.
Proposition 1. If it is optimal to sell the product in the region, then it is optimal to do so for the entire time horizon.

Since we have assumed that the market for the manufacturer’s product does not grow if it is not made available on the nanostores’ shelves, there is no benefit in delaying market entry. Also, because we have assumed positive market growth when the product is sold, profitability only improves over time so that it is never optimal to stop selling the product. However, it may be optimal to sell the product initially at a loss, before the market reaches a critical mass, as will be illustrated below. Also, it may be optimal to change channel strategy over time. Our next result shows that, if it is optimal to sell, there should be at most one switch between the direct and wholesale channel strategies during the time horizon.

Proposition 2. If the optimal channel policy includes both the direct and wholesale channel strategies, then it is optimal to switch between them at most once. Also, the direct channel strategy is used before the wholesale channel strategies during the time horizon.

We refer to $\frac{M_D \beta_D}{\alpha_D}$ and $\frac{M_W \beta_W}{\alpha_W}$ as the growth-adjusted profitability of the direct channel and wholesale channel strategy respectively. This measure is obtained by dividing the channel’s gross profitability (defined in §3.1) by the channel demand growth rate. According to Proposition 2, the manufacturer who uses both channel strategies over the time horizon should enter the market with the strategy with the lowest growth-adjusted profitability then switch to the strategy with the highest growth-adjusted profitability. Mathematically, this condition emerges from the comparison of the profit functions from two switching policies. First, consider the following channel policy: $N = 2$, $S = (W, D)$ and $t = (t_1, T)$, that is, the manufacturer enters the market at time zero using the wholesale channel strategy then switches to the direct channel strategy at time $t_1 > 0$ until the end of the time horizon. The profit of this policy can be written as:

$$\Pi(S, t) = x y \delta \left( \frac{M_W \beta_W}{\alpha_W} \log \frac{e^{b + e^t} a_t + e^t}{1 + e^b} + \frac{M_D \beta_D}{\alpha_D} \log \frac{e^{b + e^t} a_t + e^t}{e^{b + e^t} a_t} \right) - k_D (T - t_1) - F_W - F_D.$$

Now consider the reverse channel policy: $N = 2$, $S' = (D, W)$ and $t' = (T - t_1, T)$ such that the manufacturer enters the market at time zero using the direct channel strategy then switches to the wholesale channel strategy at time $T - t_1$ until the end of the time horizon (the switching time is chosen so that the length of time to use each channel strategy is the same with policy $(S, t)$). The profit of this policy can be written as:

$$\Pi(S', t') = x y \delta \left( \frac{M_D \beta_D}{\alpha_D} \log \frac{e^{b + e^t} a_t + e^t}{1 + e^b} + \frac{M_W \beta_W}{\alpha_W} \log \frac{e^{b + e^t} a_t + e^t}{e^{b + e^t} a_t} \right) - k_D (T - t_1) - F_W - F_D.$$

The difference in profits between the two policies is given by:

$$\Pi(S, t) - \Pi(S', t') = \left( \frac{M_D \beta_D}{\alpha_D} - \frac{M_W \beta_W}{\alpha_W} \right) xy \delta \log \left( \frac{1 + e^b}{e^{a t} + e^{a t_1}} \right) \left( \frac{e^{b + a t_1} + e^{a t + a T}}{e^{b + a t_1} + e^{a t + a T}} \right)$$

where the logarithmic term is positive because the numerator is greater than the denominator with a difference $e^b (e^{a T} - e^{a t_1}) (e^{a t_1} - 1)$. It follows that the first policy (W followed by D) yields a
higher cumulative profit than the second policy (D followed by W) if and only if \( \frac{M_D \beta_D}{\alpha_D} > \frac{M_W \beta_W}{\alpha_W} \).

We further discuss the intuition beyond this inequality condition, with Propositions 3 to 5 below.

From Proposition 2, there are only five possible optimal channel policies: no selling (O), direct-only (D), wholesale-only (W), direct channel then switching to the wholesale channel (DW) and wholesale channel then switching to the direct channel (WD).

Consider the DW channel policy and let \( t_W \) denote the switching time between the direct and wholesale channel strategies. The profit generated by the DW channel policy as a function of \( t_W \in [0, T] \) is as follows:

\[
\Pi^{DW}(t_W) = \int_0^{t_W} \pi_D(t)dt + \int_{t_W}^T \pi_W(t)dt - F_D I_{\{t_W>0\}} - F_W I_{\{t_W<T\}}
\]

\[
= xy\delta \left( \frac{M_D \beta_D}{\alpha_D} \log \left[ \frac{e^{b + e^{\alpha_D t_W}}}{1 + e^b} \right] + \frac{M_W \beta_W}{\alpha_W} \log \left[ \frac{e^{b + e^{\alpha_W t_W + \alpha_D (T-t_W)}}}{e^b + e^{\alpha_D t_W}} \right] \right) - k_D t_W - F_D I_{\{t_W>0\}} - F_W I_{\{t_W<T\}}.
\]

Note that the W channel policy is obtained when \( t_W = 0 \) and the D channel policy is obtained when \( t_W = T \).

Consider the WD channel policy and let \( t_D \) denote the switching time between the wholesale and direct channel strategies. The profit generated by the WD channel policy as a function of \( t_D \in [0, T] \) is as follows:

\[
\Pi^{WD}(t_D) = \int_0^{t_D} \pi_W(t)dt + \int_{t_D}^T \pi_D(t)dt - F_D I_{\{t_D<T\}} - F_W I_{\{t_D>0\}}
\]

\[
= xy\delta \left( \frac{M_W \beta_W}{\alpha_W} \log \left[ \frac{e^{b + e^{\alpha_W t_D}}}{1 + e^b} \right] + \frac{M_D \beta_D}{\alpha_D} \log \left[ \frac{e^{b + e^{\alpha_D t_D + \alpha_W (T-t_D)}}}{e^b + e^{\alpha_D t_D}} \right] \right) - k_D (T-t_D) - F_W I_{\{t_D>0\}} - F_D I_{\{t_D<T\}}.
\]

Note that the W channel policy is obtained when \( t_D = T \) and the D channel policy is obtained when \( t_D = 0 \).

Though they are not necessarily concave, the two functions \( \Pi^{DW}(t_W) \) and \( \Pi^{WD}(t_D) \) are well-behaved. In particular, we can show they have at most one or two interior local maxima on \([0, T]\) as stated in the following lemma.

**Lemma 3.** When \( \frac{M_D \beta_D}{\alpha_D} < \frac{M_W \beta_W}{\alpha_W} \), \( \Pi^{DW}(t_W) \) has at most one interior local maximum on \([0, T]\). When \( \frac{M_D \beta_D}{\alpha_D} \geq \frac{M_W \beta_W}{\alpha_W} \), \( \Pi^{WD}(t_D) \) has at most two interior local maxima on \([0, T]\) and these two interior local maxima exist only when \( \alpha_D > 2 \alpha_W \) and \( M_W < 0 \).

In both cases, the global maximum can be found by comparing the interior local maximum or maxima (given existence) with the values of the profit function at 0 and T, which correspond to the one-channel policies. Let \( t^*_W \) denote the switching time which maximizes \( \Pi^{DW}(t_W) \) for \( t_W \in [0, T] \).
Let $t^*_D$ denote the switching time which maximizes $\Pi^{WD}(t_D)$ for $t_D \in [0, T]$. From Proposition 2, we get the following characterization of the optimal profit.

**Corollary 1.** If $\frac{M_D\beta_D}{\alpha_D} < \frac{M_W\beta_W}{\alpha_W}$, the optimal profit is $\max\{\Pi^{DW}(t^*_W), 0\}$; otherwise, it is $\max\{\Pi^{WD}(t^*_D), 0\}$.

Based on our previous discussion, $M_D\beta_D - \frac{k_D}{\alpha_D}$ is the gross profitability after accounting for the per-store fixed logistics cost of using the direct channel strategy, which can be interpreted as the net profitability of using the direct channel strategy when the market size of 1 is fully realized. When $M_D\beta_D - \frac{k_D}{\alpha_D} \leq M_W\beta_W$, the wholesale channel strategy is more profitable in net terms than the direct channel for every possible value of the market size; otherwise, there exists a threshold demand value such that the wholesale channel is more profitable than the direct channel when demand is below this threshold and less profitable when it is above. We compare the two channel strategies in terms of gross profitability and growth-adjusted profitability in Table 1.

<table>
<thead>
<tr>
<th>Gross Profitability</th>
<th>Growth-Adjusted Profitability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_D\beta_D - \frac{k_D}{\alpha_D} \leq M_W\beta_W$</td>
<td>Case I</td>
</tr>
<tr>
<td>$M_D\beta_D - \frac{k_D}{\alpha_D} &gt; M_W\beta_W$</td>
<td>Case II</td>
</tr>
</tbody>
</table>

Table 1: Four possible cases by gross profitability and growth-adjusted profitability

The four cases we identify (remember that we have assumed $\alpha_D > \alpha_W$) are used to characterize the optimal policy as a function of the length of the time horizon $T$. We build up intuition by considering three settings sequentially: (i) no fixed channel-building investment costs and no direct channel fixed logistics cost i.e., $F_D = F_W = k_D = 0$; (ii) no fixed channel-building investment costs but a positive direct channel fixed logistics cost, i.e., $F_D = F_W = 0$ and $k_D > 0$ and (iii) positive fixed channel-building investment costs and direct channel fixed logistics cost i.e., $F_D \geq F_W > 0$ and $k_D > 0$. We first study the setting with no fixed costs and present the solution in Proposition 3.

**Proposition 3.** Suppose that $\max\{M_W\beta_W, M_D\beta_D\} > 0$ and $F_D = F_W = k_D = 0$. The optimal channel policy varies with the length of the time horizon as shown in Figure 5.

<table>
<thead>
<tr>
<th>Case I</th>
<th>[ T = 0 ]</th>
<th>W</th>
<th>DW</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cases II &amp; IV</td>
<td>[ T = 0 ]</td>
<td>D</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5: Optimal channel policy as a function of the length of the time horizon $T$ when $F_D = F_W = k_D = 0$. 

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Note that Case III does not occur when \( k_D = 0 \). From Figure 5, we see that, when the direct channel has lower gross profitability (Case I), the optimal policy differs based on the length of the time horizon. For relatively short time horizons, it is optimal to follow the W channel policy, while for relatively long time horizons, the optimal channel policy is DW, which we refer to as the “Grow Fast then Back Down” policy: the manufacturer first expands the market at a fast pace using the direct channel (possibly at a loss) then switches to the wholesale channel to exploit the expanded market. In contrast, in Cases II and IV, the direct channel is more profitable in net terms and leads to higher growth, so the D channel policy is optimal for any length of the time horizon.

Next, we consider the case of positive direct channel fixed logistics cost but zero fixed channel-building investment costs, i.e., \( F_D = F_W = 0 \) and \( k_D > 0 \). Proposition 4 shows how the optimal policy varies with the length of the time horizon \( T \).

**Proposition 4.** Suppose that \( \max\{M_W \beta_W, M_D \beta_D - \frac{k_D}{\sigma_D}\} > 0 \), \( F_D = F_W = 0 \) and \( k_D > 0 \). The optimal channel policy varies with the length of the time horizon as shown in Figure 6.

![Figure 6: Optimal channel policy as a function of the length of the time horizon T when \( F_D = F_W = 0 \) and \( k_D > 0 \).](image)

The existence of a positive fixed logistics cost \( k_D \) lowers the appeal of the direct channel strategy, which now requires a sufficient market size and gross profitability in order to be profitable. In Cases I and III, when the wholesale channel strategy is always more profitable than the direct channel strategy in net terms, the direct channel strategy should not be used as a single channel throughout the entire horizon or as a post-switch channel prior to the end of the horizon because of its poor net profitability. Instead, using the direct channel can add value to expand the market for switching to the more profitable wholesale channel to accumulate a higher profit overall. However, such a market expansion effect of the direct channel is only pronounced in Case I when the direct channel growth potential exceeds the threshold \( \frac{M_W \beta_W}{\sigma_W} \). So, it is always optimal to use the wholesale channel throughout the entire horizon for Case III. Further, for Case I, such a Grow Fast then Back
Down policy can only be optimal for relatively large horizons, which grant sufficient time for the manufacturer not only to expand the market but also to exploit the expanded market.

