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Optimal control of interacting particle systems

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Abstract

In this paper, the behaviour of an interacting particle system is investigated. The aim of this paper is to let a set of controls guide a group of particles to a certain destination while minimizing a suitable cost functional. The corresponding optimization problem will be defined and existence of solutions to this optimization problem will be proven. Furthermore, the first-order necessary conditions for optimality are derived with the help of the method of Lagrange multipliers. These conditions are rewritten to an initial value problem of which existence of solutions will be proven. Thereafter, this initial value problem is transformed into a closed-loop control system that is used to visualize the behaviour of the particles and the controls.
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1 Introduction

Suppose that there is a flock of sheep grazing on a grass field, but there is not much grass left anymore. Hence, the sheep need to be transferred to another grass field. This can for example be done by herding dogs. However, there are many different ways to do this: the dogs could for example cluster together and steer the sheep from a specific angle to the new grass field or they could divide themselves over the flock of sheep to steer them in the right direction. Of course, not all methods are equally efficient. The most efficient way of accomplishing this task can be investigated by modelling this situation. Interacting particle systems are suited for this, since they can describe real-world behaviour in an accurate manner. Therefore, research focuses more and more on the behaviour of these systems [6].

This paper considers an interacting particle system containing \( N \in \mathbb{N} \) particles and \( M \in \mathbb{N} \) controls with \( N \gg M \). Let \( d \geq 1 \) be the dimension. The time interval considered is \([0, T] \subset \mathbb{R} \) with \( T > 0 \). The goal in this paper is to let the set of controls (the dogs) guide the group of particles (the sheep) to a certain position while minimizing a suitable cost functional, which will be defined in section 2.3.

The initial positions of both the particles and the controls are fixed, i.e. \( \forall i = 1, ..., N \) and \( l = 1, ..., M \). More information about the spaces can be found in section 2.2 and in Appendix A.I.