In Cases II and IV, even though the direct channel strategy has higher gross profitability than the wholesale channel, \( M_D\beta_D > M_W\beta_W \), it requires fixed logistics investments and therefore could have lower net profitability (defined as the gross profitability minus the fixed logistics costs then divided by the demand) when the market size is small. Yet we see from Figure 6 that, when the time horizon is very long, it is optimal to use only the direct channel strategy as there is enough time to reap the benefits from growing the market at a fast pace, even if it means doing so initially at a loss. In contrast, if the time horizon is very short, there is not enough time for the market to grow sufficiently to make the direct channel profitable net of logistics costs; therefore, it is optimal to use the W channel policy unless the wholesale channel gross profitability is negative (only possible in Case IV). For time horizons of intermediate length, a switching channel policy may be optimal. In Case II, the optimal channel policy is to Grow Fast then Back Down. In Case IV, the optimal policy is WD, which we refer to as the “Grow Slow then Upgrade” policy where the manufacturer first expands the market slowly using the wholesale channel at a small profit (as the direct channel is not yet profitable due to the fixed logistics costs) then switches to the direct channel to benefit from high demand growth and high gross profitability.

Finally, we analyze the setting where all fixed costs are positive, i.e., \( F_D \geq F_W > 0 \) and \( k_D > 0 \). Proposition 5 establishes how the optimal policy varies with the length of the time horizon \( T \).

**Proposition 5.** Suppose that \( \max\{ M_W\beta_W, M_D\beta_D - \frac{k_D}{M_W} \} > 0 \), \( F_D \geq F_W > 0 \) and \( k_D > 0 \). The optimal channel policy varies with the length of the time horizon as shown in Figure 7.

<table>
<thead>
<tr>
<th>Case</th>
<th>Policy</th>
<th>( T = 0 )</th>
<th>( T = T' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>( \emptyset )</td>
<td>( W )</td>
<td>( D )</td>
</tr>
<tr>
<td></td>
<td>( W )</td>
<td>( D )</td>
<td>( W )</td>
</tr>
<tr>
<td></td>
<td>( D )</td>
<td>( D )</td>
<td>( W )</td>
</tr>
<tr>
<td>II</td>
<td>( \emptyset )</td>
<td>( W )</td>
<td>( D )</td>
</tr>
<tr>
<td></td>
<td>( W )</td>
<td>( D )</td>
<td>( W )</td>
</tr>
<tr>
<td></td>
<td>( D )</td>
<td>( D )</td>
<td>( W )</td>
</tr>
<tr>
<td>III</td>
<td>( \emptyset )</td>
<td>( W )</td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>( \emptyset )</td>
<td>( W )</td>
<td>( WD )</td>
</tr>
<tr>
<td></td>
<td>( WD )</td>
<td>( D )</td>
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<td></td>
<td>( D )</td>
<td>( D )</td>
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</tbody>
</table>

Table: Optimal channel policy as a function of the length of the time horizon \( T \) when \( F_D \geq F_W > 0 \) and \( k_D > 0 \).

From Figure 7, we see that the existence of positive fixed channel-building investment costs...
imposes a minimum time horizon length only beyond which it is profitable to enter the market: in all four cases, if the time horizon is too short, it is not possible to recoup the fixed channel-building investment costs of using either strategy and therefore the manufacturer does not sell her product to the region. In Cases II, III and IV, we see that the structure of the optimal policy as a function of $T$ is the same as in the absence of fixed channel-building investment costs. In contrast, in Case I, the existence of positive fixed channel-building investment costs may create situations (for intermediate values of the time horizon) where the D policy is optimal. This occurs when the time horizon is long enough to eventually make the direct channel sufficiently profitable (net of logistics costs) but at the same time not long enough to justify incurring the channel-building investment cost to switch to the wholesale channel strategy even if it is more (grossly) profitable.

To conclude our analysis, below we present a proposition to show how the optimal channel switching time varies with the length of the time horizon $T$.

**Proposition 6.** For the DW policy, the optimal switching time from the direct channel to the wholesale channel $t^*_w$ is increasing in $T$. For the WD policy, the optimal switching time from the wholesale channel to the direct channel $t^*_d$ is decreasing in $T$ and the proportion of time using the direct channel $\frac{T-t^*_d}{T}$ is increasing in $T$.

In both switching policies, as the time horizon $T$ increases, the manufacturer should use the direct channel strategy for a longer period of time as larger time horizons give the direct channel strategy more opportunity to expand the market at a faster pace. For the DW policy, numerically, in Case I, we consistently find that the optimal proportion of time using the direct channel $\frac{t^*_w}{T}$ is first increasing then decreasing in $T$. In this case, the direct channel is always less profitable in net terms than the wholesale channel regardless of the horizon length such that the use of the direct channel serves merely to expand the market. Intuitively, as $T$ increases, the marginal market expansion effect of using the direct channel is first intensifying then shrinking as the demand grows convexly initially then concavely until saturation. Therefore, as $T$ increases, the manufacturer should use the direct channel for a larger proportion then for a smaller proportion of the time. In Case II, numerically, we always observe that the optimal proportion of time using the direct channel $\frac{t^*_w}{T}$ is increasing in $T$. In this case, the direct channel eventually becomes more profitable as $T$ increases which renders the direct channel a dominant position against the wholesale channel in both demand growth and profitability. Therefore, as $T$ increases, the manufacturer should use the direct channel for a larger proportion of time until the direct channel takes over the entire time horizon. For the WD policy, as $T$ increases, the manufacturer should always dedicate a larger proportion of time to use the direct channel since the earlier the channel is switched, the market is expanded at a higher growth rate again due to the S-shape growth curve.

Figure 8 illustrates Proposition 6 when $F_D = F_W = 0$ and $k_D > 0$. In the first graph (Case I), the
region where the DW switching channel policy is optimal is to the right of the vertical dashed line. On the second graph (Case II), it is between the two dashed lines. Finally, on the third graph, the WD switching channel policy is also to be found for time horizon values which are between the two vertical dashed lines. For Cases I and II, when the DW policy is optimal, $t_{DW}^*(T)$ is increasing in $T$, which means that it is optimal to switch from the direct channel to the wholesale channel strategy relatively later as the length of the time horizon increases; for Case IV, when the WD policy is optimal, $t_{WD}^*(T)$ is decreasing in $T$, which means it is optimal to switch from the wholesale channel to the direct channel proportionally sooner for longer horizons. We see, in Cases II and IV, as the time horizon $T$ increases, the manufacturer should use the direct channel strategy for a longer proportion of the time, while in Case I, the proportion of time using the direct channel strategy is increasing initially then decreasing.

Figure 8: Optimal switching time as a function of the length of the time horizon $T$ when $F_D = F_W = 0$ and $k_D > 0$.

In summary, we see that the optimal channel policy depends on the relative values of the growth-adjusted profitability and the gross profitability of both channels. Using these two profitability measures, we are able to fully characterize the impact of the time horizon on the optimal channel policy by considering Cases I, II, III and IV. We see that, for short time horizons, it is generally optimal to use the W channel policy as it has lower fixed costs. On the other hand, for long time horizons, there is enough time to recoup the higher fixed costs required to implement the direct channel strategy and benefit from the higher demand growth rate it generates. For the time horizons with intermediate length, a switching channel policy may be optimal, that is, the firm should use one channel then switch to the other. When the wholesale channel has higher growth-adjusted profitability than the direct channel, the manufacturer who uses both channels should Grow Fast then Back Down, that is, first expand the market at a fast pace using the direct channel
(possibly at a loss) then switch to the wholesale channel to exploit the expanded market. In contrast, when the direct channel has higher growth-adjusted profitability than the wholesale channel, the manufacturer who uses both channels should Grow Slow then Upgrade, that is, first earn limited profits while expanding the market slowly using the wholesale channel then switch to the direct channel to benefit from high demand growth and high gross profitability.

4 Numerical Results

In this section, we numerically illustrate the parametric sensitivity of the optimal channel policy and evaluate the performance of several heuristic policies. We use a base parameter set adapted from the data of a CPG company in the city of Bogota, Colombia, to conduct the analysis. According to the manufacturer’s data set, in the fiscal year 2014, the city-wide sales revenues from all the products supplied directly to nanostores increased by 16.2% while sales by the wholesale channel grew by 5%. However, the city-wide operating margin of the wholesale channel was 9.2% higher than that of the direct channel strategy.

In Online Appendix B, we provide background on how we used the company’s numbers to set the base parameter values: $xy = 1, \delta = 100, T = 2, b = 1, k_D = 2000, a_D = 3, a_W = 1, M_D\beta_D = 300, M_W\beta_W = 500, F_D = 2000, F_W = 1000$. Under this base case, it takes around 1.07 years and 3.20 years to penetrate 90% of the market potential through using the direct channel and the wholesale channel respectively.

Given these parameters, we have $M_D\beta_D \leq M_W\beta_W$ and $M_D\beta_D - \frac{k_D}{xy} \leq M_W\beta_W$ (Case I from Table 1). After optimization, we see it is optimal for the manufacturer to switch from the direct channel to the wholesale channel (the DW policy) and the switch occurs at $t^*_W = 0.69$.

In Figure 9, we vary the gross profitability fraction $\frac{M_D\beta_D}{M_W\beta_W}$ and the ratio of demand growth rate $\frac{a_D}{a_W}$ by varying $a_D$ in $[1, 3]$ and $M_D\beta_D$ in $[100, 900]$ while keeping all other parameters fixed. By doing so, we generated 64,160 instances which span all four cases listed in Table 1. Panels (a) and (b) represent the optimal policy when the time horizon is equal to 1 and 2 respectively.

We see that the DW policy is optimal when the gross profitability of the wholesale channel is larger than that of the direct channel but the direct channel provides a much higher demand growth rate. In these instances, the role of the direct channel is to initially expand the market, then the firm switches to the more profitable wholesale channel when facing a well-developed market. If the direct channel is more profitable and offers a significantly higher growth rate, it should be used exclusively. Finally, when the wholesale channel profitability is high and the difference in growth rates is small, it is best to use only the wholesale channel. Note the “W is optimal” region includes instances where the direct channel has both higher net profitability and demand growth rate (Cases II and IV). This can be explained by the fixed-costs effect. Comparing Panels (a) and (b) of Figure
9, we see that the region where the DW policy is optimal expands when the time horizon increases as the firm has more time to develop the market through direct store visits.

Next, we consider four heuristic policies: (i) the W heuristic, by which the manufacturer uses solely the wholesale channel, (ii) the D heuristic, by which she uses only the direct channel, (iii) the WD-half heuristic, where she switches from the wholesale channel to the direct channel at the halfway point of the time horizon and (iv) the DW-half heuristic, where she switches from the direct channel to the wholesale channel at the halfway point of the time horizon.

We vary parameters $b$, $T$, $M_D\beta_D$ and $\alpha_D$ one at a time in order to evaluate the optimality gap of the heuristic policies, which is calculated as $OG = \frac{\Pi_{opt}^t - \Pi^h}{\Pi_{opt}^t}$ where $\Pi_{opt}^t$ represents the optimal profit and $\Pi^h$ represents the profit of a heuristic policy, except when $\Pi_{opt}^t$ is equal to zero, in which case, the optimality gap is equal to zero. The results are illustrated in Figure 10.

All the instances considered on Figure 10 fall into Case I from Table 1 so that the DW policy is potentially optimal, i.e., the manufacturer should switch from the direct channel to the wholesale channel if necessary. All four graphs show that the DW-half heuristic generally performs best as it chooses the right sequence of channel strategies; on the other hand, the WD-half heuristic generally performs worst as choosing the wrong sequence of channel strategies. In comparison, the two single-channel policies tend to perform well for extreme instances but perform very poorly in the opposite extreme instances because they fail to capture the trade-off between the two profitability metrics. This shows that it is more critical to refer to the growth-adjust profitability metric to devise the channel policy as it factors in both the gross profitability and also future growth potential.
5 Conclusions

Manufacturers serving the nanostore channel in emerging megacities face a tough challenge. Based on empirical evidence, serving nanostores directly will grow demand faster but can be very expensive, especially if the density of stores is low in a particular neighbourhood. In practice, many manufacturers choose a blanket channel strategy for a city, that is, serving the entire city either directly or via wholesale; in this paper, we provide a model that captures the channel choice decision for a neighborhood in such a city.

Since the channel choice affects demand growth over time, we show that it is important to include demand growth differences in the channel selection decision. The channel selection decision then develops into a channel selection policy: not only does a manufacturer decide how to enter into a market, but this decision is also linked to switches in strategy at a later moment in time.

The structure of the optimal channel selection and switching policy is complex. However, a number of interesting general insights emerge from our analysis. The channel selection depends on the time horizon of the manufacturer making the channel selection decision. For companies that evaluate their decisions over a relatively short horizon, it is generally optimal to choose a wholesale-only policy. Using this strategy, the high costs of setting up a direct channel are avoided, albeit at the expense of limited market-growth. For companies that are in the market for the long
term, such as many local incumbents, it is often optimal to choose a direct-only policy. Our results however also show a clear alternative, namely to switch between the two channel strategies at some point in time, either starting in the direct channel and then switching to wholesale, or doing this the other way around. Such a switching policy is optimal for intermediate time horizon, and dependent on the so-called growth adjusted profitability. The growth adjusted profitability is based on the gross profitability of a channel (determined by the gross margin and the logistics cost) but adjusted for the demand growth pace of the channel. A manufacturer may hence choose to go first direct and then wholesale, i.e., Grow Fast, then Back Down, that is, first expand the market at a fast pace using the direct channel (possibly at a loss), and then switch to the wholesale channel to exploit the developed market. This policy is generally optimal for intermediate time horizons if the growth-adjusted profitability of the wholesale channel is larger than the growth-adjusted profitability of the direct channel. In the reverse case, Grow Slow, then Upgrade is generally optimal, that is, first earn limited profits while slowly growing the market using the wholesale channel, and then switch to the direct channel to benefit from high demand growth and high gross profitability.

In general, our study provides insights into the channel strategies that manufacturers could deploy when serving urban nanostores in emerging markets. Our work enables manufacturers to develop distribution strategies where the channel changes after some time. Given the large diversity in neighborhoods in terms of income and nanostores density, manufacturers could deploy such strategies in a heterogeneous manner, making the key trade-off in different neighborhoods.

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References


A Travel distance calculation in the direct channel strategy

To calculate the total distance traveled by all the delivery vehicle trips in the selling region when using the direct channel strategy, we use the half-width routing formulation from Daganzo (1987), page 174. According to Equation (1) from this paper (and using our notation), the total distance traveled by one delivery vehicle making $S$ stops over a rectangular region with length $y$ and width $x$, when the depot is located at a distance of $\varphi$ from the bottom edge of the region, can be approximated by $2\varphi + 2y + \frac{Sx}{6}$. In this expression, the first term is the distance from the depot to the edge of the region. The second term is the vertical distance traveled within the region and the third term is the horizontal distance. In our problem, it takes $\Omega_{D(t)}$ vehicle trips to serve the $xy\delta$ customers in the selling region. We assume that this is implemented by dividing the selling region in rectangular sub-regions along the vertical dimension as shown in Figure 11: the first vehicle trip serves the bottom part of the rectangle, from the bottom edge to a vertical distance of $\varphi + y \frac{C_D}{\Omega_{D(t)}}$ from the depot, the second vehicle trip covers a rectangle which starts from a distance of $\varphi + y \frac{C_D}{\Omega_{D(t)}}$ from the depot to a distance of $\varphi + 2y \frac{C_D}{\Omega_{D(t)}}$, etc. As such, each vehicle trip covers a rectangle of height $y \frac{C_D}{\Omega_{D(t)}}$ and width $x$. The first vehicle has to drive a distance of $\varphi$ to get to its serving region, the second vehicle drives a distance $\varphi + y \frac{C_D}{\Omega_{D(t)}}$ to get there, etc.

![Figure 11: Visual representation of delivery trips using the direct channel strategy.](image_url)

The total distance traveled by the first vehicle is $2\varphi + 2y \frac{C_D}{\Omega_{D(t)}} + \frac{x^2 y \delta C_D}{6 \Omega_{D(t)}}$. The total distance traveled by the second vehicle is $2\left(\varphi + y \frac{C_D}{\Omega_{D(t)}}\right) + 2y \frac{C_D}{\Omega_{D(t)}} + \frac{x^2 y \delta C_D}{6 \Omega_{D(t)}}$. The total distance traveled by the last vehicle (which is the $\frac{\Omega_{D(t)}}{C_D}$-th vehicle) is: $2 \left[\varphi + y \frac{C_D}{\Omega_{D(t)} \left(\frac{\Omega_{D(t)}}{C_D} - 1\right)}\right] + 2y \frac{C_D}{\Omega_{D(t)}} + \frac{x^2 y \delta C_D}{6 \Omega_{D(t)}}$. Sum-
ming up the travel distances of all the vehicles, we get the total distance traveled as:

\[
2 \left[ \frac{\Omega_D(t)}{C_D} \phi + \frac{C_D}{\Omega_D(t)} \left( 0 + 1 + 2 + \ldots + \frac{\Omega_D(t)}{C_D} - 1 \right) \right] + 2y \frac{C_D}{\Omega_D(t)} \frac{\Omega_D(t)}{C_D} + \frac{x^2 y \delta C_D}{6 \Omega_D(t)} \frac{\Omega_D(t)}{C_D}
\]

\[
= 2 \left[ \frac{\Omega_D(t)}{C_D} \phi + \frac{C_D}{\Omega_D(t)} \frac{1}{2} \left( \frac{\Omega_D(t)}{C_D} - 1 \right) \frac{\Omega_D(t)}{C_D} \right] + 2y + \frac{x^2 y \delta}{6}
\]

\[
= 2 \frac{\Omega_D(t)}{C_D} \phi + y \left( \frac{\Omega_D(t)}{C_D} - 1 \right) + 2y + \frac{x^2 y \delta}{6}
\]

\[
= \frac{\Omega_D(t)}{C_D} (2 \phi + y) + y \left( 1 + \frac{x^2 \delta}{6} \right).
\]

### B Base parameter values for the numerical study

According to Mejía-Argueta et al. (2017), the city of Bogota counts about 100,000 stores in the urban area of the city. The manufacturer who provided us with the dataset (and who cannot be named for confidentiality reasons) is a relatively large publicly-traded company which operates mostly in Latin America selling packaged food products such as candy, cookies, pasta, sauces, juices, ice cream, etc. It sells to both modern and traditional retail stores. In 2014, the company’s share of revenues in the city of Bogota generated from modern retail stores (e.g., convenience store chains and large supermarkets) was 38% versus 62% for traditional retail stores. For the sales from traditional retail, 15% directly came from 42,500 nanostores in the urban city area with an average density of 138 stores per km$^2$, and 47% came from around 800 wholesalers. It is opaque how many nanostores that each wholesaler is serving; however, a confident guess is the number ranges from 20 to 50, as we have observed in most emerging markets that wholesalers in the CPG sector are small sized and each wholesaler only serves a few dozens of nanostores in close proximity. Therefore, in this numerical study, we use a base geographic region with an area of 1 km$^2$ and a density of 100 stores per km$^2$; namely, the base values of $xy$ and $\delta$ are 1 and 100 respectively.

According to the manufacturer’s data set, in the fiscal year 2014, the city-wide sales revenues from all the products supplied directly to nanostores increased by 16.2% while sales by wholesalers grew by 5%. Assuming the revenue increase was driven by market size growth, then the market size grew three times faster when using the direct channel than using the wholesale channel. Given an assumption of the product launch time, we can use a base value of $b$ to induce the corresponding values of $\alpha_D$ and $\alpha_W$. However, in order to generate a full spectrum of product attributes, we loosely use a base value of $\alpha_D = 3$ and a base value of $\alpha_W = 1$ as one third of $\alpha_D$. We also use the base value of $b$ as 1, at which it takes around 1.07 years and 3.20 years to penetrate 90% of the market potential through using the direct channel and the wholesale channel respectively. Next, we use these base values to estimate the margin and cost parameters.
For the city-wide direct channel operations, the manufacturer estimates the cost of goods sold at 56% of sales revenues and the (variable+fixed) cost of logistics at 5.1% of sales revenues. Hence, the manufacturer calculates the operating margin as 38.9% of sales revenues. On average, the city-wide operating margin of the wholesale channel is 9.2% higher than that of the direct channel strategy on a monthly basis in the year of 2014. Further, according to a sample of 942 stores in 4 regions, the annual per-store average revenue is around 384 US dollars so that the average annual per-store net profit of using the direct channel is around 149 US dollars. Then, we can inflate this number by 9.2% as the average annual per-store net profit of using the wholesale channel, which is 163 US dollars. Assume the new product was launched at the beginning of the year 2014. Then, the annual profit of using the wholesale channel was:

$$\Pi^W = M_W \beta_W \delta \gamma \int_0^1 \frac{1}{1 + e^{b-a_W t}} dt - F_W.$$ 

Let $F_W = 1,000$ as the base value of the fixed investment cost to set up using the wholesale channel. Then, the base value of $M_W \beta_W$ is around 500 US dollars. When using the direct channel, the annual profit was:

$$\Pi^D = \underbrace{\left[p_D - c - z\right]}_{M_D} \beta_D \delta \gamma \int_0^1 \frac{1}{1 + e^{b-a_D t}} dt - k_D - F_D$$

where $z$ is the variable logistics cost and $k_D$ is the fixed logistics cost. For simplicity, let $z = 0$; then as 5.1% of the sales revenues, we roughly take the base value of $k_D$ as 2,000. Further, let $F_D = 2,000$; then we obtain the base value of $M_D \beta_D$ as around 300. Note that the estimated base values of $M_W \beta_W$ and $M_D \beta_D$ are the gross margins of per unit normalized demand. Finally, let the base value of the time horizon $T$ be 2.

### C Proofs

**Proof.** Proof of Lemma 1

From Equation (3), we see that, if $\pi_W > \pi_D$ ($\pi_W < \pi_D$), using the wholesale channel strategy generates more (less) profit per unit of time than using the direct channel strategy. Since these values do not vary over time, and switching channel policy is costly, it is never optimal to do so. Selling via the wholesale channel strategy during the entire time horizon generates cumulative profit $\Pi(W) = T \pi_W - F_W$ while selling via the direct channel strategy during the entire time horizon generates cumulative profit $\Pi(D) = T \pi_D - F_D$. The optimal channel policy is the channel strategy which generates the highest cumulative profits, provided this value is positive; otherwise, it is optimal not to sell the product. □
Proof. Proof of Lemma 2

From Equations (4) and (5), we see that \( \Pi(D) \geq \Pi(W) \) is equivalent to \( M_D \beta_D x y \delta \lambda T - F'_D \geq M_W \beta_W x y \delta \lambda T - F_W \). Since we have assumed \( F'_D \geq F_D \geq F_W \), this implies that \( M_D \beta_D \geq M_W \beta_W \).

Proof. Proof of Theorem 1

First, note that \( \Lambda_W(\delta), \Lambda_D(\delta) \) are the threshold values of \( \lambda \) such that the wholesale-only and direct-only policies break even respectively, i.e., they are obtained by solving \( \Pi(W) = 0 \) and \( \Pi(D) = 0 \) as a function of \( \lambda \) respectively. Also, \( \lambda^e(\delta) \) is the value of \( \lambda \) such that the cumulative profits of both policies are equal, i.e., the value of \( \lambda \) such that \( \Pi(W) = \Pi(D) \). All three expressions are convex in \( \delta \). On the graph on the left, we see that when \( M_D \beta_D \leq M_W \beta_W \frac{F_D + (\mu_D + \frac{1}{2})yT}{T_W} \), for a fixed value of \( \delta \), the wholesale-only policy always breaks even first, so that there are intermediate values of the store density \( \delta \) such that it is optimal. On the graph on the right, the same is true for values of \( \delta \) above threshold \( \delta_\lambda \), which is the unique positive value of \( \delta \) which solves \( \Lambda_D(\delta) = \Lambda_W(\delta) \), i.e. such that the two channel policies break even for the same value of \( \lambda \). For \( \delta \leq \delta_\lambda \), the direct-only policy breaks even first so there is no range of \( \delta \) value where it is optimal to use the wholesale channel strategy.

Proof. Proof of Theorem 2

The result follows directly from the fact that \( T_W \) is the value of the horizon length so that the wholesale-only policy breaks even, that is, such that \( \Pi(W) = 0 \) conditional on \( \pi_W > 0 \). Similarly, \( T_D \) is the value of the horizon length so that the direct-only policy breaks even, that is, such that, \( \Pi(D) = 0 \) conditional on \( \pi_D > 0 \). Finally, \( T^e = \frac{F_D - F_W}{\pi_D - \pi_W} \) is the value of the time horizon so that the cumulative profit of the wholesale-only policy is equal to that of the direct-only policy, which is positive if \( \pi_D > \pi_W \) since we assume that \( F_D > F_W \). When \( \pi_D > \pi_W > 0 \), there exists a range of values for \( T \) for which the wholesale-only policy is optimal when it breaks even first i.e., when \( T_W \leq T_D \).

Proof. Proof of Theorem 3

Note that \( \beta_{W}^* \) is the threshold value of \( \beta_W \) such that \( \Pi(W) = 0 \) and \( \beta_D^* \) is the threshold value of \( \beta_D \) such that \( \Pi(D) = 0 \). Also \( \beta_D^e(\beta_W) \) is the value of \( \beta_D \) such that \( \Pi(D) = \Pi(W) \) for a given value of \( \beta_W \). Finally, \( \beta_D^e \) is the value of \( \beta_W \) such that \( \beta_D^e(\beta_W) = \beta_W \).

In Figures 4(a), (b) and (c), all three possible policies can be optimal because either the slope of \( \beta_D^e \) is greater than 1 or the point \( (\beta_W^*, \beta_D^*) \) where both policies yield exactly zero profits is in the upper triangular region. In Figure 4(d), these two conditions are not satisfied, therefore there is no region where the wholesale-only policy is optimal.

Proof. Proof of Proposition 1
First, note that not selling the product leads to zero profit so that, if it is optimal to sell the product at any time during the time horizon, this must mean that it is possible to earn positive profit over the entire horizon. This, in turn, implies that the instant profit rate must be positive by the end of the selling period given that the market size only grows over time when the product is being sold (via either channel). As a result, it cannot be optimal to stop selling the product as it would generate zero profit instead of positive profits. Also, remember that we have assumed that the market does not grow when the product is not being sold by the manufacturer; this implies that delaying entry to the market is equivalent to shortening the total time horizon and therefore this cannot be optimal for the same reason. □

Proof. Proof of Proposition 2

From Proposition 1, if it is optimal to sell to the region, the manufacturer only uses the wholesale and direct channel strategies, possibly switching multiple times between them throughout the entire time horizon. So suppose we have a channel policy \((S, t)\) with only these two selling channel strategies, i.e., such that \(S_{i+1} = D\) and \(S_{i+2} = W\) for \(i = 0, ..., N−2\). In other words, the manufacturer uses the direct channel strategy from time \(t_i\) to \(t_{i+1}\) then the wholesale channel strategy from time \(t_{i+1}\) until \(t_{i+2}\) for some \(i \in \{0, ..., N−2\}\). Let \(\lambda(t_i) = \frac{1}{1−e^{−C}}\) where \(C = \sum_{j=1}^{N} aS_j(t_j−t_{j−1})\). The profit generated by this policy from time \(t_i\) to time \(t_{i+2}\) is:

\[
\Pi = \int_{t_i}^{t_{i+2}} \pi_D(t)dt + \int_{t_{i+1}}^{t_{i+2}} \pi_W(t)dt - F_D - F_W
\]

\[
= \frac{M_W \beta_W}{\alpha_W} \log \left[ \frac{e^{b+\alpha_D t_i} + e^{C+\alpha_D t_{i+1}+\alpha_W(t_{i+2}-t_{i+1})}}{e^{b+\alpha_W t_i} + e^{C+\alpha_D t_{i+1}+\alpha_W(t_{i+2}-t_{i+1})}} \right] + k_D(t_{i+1} - t_i)
\]

\[
+ \frac{M_D \beta_D}{\alpha_D} \log \left[ \frac{e^{b} + e^{C+\alpha_D (t_{i+1}-t_i)}}{e^{b} + e^{C}} \right] - F_D - F_W.
\]

Now consider an alternative channel policy \((S', t')\) such that \(t'_k = t_k\) for \(k = 1, ..., i, i+2, ..., N\), \(t'_{i+1} = t_i + t_{i+2} - t_{i+1}\), \(S'_k = S_k\) for \(k = 1, ..., i, i+3, ..., N\), \(S'_{i+1} = W\) and \(S'_{i+2} = D\). In other words, the alternative channel policy switches the order of the wholesale channel strategy and the direct channel strategy between time \(t_i\) and \(t_{i+2}\), keeping the length of all intervals the same as in policy \((S, t)\). The profit generated by this policy from time \(t_i\) to time \(t_{i+2}\) is:

\[
\Pi' = \int_{t_i}^{t_{i+2}} \pi_W(t)dt + \int_{t_{i+1}}^{t_{i+2}} \pi_D(t)dt - F_DI_{i=N−2} - F_WI_{i=0}
\]

\[
= \frac{M_D \beta_D}{\alpha_D} \log \left[ \frac{e^{b+\alpha_W t_{i+1}} + e^{C+\alpha_W t_{i+2}+\alpha_D(t_{i+1}-t_i)}}{e^{b+\alpha_W t_{i+1}} + e^{C+\alpha_W t_{i+2}}} \right] - k_D(t_{i+1} - t_i)
\]

\[
+ \frac{M_W \beta_W}{\alpha_W} \log \left[ \frac{e^{b} + e^{C+\alpha_W(t_{i+2}-t_{i+1})}}{e^{b} + e^{C}} \right] - F_DI_{i=N−2} - F_WI_{i=0}.
\]

Note that we may save on the fixed costs because switching the order of the channel strategies
makes them coincide with the strategies used in intervals \([t_{i-1}, t_i]\) when \(i > 0\) and \([t_{i+2}, t_{i+3}]\) when \(i < N - 2\). Since the profit generated before \(t_i\) and after \(t_{i+2}\) are the same under both policies, we have:

\[
\Pi(S', t') - \Pi(S, t) \geq x y \delta C \log \left[ \frac{(e^b + e^c) \left( e^{b + \delta t_l + \delta w t_{l+1}} + e^{c + \delta t_l + \delta w t_{l+1}} \right)}{(e^{b + \delta t_l + e^{\delta t_{l+1}}} + e^{c + \delta t_l + e^{\delta t_{l+1}}})} \left( \frac{M_D \beta_D}{\alpha_D} - \frac{M_W \beta_W}{\alpha_W} \right) \right]
\]

where the logarithmic term is positive because the numerator is greater than the denominator of the fraction with a difference \(e^{b+c} \left( e^{\delta t_{l+1}} - e^{b t_l} \right) \left( e^{\delta w t_{l+2}} - e^{a w t_{l+1}} \right)\). From this, we see that, when \(\frac{M_D \beta_D}{\alpha_D} > \frac{M_W \beta_W}{\alpha_W}\), the alternate policy generates more profit than the original policy.

We have shown that, for every pair of adjacent intervals, it is optimal for the manufacturer to switch from the direct (wholesale) channel strategy to the wholesale (direct) channel strategy if and only if \(\frac{M_D \beta_D}{\alpha_D} < \frac{M_W \beta_W}{\alpha_W}\). Hence, this directly implies that there can be at most one switch between the two selling channel strategies, that is, it is optimal to have \(N \leq 2\).

**Proof.** Proof of Lemma 3

The second derivative of \(\Pi^{DW}(t_W)\) is as follows:

\[
\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2} = x y \delta D e^{\alpha_D t_W - b} \left( e^{\alpha_W (T - t_W) \frac{M_W \beta_W (\alpha_D - \alpha_W)^2}{\alpha_W D}} \left( 1 + e^{\alpha_W (T - t_W) - b} \right)^2 \left( -\alpha_D + \frac{M_D \beta_D}{\alpha_D} - \frac{M_D \beta_D}{\alpha_W} \right) \right)
\]

where the first derivative of the sign-determinant term \(A^{DW}\) is given by:

\[
\frac{\partial A^{DW}}{\partial t_W} = e^{\alpha_W (T - t_W)} \frac{\left( 1 + e^{-b + \alpha_W t_W} \right)}{\left( 1 + e^{-b + \alpha_D t_W + \alpha_W (T - t_W)} \right)^2} \left( e^{(\alpha_D - \alpha_W) t_W - 2b} (a_W e^{\alpha_D t_W + \alpha_W T} + e^b (e^{\alpha_W T} - e^{\alpha_W t_W}) (a_W - 2 \alpha_D)) - a_W \right).
\]

Obviously, the number of sign changes of \(\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2}\) is determined by the term \(B^{DW}\). Note that \(\frac{\partial A^{DW}}{\partial t_W}\) is negative at \(t_W = 0\) since at which \(B^{DW}\) is decreasing in \(\alpha_D\) and it is negative at \(\alpha_D = a_W\). The first derivative of \(B^{DW}\) in \(t_W\) is:

\[
\frac{\partial B^{DW}}{\partial t_W} = e^{(\alpha_D - \alpha_W) t_W - 2b} (2 \alpha_D - a_W) \left( a_D e^{\alpha_W t_W + b} + a_W e^{\alpha_D t_W + \alpha_W T} - a_W e^{\alpha_W T + b} \right)
\]

where the third multiplying part (in brackets) is increasing in \(t_W\) and is positive at \(t_W = T\), i.e., \(\frac{\partial B^{DW}}{\partial t_W}\) is either positive for \(t_W \in [0, T]\) or negative for low values of \(t_W\) and positive for large values of \(t_W\). This means \(B^{DW}\) is either increasing in \(t_W \in [0, T]\) or is decreasing in low values of \(t_W\) and increasing in large values of \(t_W\). Since \(\frac{\partial A^{DW}}{\partial t_W}\) is negative at \(t_W = 0\), \(\frac{\partial A^{DW}}{\partial t_W}\) is either negative for \(t_W \in [0, T]\) or is negatively initially then positive in \(t_W\). Therefore, \(A^{DW}\) is either decreasing in \(t_W \in [0, T]\) or is decreasing then increasing in \(t_W\), which implies \(\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2}\) has at most two sign.
changes. When there is no sign change, $\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2}$ is either positive or negative, so $\Pi^{DW}(t_W)$ is either convex or concave in $t_W$. In this case, there is at most one interior local maximum. When there is one sign change, $\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2}$ either switches from positive to negative or the other way around, which implies $\Pi^{DW}(t_W)$ is either convex then concave in $t_W$ or the other way around. In this case, there is at most one interior local maximum. When there are two sign changes, $\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2}$ is positive for low values of $t_W$, negative for intermediate values of $t_W$ and positive for large values of $t_W$, which indicates $\Pi^{DW}(t_W)$ is convex then concave and finally convex in $t_W$. In this case, considering possible existence of inflection points, there is at most one interior local maximum.

Similarly, the second derivative of $\Pi^{WD}(t_D)$ is as follows:

$$\frac{\partial^2 \Pi^{WD}(t_D)}{\partial t_D^2} = \frac{xy^{\delta} M_D \beta_D (\alpha_D - a_W)^2}{\alpha_D (1 + e^{b-a_W t_D})^2 e^{a_W t_D}} \left( \frac{e^{s(t_D-T)} \left(1 + e^{b-a_W t_D}\right)^2}{1 + e^{b-a_W T+(a_D-a_W) t_D}} \right) \left( \left(1 + e^{b-a_W T+(a_D-a_W) t_D}\right)^2 \right)$$

where the first derivative of the sign-determinant term $A^{WD}$ in $t_D$ is given by:

$$\frac{\partial A^{WD}}{\partial t_D} = \frac{e^{s(t_D-T)} \left(1 + e^{b-a_W t_D}\right)^2}{\left(1 + e^{b-a_W T+(a_D-a_W) t_D}\right)^3} \left( \left(1 + e^{b-a_W T+(a_D-a_W) t_D}\right)^2 \right)$$

Obviously, the number of sign changes of $\frac{\partial^2 \Pi^{WD}(t_D)}{\partial t_D^2}$ is determined by the term $B^{WD}$. The first derivative of $B^{WD}$ in $t_D$ is:

$$\frac{\partial B^{WD}}{\partial t_D} = e^{b-a_D T-2a_W t_D} \left(2a_D - \alpha_D\right) \left(\alpha_D e^{b+a_D t_D} + (\alpha_D - a_W) e^{a_D+a_W t_D} + a_W e^{a_D+a_W t_D}\right)$$

which is negative if $\alpha_D > 2a_W$ and positive if $\alpha_D < 2a_W$. Then, if $\alpha_D \neq 2a_W$, $B^{WD}$ is either decreasing or increasing in $t_D \in [0, T]$. If $\alpha_D = 2a_W$, we have $B^{WD} = 2 \left(1 - e^{2(b-a_W T)}\right) a_W$, which is either positive or negative. Therefore, $\frac{\partial A^{WD}}{\partial t_D}$ can be positive, negative, positive then negative, or negative then positive in $t_D \in [0, T]$. This means $A^{WD}$ can be increasing, decreasing, increasing then decreasing, or decreasing then increasing in $t_D \in [0, T]$, which implies $\frac{\partial^2 \Pi^{WD}(t_D)}{\partial t_D^2}$ can have at most two sign changes. If there is no or only one sign change, there is at most one interior local maximum. Given two sign changes, considering possible inflection points, there is at most one interior local maximum if $\frac{\partial^2 \Pi^{WD}(t_D)}{\partial t_D^2}$ is positive, negative then positive in $t_D \in [0, T]$; there are at most two interior local maxima if $\frac{\partial^2 \Pi^{WD}(t_D)}{\partial t_D^2}$ is negative, positive and then negative in $t_D \in [0, T]$. When two such values exist, it must hold $\alpha_D > 2a_W$ and $M_W < 0$ since under which $A^{WD}$ is possibly increasing then decreasing in $t_D$, and

$$\left. \frac{\partial^2 \Pi^{WD}(t_D)}{\partial t_D^2} \right|_{t_D=T} = \frac{xy^{\delta} e^{b-a_W T} (M_W \beta_D + M_D \beta_D (a_D - 2a_W))}{(1 + e^{b-a_W T})^2}$$
is possibly negative. □

**Proof.** Proof of corollary 1

Proof omitted. □

**Proof.** Proof of Proposition 3

When \( F_D = F_W = k_D = 0 \), the cases to be considered are Case I where \( M_D \frac{\beta_D}{\alpha_D} \leq \frac{M_W \beta_W}{\alpha_W} \) and \( M_D \beta_D \leq M_W \beta_W \), Case II where \( M_D \frac{\beta_D}{\alpha_D} \leq \frac{M_W \beta_W}{\alpha_W} \) and \( M_D \beta_D > M_W \beta_W \) and Cases IV where

\[
M_D \frac{\beta_D}{\alpha_D} > \frac{M_W \beta_W}{\alpha_W} \quad \text{and} \quad M_D \beta_D \geq M_W \beta_W.
\]

In Cases III & IV, by Corollary 1, we need to consider \( \Pi^{WD}(t_D) \). We have:

\[
\frac{\partial \Pi^{WD}(t_D)}{\partial t_D} = xy \alpha_W \left( \frac{M_D \beta_D \frac{\alpha_W - \alpha_D}{\alpha_W} \frac{1 + e^{b - \alpha_D t + (\alpha_D - \alpha_W) t_D}}{1 + e^{b - \alpha_W t_D}}}{1 + e^{b - \alpha_D t + (\alpha_D - \alpha_W) t_D}} - \frac{M_D \beta_D \frac{\alpha_D}{\alpha_D} - \frac{M_W \beta_W}{\alpha_W}}{1 + e^{b - \alpha_W t_D}} \right)
\]

which is negative since \( M_D \frac{\beta_D}{\alpha_D} > \frac{M_W \beta_W}{\alpha_W} \), \( M_D \beta_D > 0 \) (since \( \max \{ M_W \beta_W, M_D \beta_D \} > 0 \) and \( \alpha_D > \alpha_W \). Therefore, \( \Pi^{WD} \) is decreasing in \( t_D \) so that the unique optimal solution is \( t_D^* = 0 \), that is, the D policy is optimal for any length of the time horizon.

In Cases I and II, by Corollary 1, we need to consider \( \Pi^{DW}(t_W) \). For Case II, we have:

\[
\frac{\partial \Pi^{DW}(t_W)}{\partial t_W} = xy \alpha_D \left( \frac{1 + e^{t_W + \alpha_W (T - t_W) - b}}{1 + e^{t_W + \alpha_W (T - t_W) - b}} \left( \frac{M_W \beta_W}{\alpha_W} - \frac{M_D \beta_D}{\alpha_D} \frac{1 + e^{b - \alpha_D t + (\alpha_D - \alpha_W) t_W}}{1 + e^{b - \alpha_W t_W}} - \frac{M_D \beta_D - M_W \beta_W}{\alpha_D} \right) \right)
\]

\[
\geq xy \alpha_D \left( \frac{M_W \beta_W}{\alpha_W} - \frac{M_D \beta_D}{\alpha_D} \frac{1 + e^{b - \alpha_D t + (\alpha_D - \alpha_W) t_W}}{1 + e^{b - \alpha_W t_W}} + \frac{M_D \beta_D - M_W \beta_W}{\alpha_D} \right)
\]

\[
= \frac{xy \delta (M_D \beta_D - M_W \beta_W)}{1 + e^{b - \alpha_W T - (\alpha_D - \alpha_W) t_W}} > 0
\]

where the first inequality is because \( M_W \frac{\beta_W}{\alpha_W} \geq M_D \frac{\beta_D}{\alpha_D} \) and \( e^{t_W + \alpha_W (T - t_W) - b} \geq e^{b - \alpha_W T} \) and the second inequality is because \( M_D \beta_D > M_W \beta_W \). Therefore, \( \Pi^{DW}(t_W) \) is increasing in \( t_W \) so that the optimal solution in Case II is \( t_W^* = T \), that is, the D policy is optimal for any length of the time horizon.

In Case I, where \( M_D \beta_D < M_W \beta_W \), we evaluate the first derivative of \( \Pi^{DW}(t_W) \) at \( t_W = 0 \) and \( t_W = T \) as:

\[
\left. \frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \right|_{t_W=0} = xy \alpha_D \left( \frac{M_W \beta_W}{\alpha_W} - \frac{M_D \beta_D}{\alpha_D} \frac{1 + e^{b - \alpha_D t + (\alpha_D - \alpha_W) t_W}}{1 + e^{b - \alpha_W t_W}} + \frac{M_D \beta_D - M_W \beta_W}{\alpha_D} \right)
\]

\[
> \frac{xy \delta (M_D \beta_D - M_W \beta_W)}{1 + e^{b - \alpha_D t + (\alpha_D - \alpha_W) t_W}} + \frac{M_D \beta_D - M_W \beta_W}{\alpha_D}
\]

\[
\left. \frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \right|_{t_W=T} = \frac{xy \delta (M_D \beta_D - M_W \beta_W)}{1 + e^{b - \alpha_D t + (\alpha_D - \alpha_W) T}}
\]
where the inequality is because \( M_W \beta_W > 0 \) (due to \( \frac{M_D \beta_D}{a_D} \leq \frac{M_W \beta_W}{a_W} \) and \( \max\{M_W \beta_W, M_D \beta_D\} > 0 \). We first see \( \frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \bigg|_{t_W=0} \) can be positive or negative. We also see \( \frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \bigg|_{t_W=T} \) is negative for \( M_D \beta_D < M_W \beta_W \). Therefore, the D policy is never optimal. Further since we always have \( \frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \bigg|_{t_W=T} > \frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \bigg|_{T=T_1} \) for \( t_W \leq T_1 \) for any \( T_1 \) and \( T_2 > T_1 \), the interior local maximum \( \hat{t}_W(T) \) always exists for the time horizon beyond a certain threshold and \( \hat{t}_W(T) \) is increasing in \( T \).

Next, we show that the first derivative of \( \Pi^{DW}(t_W) \) increases in \( T \) faster at its interior local maximum (if it exists) than at zero in Case I. We have:

\[
\frac{\partial \Pi^{DW}(\hat{t}_W(T))}{\partial T} = \frac{xy\delta M_W \beta_W}{1 + e^{b-a_W T - (a_D-a_W)\hat{t}_W}}
\]

\[
\frac{\partial \Pi^{DW}(0)}{\partial T} = \frac{xy\delta M_W \beta_W}{1 + e^{b-a_W T}}
\]

where \( \frac{\partial \Pi^{DW}(\hat{t}_W(T))}{\partial T} > \frac{\partial \Pi^{DW}(0)}{\partial T} > 0 \) because \( M_W \beta_W > 0 \) as shown previously and \( e^{b-a_W T} > e^{b-a_W T - (a_D-a_W)\hat{t}_W} \). Because \( \lim_{T \to 0} \frac{\partial \Pi(t_W)}{\partial t_W} = \frac{xy\delta(M_D \beta_D - M_W \beta_W)}{1+e^{a_D t_W}} < 0 \) for \( M_D \beta_D < M_W \beta_W \), for low \( T \), the first derivative in \( t_W \) is always negative, implying that \( \Pi^{DW}(t_W) \) is decreasing. After the time horizon is beyond some certain threshold, the interior local maximum \( \hat{t}_W(T) \) always exists.

Initially, \( \Pi^{DW}(t_W) \) may be maximized at 0 but since its first derivative increases in \( T \) faster at its interior local maximum than at zero, there always exists a threshold value \( T_i \) such that \( \Pi^{DW}(t_W) \) is maximized at zero for \( T \leq T_i \) and maximized at its unique interior local maximum for \( T \geq T_i \).

Hence, the optimal channel policy is \( W \) for low values of \( T \leq T_i \) and is \( DW \) for \( T \geq T_i \). \( \square \)

**Proof.** Proof of Proposition 4

When \( F_D = F_W = 0 \) and \( k_D > 0 \), the first derivatives of \( \Pi^{WD}(t_D) \) and \( \Pi^{DW}(t_W) \) are as follows:

\[
\frac{\partial \Pi^{WD}(t_D)}{\partial t_D} = xy\delta a_W \left( \frac{M_D \beta_D a_w - a_D}{1 + e^{b-a_D T + (a_D-a_w)T_D}} - \frac{M_D \beta_D}{a_D} \right) + k_D
\]

\[
\frac{\partial \Pi^{DW}(t_W)}{\partial t_W} = xy\delta a_D \left( \frac{M_W \beta_W a_w - M_D \beta_D a_D}{1 + e^{a_D T + a_D(T-W)}} - \frac{M_W \beta_W}{a_D} \right) - k_D.
\]

In Cases I and II, by Corollary 1, we need to consider \( \Pi^{DW}(t_W) \). We have:

\[
\frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \bigg|_{t_W=0} = \begin{cases} 
xy\delta a_D \left( \frac{M_W \beta_W a_w - M_D \beta_D a_D}{1 + e^{-b}} - \frac{M_W \beta_W a_D - a_D}{1 + e^{a_D T - b}} + \frac{M_D \beta_D - M_W \beta_W}{a_D} \right) & - k_D \\
xy\delta (M_D \beta_D - M_W \beta_W) & - k_D
\end{cases}
\]

\[
\frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \bigg|_{t_W=T} = \begin{cases} 
xy\delta (M_D \beta_D - M_W \beta_W) - k_D \\
xy\delta (M_D \beta_D - M_W \beta_W) & - k_D
\end{cases}
\]
where the inequality is because $M_W \beta_W > 0$ (due to $M_D \frac{\beta_D}{\alpha_D} \leq \frac{M_W \beta_W}{\alpha_W}$ and max($M_W \beta_W, M_D \beta_D - \frac{k_0}{xy^0}) > 0$).

In Case I, the first derivative is always negative at $t_W = T$ for finite $T$ since $M_D \beta_D - \frac{k_0}{xy^0} \leq M_W \beta_W$ by assumption. So, the D policy is never optimal for any finite time horizon. By arguments similar to the proof of Proposition ??, we can prove that $\Pi^{DW}(t_W)$ increases in $T$ faster at its interior local maximum (if it exists) than at zero and conclude that the optimal channel policy is $W$ for low values of $T \leq T_I$ and is $DW$ for $T \geq T_I$.

In Case II, $\frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \bigg|_{t_W=0}$ can be positive or negative and $\frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \bigg|_{t_W=T}$ may be negative for low values of $T$ but is positive for large values of $T$. The interior local maximum $\hat{t}_W (T)$ may not exist, for example, when $\frac{xy \delta(M_D \beta_D - M_W \beta_W)}{1 + e^\delta} - k_D \geq 0$, we have:

\[
\frac{\partial \Pi^{DW}(t_W)}{\partial t_W} = xy \delta_D \left( \frac{M_W \beta_W}{\alpha_W} - \frac{M_D \beta_D}{\alpha_D} \right) \left( 1 + e^{\delta_D t_W} - b \right) - k_D
\]

\[
\geq xy \delta_D \left( \frac{M_W \beta_W}{\alpha_W} - \frac{M_D \beta_D}{\alpha_D} \right) \left( 1 + e^{\delta_D t_W} - b \right) - k_D
\]

\[
= \frac{xy \delta(M_D \beta_D - M_W \beta_W)}{1 + e^{\delta_D t_W}} - k_D > \frac{xy \delta(M_D \beta_D - M_W \beta_W)}{1 + e^\delta} \geq 0
\]

where the first inequality is because $M_W \beta_W > 0$ (due to $\frac{M_D \beta_D}{\alpha_D} \leq \frac{M_W \beta_W}{\alpha_W}$ and max($M_W \beta_W, M_D \beta_D - \frac{k_0}{xy^0}) > 0$) and $\alpha_D > \alpha_W$, and the second inequality is because $M_D \beta_D - M_W \beta_W > \frac{k_0}{xy^0} > 0$ by the definition of Case II.

We follow up to claim, if $\hat{t}_W (T)$ exists, it must exist within a continuous time horizon interval and it is increasing in $T$. To prove this, we refer to the proof of Lemma 3 and characterize the impact of $T$ on the number of sign changes of the second derivative of $\Pi^{DW}(t_W)$ in $t_W$:

\[
\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2} = xy \delta_D e^{\delta_D t_W} a_D \left( \frac{e^{\delta_D (T-t_W)} M_W \beta_W (a_D-a_W)}{\alpha_D} \right) - \frac{\alpha_D (M_W \beta_W - M_D \beta_D)}{\alpha_D} \left( 1 + e^{\delta_D t_W} - b \right)^2
\]

We begin to evaluate $\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2}$ when $T$ approaches 0 and at $t_W = T$ as follows:

\[
\lim_{T \to 0} \frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2} = \frac{xy \delta e^b (M_D \beta_D a_D - M_W \beta_W (2a_D - a_W))}{(1 + e^b)^2}
\]

\[
\frac{\partial^2 \Pi^{DW}(t_W)}{\partial t_W^2} \bigg|_{t_W=T} = \frac{xy \delta e^b (M_D \beta_D a_D - M_W \beta_W (2a_D - a_W))}{(e^b + e^{a_D T})^2}
\]

of which the signs are determined by the term $M_D \beta_D a_D - M_W \beta_W (2a_D - a_W)$. Next, we split the analysis into two scenarios: (i) $M_D \beta_D < M_W \beta_W a_D a_W$, and (ii) $M_D \beta_D \geq M_W \beta_W a_D a_W$. 

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For (i), $\Pi^{DW}(t_W)$ is concave in some small neighborhood with $t_W = T$ on the perimeter for any $T$. According to the proof of Lemma 3, this is the only possible concave segment of the profit function. We assume the unique interior local maximum $\hat{t}_W(T)$ exists for horizon $T'$. Then, the profit function is decreasing in $(\hat{t}_W(T'),T']$. Since $\frac{\partial \Pi^{DW}(t_W)}{\partial t_W}$ is increasing in $T$ for $t_W < \hat{t}_W(T')$, $\hat{t}_W(T)$ keeps in existence and increases in $T > T'$ until the profit function is increasing in the concave segment. Since $T'$ applies to any horizon such that $\hat{t}_W(T)$ exists, $\hat{t}_W(T)$ is increasing in a continuous range of $T$ given its existence.

For (ii), $\Pi^{DW}(t_W)$ is convex in $t_W$ for very low values of $T$ because $\lim_{T\to 0} \frac{\partial \Pi^{DW}(t_W)}{\partial t_W} \geq 0$ and

$$\frac{\partial A^{DW}}{\partial T} = \frac{M_W \beta_W (a_D - a_W)^2 e^{b + a_W t_W + T} (e^{b + a_W t_W} - e^{a_D t_W + a_W T})}{a_D (e^{a_D t_W + a_W T} + e^{b + a_W t_W})^3}$$

is positive for very low values of $T$. Since the sign-determinant term of $\frac{\partial A^{DW}}{\partial T}$ is decreasing in $T$, as $T$ increases, the profit function may become concave in a segment and stay concave in this segment thereafter. If this occurs, according to the proof of Lemma 3, this is the only concave segment of the profit function. Similar to (i), we can show $\hat{t}_W(T)$ exists and increases in a continuous range of $T$ given its existence.

In Case II, in general, we consider all $W$, $D$ and $DW$ policies to be potentially optimal. We have:

$$\frac{\partial \Pi^{DW}(\hat{t}_W(T))}{\partial T} = \frac{xy \delta M_W \beta_W}{1 + e^{b - a_W T - (a_D - a_W) \hat{t}_W}}$$ (6)

$$\frac{\partial \Pi^{DW}(0)}{\partial T} = \frac{xy \delta M_W \beta_W}{1 + e^{b - a_W T}}$$ (7)

$$\frac{\partial \Pi^{DW}(T)}{\partial T} = \frac{xy \delta M_D \beta_D}{1 + e^{b - a_W T} - k_D}$$ (8)

where (8) may be less than (6) and (7) for low values of $T$ but are greater than these two for large values of $T$ because $M_D \beta_D - \frac{k_D}{xy} > M_W \beta_W > 0$ as shown previously and $b - a_D T < b - a_W T - (a_D - a_W) \hat{t}_W < b - a_W T$. In addition, it is easy to see that $\frac{\partial \Pi^{DW}(\hat{t}_W(T))}{\partial T} > \frac{\partial \Pi^{DW}(0)}{\partial T} > 0$ for any length of the time horizon (given the existence of $\hat{t}_W(T)$). Therefore, there exist thresholds $T_{II}^1 \geq 0$ and $T_{II}^2 \geq T_{II}^1$ such that the $W$ policy is optimal for $T \leq T_{II}^1$, the $DW$ policy is optimal for $T \in (T_{II}^1, T_{II}^2]$ and the $D$ policy is optimal for $T > T_{II}^2$.

In Cases III and IV, by Corollary 1, we need to consider $\Pi^{WD}(t_D)$. When we evaluate the first
derivative at \( t_D = 0 \) and \( t_D = T \), we get:

\[
\frac{\partial \Pi^{WD}(t_D)}{\partial t_D} \bigg|_{t_D=0} = k_D + xy\delta\alpha W \left( \frac{M_D\beta_D a_W - a_D}{a_D} \frac{a_D}{a_W} \left( 1 + e^{b - a_D T} \right) \right) - \frac{M_D\beta_D}{a_D} - \frac{M_W\beta_W}{a_W}
\]

\[
< k_D + xy\delta\alpha W \left( \frac{M_D\beta_D a_W - a_D}{a_D} \frac{a_D}{a_W} \left( 1 + e^{b} \right) \right)
\]

\[
= \frac{xy\delta (M_W\beta_W - M_D\beta_D)}{1 + e^{b}} + k_D
\]

\[
\frac{\partial \Pi^{WD}(t_D)}{\partial t_D} \bigg|_{t_D=T} = \frac{xy\delta (M_W\beta_W - M_D\beta_D)}{1 + e^{b - a_W T}} + k_D
\]

where the inequality is because \( M_D\beta_D > 0 \) (due to \( \frac{M_D\beta_D}{a_D} > \frac{M_W\beta_W}{a_W} \) and \( \max\{M_D\beta_D - \frac{k_D}{xy\delta}, M_W\beta_W\} > 0 \)) and \( \alpha_D > \alpha_W \).

For Case III, we have \( \frac{\partial \Pi^{WD}(t_D)}{\partial t_D} \bigg|_{t_D=T} > 0 \) for any finite \( T \) because \( M_D\beta_D - \frac{k_D}{xy\delta} \leq M_W\beta_W \). For \( t_D < T \), we have:

\[
\frac{\partial \Pi^{WD}(t_D)}{\partial t_D} = xy\delta\alpha W \left( \frac{M_D\beta_D a_W - a_D}{a_D} \frac{a_D}{a_W} \left( 1 + e^{b - a_D T + (a_D - a_W) t_D} \right) \right) + k_D
\]

\[
> xy\delta\alpha W \left( \frac{M_D\beta_D a_W - a_D}{a_D} \frac{a_D}{a_W} \left( 1 + e^{b - a_W T} \right) \right) - \frac{M_D\beta_D}{a_D} - \frac{M_W\beta_W}{a_W}
\]

\[
= \frac{xy\delta (M_W\beta_W - M_D\beta_D)}{1 + e^{b - a_D T + (a_D - a_W) T}} + k_D
\]

\[
> xy\delta (M_W\beta_W - M_D\beta_D) + k_D \geq 0
\]

where the first inequality is because \( b - \alpha_D T + (a_D - a_W) t_D < b - \alpha_W T_D \) and \( \frac{M_D\beta_D}{a_D} > \frac{M_W\beta_W}{a_W} \), the second to the last inequality is due to \( M_D\beta_D - \frac{k_D}{xy\delta} \leq M_W\beta_W \) as shown previously and the last inequality is due to \( M_D\beta_D - \frac{k_D}{xy\delta} \leq M_W\beta_W \) in Case III. Therefore, \( \Pi^{WD}(t_D) \) is increasing in \( t_D \) and, as a result, the \( W \) policy is optimal for any length of the horizon.

In Case IV, \( \frac{\partial \Pi^{WD}(t_D)}{\partial t_D} \bigg|_{t_D=0} \) can be negative or positive, and \( \frac{\partial \Pi^{WD}(t_D)}{\partial t_D} \bigg|_{t_D=T} \) may be positive initially and then is negative for large values of \( T \) due to \( M_D\beta_D - \frac{k_D}{xy\delta} > M_W\beta_W \). The profit function has at most two concave segments (see the proof of Lemma 3) either of which possibly admits an interior local maximum for different length of the time horizon. However, the profit function may not admit any interior local maximum for any length of the time horizon, for example, when
\[
\frac{xy \delta (M_W b_D - M_D b_D)}{1 + \epsilon^b} + k_D \leq 0, \text{ we have:}
\]
\[
\frac{\partial \Pi^{WD}(t_D)}{\partial t_D} = xy \delta \alpha_W \left( \frac{M_D b_D \alpha_W - \alpha_D}{\alpha_D} - \frac{M_W b_W \alpha_W}{\alpha_W} \right) \left( \frac{M_D b_D \alpha_W - \alpha_D}{\alpha_D} - \frac{M_W b_W \alpha_W}{\alpha_W} \right) + k_D
\]
\[
\leq xy \delta \alpha_W \left( \frac{M_D b_D \alpha_W - \alpha_D}{1 + \epsilon^b - \alpha_w t_D} - \frac{M_W b_W \alpha_W}{1 + \epsilon^b - \alpha_w t_D} \right) + k_D
\]
\[
= \frac{\partial \Pi^{WD}(t_D)}{\partial t_D} = \frac{xy \delta (M_W b_W - M_D b_D)}{1 + \epsilon^b - \alpha_w t_D} + k_D
\]
\[
\leq \frac{xy \delta (M_W b_W - M_D b_D)}{1 + \epsilon^b} + k_D \leq 0
\]

where the first inequality is because \( M_D b_D > 0 \) as shown previously, \( \alpha_D > \alpha_W \) and \( \epsilon^{b - \alpha_D T + (\alpha_D - \alpha_w) T} < \epsilon^b - \alpha_w t_D \), and the second inequality is because \( M_D b_D > M_D b_D - \frac{k_D}{x y \delta} > M_W b_W \).

Next, given existence, we show the property of the interior local maximum (maxima) in \( T \) and also characterize the optimal policy change in \( T \). We refer to the proof of Lemma 3 and characterize the impact of \( T \) on the number of sign changes of the second derivative of \( \Pi^{WD}(t_D) \) in \( t_D \):

\[
\frac{\partial^2 \Pi^{WD}(t_D)}{\partial t_D^2} = \frac{xy \delta e^b (M_W b_W \alpha_W + M_D b_D (\alpha_D - 2 \alpha_W))}{1 + \epsilon^b} + \frac{x y \delta \epsilon^b (M_W b_W \alpha_W + M_D b_D (\alpha_D - 2 \alpha_W))}{1 + \epsilon^b}
\]

of which the signs are determined by the term \( M_W b_W \alpha_W + M_D b_D (\alpha_D - 2 \alpha_W) \). Next, we split the analysis in four scenarios: (i) \( \alpha_D \leq 2 \alpha_W \) and \( M_W b_W \alpha_W + M_D b_D (\alpha_D - 2 \alpha_W) \geq 0 \), (ii) \( \alpha_D \leq 2 \alpha_W \) and \( M_W b_W \alpha_W + M_D b_D (\alpha_D - 2 \alpha_W) < 0 \), (iii) \( \alpha_D > 2 \alpha_W \) and \( M_W b_W \alpha_W + M_D b_D (\alpha_D - 2 \alpha_W) \geq 0 \), and (iv) \( \alpha_D > 2 \alpha_W \) and \( M_W b_W \alpha_W + M_D b_D (\alpha_D - 2 \alpha_W) < 0 \).

For (i), according to the proof of Lemma 3, there is at most one concave segment, which possibly admits one interior local maximum. For low values of \( T \), \( \Pi^{WD}(t_D) \) is convex in \( t_D \) because \( \lim_{T \to 0} \frac{\partial^2 \Pi^{WD}(t_D)}{\partial t_D^2} \geq 0 \) and

\[
\frac{\partial A^{WD}}{\partial T} = \frac{\alpha_D e^{\alpha_D (t_D + T)} \left( e^b + e^{\alpha_W T} \right)^2 \left( e^{b + \alpha_D T} - e^b \right)}{\left( e^{b + \alpha_D T} + e^{\alpha_D T} \right)^3}
\]

is positive for very low values of \( T \). We see the sign of \( \frac{\partial A^{WD}}{\partial T} \) is determined by the term \( e^{b + \alpha_D T} - e^b \), which is decreasing in \( T \). Therefore, \( A^{WD} \) can become negative so that \( \Pi^{WD}(t_D) \) can
become concave in one segment of $t_D$. We assume the unique interior local maximum $\hat{t}_D(T)$ exists for horizon $T'$. This implies the profit function is increasing then decreasing in the neighborhoods centered at $\hat{t}_D(T')$. Since $\frac{\partial \Pi_{WD}(t)}{\partial t}$ is decreasing in $T > T'$ for $t_D < \hat{t}_D(T')$, $\hat{t}_D(T)$ keeps in existence and decreases in $T > T'$ until $\Pi_{WD}(t_D)$ is decreasing in the unique concave segment starting from $t_D = 0$. Since $T'$ applies to any horizon of $T$ such that $\hat{t}_D(T)$ exists, given its existence, $\hat{t}_D(T)$ exists and decreases in a continuous range of $T$.

For (ii), there also exists at most one concave segment since $α_D ≤ 2α_W$. Further, $\Pi_{WD}(t_D)$ is concave in neighborhoods with $t_D = T$ on the perimeter and it is the only concave segment for any length of the time horizon. Using the similar arguments to (i), we also can show, given its existence, the unique interior local maximum $\hat{t}_D(T)$ exists and decreases in a continuous range of $T$.

For (iii), if there are two concave segments, according to the proof of Lemma 3, $\Pi_{WD}(t_D)$ is negative and increasing in low values of $t_D$, then is positive and decreasing in intermediate values of $t_D$, and finally is negative in large values of $t_D$. This contradicts with $\frac{\partial^2 \Pi_{WD}(t_D)}{\partial t^2} |_{t_D=T} ≥ 0$ which defines (iii). Therefore, there exists at most one concave segment. Following similar arguments to (i), we can show, given its existence, the unique interior local maximum $\hat{t}_D(T)$ exists and decreases in a continuous range of $T$.

For Scenarios (i) – (iii), there is at most one interior local maximum in a continuous range of $T$, so there are three optimal policy candidates: the D policy, the W policy and the WD policy with $t_D = \hat{t}_D(T)$. We have:

\[
\frac{\partial \Pi_{WD}(\hat{t}_D(T))}{\partial T} = \frac{xyδM_Dβ_D}{1 + e^{b-a_DT+(a_D-α_W)l_D}} - k_D \quad (10)
\]

\[
\frac{\partial \Pi_{WD}(0)}{\partial T} = \frac{xyδM_Dβ_D}{1 + e^{b-a_DT}} - k_D \quad (11)
\]

\[
\frac{\partial \Pi_{WD}(T)}{\partial T} = \frac{xyδM_Wβ_W}{1 + e^{b-α_WT}} \quad (12)
\]

If $M_Wβ_W > 0$, (11) and (10) may be less than (12) for low values of $T$ and are greater than (12) for large values of $T$ because $M_Dβ_D - \frac{k_D}{2β_D} > M_Wβ_W$. Further, (11) is greater than (10) for any length of the horizon given the existence of $\hat{t}_D(T)$. Therefore, there exist thresholds $\bar{T}_1^V ≥ 0$ and $\bar{T}_2^V ≥ T_1^V$ at which the W policy is optimal for $T ≤ \bar{T}_1^V$, the WD policy is optimal for $T ∈ (T_1^V, T_2^V]$ and the D policy is optimal for $T > T_2^V$. If $M_Wβ_W < 0$, the W policy is never optimal so that there exist two thresholds $\bar{T}_1^V ≥ 0$ and $\bar{T}_2^V ≥ T_1^V$ at which the no-selling policy is optimal for $T ≤ \bar{T}_1^V$, the WD policy is optimal for $T ∈ (\bar{T}_1^V, \bar{T}_2^V]$ and the D policy is optimal for $T > \bar{T}_2^V$. Overall, if the switch policy is optimal, the optimal switch point $\bar{t}_D(T)$ is decreasing in $T$. If there is no interior local maximum for any length of the time horizon, there are two optimal policy candidates: the D policy and the W policy. Then, we have $\bar{T}_1^V = \bar{T}_2^V$.

Finally, we consider Scenario (iv), which can only occur when $M_Wβ_W < 0$ so the W only policy
is never optimal. According to Lemma 3, $\Pi_{WD}(t_D)$ possibly owns two concave segments each of which admits an interior local maximum. If there is at most one interior local maximum, similar to Scenarios (i) – (iii), we can show it exists and decreases in a continuous interval of $T$ and the optimal policy changes in $T$ in the same fashion. In the following, we only focus on the possibility of the existence of two interior local maxima $\hat{t}_D^1(T)$ and $\hat{t}_D^2(T)$ which may not co-exist for any length of the time horizon.

Since $\frac{\partial \Pi_{WD}(t_D)}{\partial t_D}\big|_{t_D=T} < 0$, $\Pi_{WD}(t_D)$ is always concave in neighborhoods with $t_D = T$ on the perimeter where $\hat{t}_D^2(T)$ emerges in a continuous interval $T \in (T'^2, T'^1)$. For low $T$, this is the only concave segment since $\lim_{T \to 0} \frac{\partial \Pi_{WD}(t_D)}{\partial t_D} < 0$. As $T$ increases, a new concave segment in the neighborhoods with $t_D = 0$ on the perimeter appears where $\hat{t}_D^1(T)$ emerges in a continuous interval $T \in (T'^1, T'^2)$. Using similar arguments to (i) – (iii), we can show both $\hat{t}_D^1(T)$ and $\hat{t}_D^2(T)$ decrease in their respective interval of $T$.

The profit function is always increasing (decreasing) at $t_D = 0$ before (after) $\hat{t}_D^1(T)$ disappears; it is always increasing (decreasing) at $t_D = T$ before (after) $\hat{t}_D^2(T)$ appears. When $\hat{t}_D^1(T)$ appears earlier than $\hat{t}_D^2(T)$, i.e., $T'^2 < T'^1$, $\hat{t}_D^2(T)$ must disappear after $\hat{t}_D^1(T)$ appears, i.e., $T'^2 > T'^1$, because otherwise the profit function is decreasing in $t_D$ for $T'^2 < T < T'^1$, where $\hat{t}_D^1(T)$ never exists. When $\hat{t}_D^2(T)$ appears no earlier than $\hat{t}_D^1(T)$, i.e., $T'^2 \geq T'^1$, $\hat{t}_D^2(T)$ may appear after and before $\hat{t}_D^1(T)$ disappears, which means these two values may not overlap in $T$. When $\hat{t}_D^1(T)$ and $\hat{t}_D^2(T)$ co-exist in the relevant continuous range of $T$, we have:

$$\frac{\partial \Pi_{WD}(\hat{t}_D^1(T))}{\partial T} = \frac{xy\delta M_D \beta_D}{1 + e^{b-a_D T + (a_D-a_W)\hat{t}_D^1}} - k_D > \frac{xy\delta M_D \beta_D}{1 + e^{b-a_D T + (a_D-a_W)\hat{t}_D^2}} - k_D = \frac{\partial \Pi_{WD}(\hat{t}_D^2(T))}{\partial T}$$

(13)

which means, as $T$ increases, the early switch policy (with the switch point $\hat{t}_D^1(T)$) approaches (continues to outperform) its counterpart if it generates a less profit (it has already generated a higher profit). In the following, we split the discussion in two sub-scenarios.

When $\hat{t}_D^2(T)$ appears earlier than or at the same time with $\hat{t}_D^1(T)$, regardless the disappearance sequence of these two values, given co-existence, the manufacturer realizes a higher profit at $\hat{t}_D^2(T)$ than at $\hat{t}_D^1(T)$ initially and then possibly the other way around due to (13). This implies the manufacturer needs to trade off the D policy with the WD policy with $t_D = \hat{t}_D^2(T)$ initially then possibly with $t_D = \hat{t}_D^1(T)$ where the relevant switch point in consideration is decreasing in $T$. Then, since (11) is always greater than (10), the optimal policy changes in $T$ in the same fashion as with Scenarios (i) – (iii) with $M_W \beta_W < 0$.

When $\hat{t}_D^2(T)$ appears after $\hat{t}_D^1(T)$, the first derivative of the profit function at $t_D = T'^2$ is exactly zero for $T = T'^2$. This implies, the profit of the second concave segment is negative for $T \leq T'^2$ because in which this segment is maximized at $t_D = T$ with a negative profit under the W policy. Given that $\hat{t}_D^2(T)$ appears before $\hat{t}_D^1(T)$ disappears, if the early switch policy (with the switch point
\( i_D^1(T) \) generates a higher profit at \( T \leq T'^2 \), it will continue to do so due to (13). Then, the policy candidates for the manufacturer are the no-selling policy, the D policy and the WD policy with the switch point \( i_D^1(T) \). Otherwise, the early switch policy generates a negative profit for \( T \leq T'^2 \). Then, for \( T > T'^2 \), due to (13), the manufacturer only needs to trade off the D policy with the switch policy with \( t_D = i_D^1(T) \) initially and then possibly with \( t_D = i_D^1(T) \) where the switch point in consideration is decreasing in \( T \). Given that \( i_D^1(T) \) appears after \( i_D^1(T) \) disappears, if the D policy (with \( t_D = 0 \)) generates a less or the same profit with the late switch policy at \( t_D = i_D^1(T) \) for \( T = T'^2 \), the no-selling policy is optimal for \( T \leq T'^2 \), and the manufacturer only needs to trade off these two policies and the no-selling policy for \( T > T'^2 \). Otherwise, the late switch policy is never optimal for \( T > T'^2 \) because the profit it generates is less and is increasing at a slower rate in \( T \) than the switch policy with switch points close to \( t_D = 0 \) in the first concave segment. Therefore, the manufacturer only needs to trade off the D policy, the early switch policy and the no-selling policy for any length of \( T \). As a summary, optimal policy changes in \( T \) in the same fashion as with Scenarios (i) – (iii) with \( W_W \beta_W < 0 \). \( \square \)

**Proof.** Proof of Proposition 5

Let \( \Pi_0^{DW}(t_W) \) and \( \Pi_0^{WD}(t_D) \) represent the profit functions when \( F_W = F_D = 0 \) and \( k_D > 0 \).

In Case I, by Corollary 1, we need to consider \( \Pi^{DW}(t_W) \). For \( t_W < T \), we have:

\[
\frac{\partial \Pi_0^{DW}(t_W)}{\partial T} = \frac{xy \delta M_W \beta_W}{1 + e^{b-a_T} - (a_D-a_W)T}
\]

which is positive since \( M_W \beta_W > 0 \) (due to \( M_W \beta_W \geq M_D \beta_D - \frac{k_D}{xy} \) and \( \max\{M_D \beta_D - \frac{k_D}{xy}, M_W \beta_W\} > 0 \)). This implies the value \( \Pi_0^{DW}(t_W) \) is increasing in \( T \) for \( t_W < T \). Therefore, when \( F_D > 0 \) and \( F_W > 0 \), there exists a time horizon threshold \( T^1 \) so that it is optimal not to sell the product for \( T \leq T^1 \) and optimal to sell otherwise.

In the proof of Proposition 4, we have shown that \( \frac{\partial \Pi_0^{DW}(t_W)}{\partial W_W} \bigg|_{t_W = T} < 0 \), which is to say, it is never optimal to use the D policy when \( F_D = F_W = 0 \). Now, when \( F_D > 0 \) and \( F_W > 0 \), the D policy is potentially optimal since it incurs a fixed cost less than the DW policy. For \( T > T^1 \), we have:

\[
\frac{\partial \Pi_0^{DW}(\hat{t}_W(T))}{\partial T} = \frac{xy \delta M_W \beta_W}{1 + e^{b-a_T} - (a_D-a_W)\hat{t}_W}
\]

(14)

\[
\frac{\partial \Pi_0^{DW}(0)}{\partial T} = \frac{xy \delta M_W \beta_W}{1 + e^{b-a_T} - k_D}
\]

(15)

\[
\frac{\partial \Pi_0^{DW}(T)}{\partial T} = \frac{xy \delta M_D \beta_D}{1 + e^{b-a_T} - k_D}
\]

(16)

where (15) is less than (14) as long as the interior local maximum \( \hat{t}_W(T) \) exists. Since \( M_D \beta_D - \frac{k_D}{xy} \leq M_W \beta_W \) and \( b - a_T > b - a_W T \), (16) is less than (15) for low and high values of \( T \) and possibly higher than (15) for intermediate values of \( T \). Similarly, given the existence of \( \hat{t}_W(T) \), since
where $M_D \beta_D - \frac{k_D}{x_0} \leq M_W \beta_W$, (16) is less than (14) for high values of $T$. Further because $b - \alpha_D T < b - \alpha_W T - (a_D - a_W)\hat{t}_W$, if (16) is less than (14) right when $\hat{t}_W(T)$ appears, (16) is less than (14) for low values of $T$ and possibly larger than (14) for intermediate values of $T$. Otherwise, (16) is larger than (14) in $T$ initially then it happens the other way around.

Given the above arguments and further because $F_W \leq F_D$, the W policy is more likely to be optimal for low and intermediate values of $T$, the D policy is more likely to be optimal for intermediate values of $T$, and the DW policy is more likely to be optimal for high values of $T$. However, technically, for $T > T^3_I$, all these three can be the optimal policy candidate immediately following the no-selling policy. We only consider the most general case where there exists horizon threshold $T^3_I \geq T^3$ such that the W policy is optimal for $T \in (T^3_I, T^3]$ due to a lower fixed investment cost and the profit function is decreasing in $t_W$ for very low $T$ (due to $\lim_{T \to 0} \frac{\partial \Pi(t_W)}{\partial t_W} = \frac{x u \delta (M_D \beta_D - M_W \beta_W)}{1 + e^{T}} - k_D < 0$). For $T > T^3_I$, the D and DW are the next optimal policy candidates. Again, for generality, we only consider the case where there exists horizon threshold $T^3_I \geq T^3$ so that the D policy is optimal for $T \in (T^3_I, T^3]$ because the profit of the D policy is possibly increasing at a higher pace than the DW policy for intermediate values of $T$ and it incurs a less fixed investment.

We follow up to show the optimal policy structure in $T$ under the most general case. Note that it is very easy to show the results of the others are simply special cases of the general structure, and, as a result, we omit the details.

For $T > T^3_I$, the W and DW are the optimal policy candidates. There are two scenarios: (i) the W policy follows to be the next optimal policy and (ii) the DW follows to be the next optimal policy.

In (i), the profit growth rate of the W policy must have outpaced that of the D policy at $T = T^3_I$, which implies the D policy is suboptimal for $T > T^3_I$. Therefore, there must exist a horizon threshold $T^4_I$ such that the W policy is optimal for $T \in (T^3_I, T^4_I]$ and the DW policy is optimal for $T > T^4_I$. Note that the above results hold only if $M_D \beta_D > 0$. If $M_D \beta_D \leq 0$, the D policy is never optimal so that we must have $T^4_I = T^3 = T^3_I$.

In (ii), the W policy is suboptimal for $T > T^3_I$ and the profit growth rate of the DW policy must have been higher than that of the D policy at $T = T^3_I$. There are two possibilities. First, for $T > T^3_I$, the profit growth rate of the DW policy in $T$ continues to be higher then lower and finally higher than that of the D policy, which implies there exist two additional thresholds $T^5_I$ and $T^6_I$ so that the DW policy is optimal for $T \in (T^3_I, T^5_I]$ and the D policy becomes optimal again for $T \in (T^5_I, T^6_I]$ and, then is outperformed again by the DW policy for $T > T^6_I$. Second, the profit growth rate of the DW policy in $T$ has been higher initially then lower and further higher for $T \leq T^3_I$. Therefore, the profit growth rate of the DW policy in $T$ continues to be higher for $T > T^3_I$, which implies the DW policy continues to be optimal so that we have $T^5_I = T^4 = T^3_I$. Note that the above results hold only if $M_D \beta_D > 0$. If $M_D \beta_D \leq 0$, the D policy is never optimal so that we must have $T^4_I = T^3 = T^3_I$.
In Case II, we have $M_D \beta_D - \frac{k_D}{\sigma_Y} > M_W \beta_W > 0$. Similar to Case I, we can show $\Pi_0^{ DW}(t_W)$ is increasing in $T$ for $t_W < T$. Therefore, when $F_D > 0$ and $F_W > 0$, there exists a time horizon threshold $T_{1I}$ so that it is optimal not to sell the product for $T \leq T_{1I}$ and optimal to sell otherwise. We see (15) is still less than (14) as long as the interior local maximum $t_W(T)$ exists, and (16) is possibly lower than (15) and (14) for low values of $T$ and higher than these two for high values of $T$. Further since $F_D \geq F_W > 0$, there exist time horizon thresholds $T_{1I}^2 \geq T_{1I}$ and $T_{1I}^3 \geq T_{1I}$ such that it is optimal to sell by the W policy for $T \in (T_{1I}, T_{1I}^2]$ due to a lower fixed investment, it is optimal to use the DW policy for $T \in (T_{1I}^2, T_{1I}^3]$ because the profit growth rate of this policy is possibly higher than that of the D policy for low values of $T$, and finally it is optimal to use the D policy for $T > T_{1I}^3$. 

In Case III, by Corollary 1, we need to consider $\Pi^{ WD}(t_D)$. According to the proof of Proposition 4, $\Pi_0^{ WD}(t_D)$ is increasing in $t_D$. Since $F_D \geq F_W$, the W policy is the only optimal policy candidate. We have:

$$\left. \frac{\partial \Pi^{ WD}(T)}{\partial T} \right|_{t_D= \lambda} = \frac{xy \delta M_W \beta_W}{1 + e^{b - a_W T}}$$

which is always positive since $M_D \beta_D > 0$ (due to $M_D \beta_D - \frac{k_D}{\sigma_Y} \leq M_W \beta_W$ and $\min\{M_D \beta_D - \frac{k_D}{\sigma_Y}, M_W \beta_W\} > 0$). Therefore, when $F_D > 0$ and $F_W > 0$, there exists a time horizon threshold $T_{1III}$ so that it is optimal not to sell the product for $T \leq T_{1III}$ and optimal to use the W policy to sell otherwise.

In Case IV, we have $M_D \beta_D - \frac{k_D}{\sigma_Y} > M_W \beta_W$ where $M_D \beta_D - \frac{k_D}{\sigma_Y}$ is positive and $M_W \beta_W$ can be negative. We have:

$$\left. \frac{\partial \Pi^{ WD}(t_D)}{\partial T} \right|_{t_D < T} = \frac{xy \delta M_D \beta_D}{1 + e^{b - a_D T + (a_D - a_W) T_D}} - k_D$$

$$\left. \frac{\partial \Pi^{ WD}(T)}{\partial T} \right|_{t_D = T} = \frac{xy \delta M_W \beta_W}{1 + e^{b - a_W T}}$$

where the first term is positive for high values of $T$ and the second term is positive if $M_W \beta_W > 0$. Therefore, when $F_D > 0$ and $F_W > 0$, there exists a time horizon threshold $T_{1IV}$ so that it is optimal not to sell the product for $T \leq T_{1IV}$ and optimal to sell otherwise. For $T > T_{1IV}$, the W, D and DW
are optimal policy candidates. We have:

$$\frac{\partial \Pi_{0}^{WD}(t_{D}(T))}{\partial T} \bigg|_{t_{D}=t_{D}(T)} = \frac{xy\delta M_{D}\beta_{D}}{1 + e^{b-a_{D}T+(a_{D}-a_{W})I_{D}}} - k_{D}$$

(17)

$$\frac{\partial \Pi_{0}^{WD}(0)}{\partial T} \bigg|_{t_{D}=0} = \frac{xy\delta M_{D}\beta_{D}}{1 + e^{b-a_{D}T}} - k_{D}$$

(18)

$$\frac{\partial \Pi_{0}^{WD}(T)}{\partial T} \bigg|_{t_{D}=T} = \frac{xy\delta M_{W}\beta_{W}}{1 + e^{b-a_{W}T}}$$

(19)

where (18) is always greater than (17). If $M_{W}\beta_{W} > 0$, (19) may be greater than (17) and (18) for low $T$ and smaller than these two for large $T$. Then, when $F_{D} \geq F_{W} > 0$, there exist horizon thresholds $T_{1IV}^{2} \geq T_{1IV}^{1}$ and $T_{1IV}^{3} \geq T_{1IV}^{2}$ so that $W$ is the optimal policy for $T \in (T_{1IV}^{1}, T_{1IV}^{2})$, WD is the optimal policy for $T \in (T_{1IV}^{2}, T_{1IV}^{3})$, and $D$ is the optimal policy for $T > T_{1IV}^{3}$. If $M_{W} \leq 0$, the $W$ policy is never optimal so that the manufacturer only needs to consider the $D$ and WD policy. Therefore, there exists one threshold $T_{1IV}^{2} \geq T_{1IV}^{1}$ such that WD is the optimal policy for $T \in (T_{1IV}^{1}, T_{1IV}^{2})$ and $D$ is the optimal policy for $T > T_{1IV}^{2}$.

Proof. Proof of Proposition 6

According to the proof of Proposition 4, when the DW policy is optimal, the optimal switching time $t_{W}^{*}$ is increasing in $T$. When the WD policy is optimal, the optimal switching time $t_{D}^{*}$ is decreasing in $T$, which further implies the proportion of time using the direct channel strategy $\frac{T-t_{D}^{*}}{T}$ is increasing in $T$. $\Box$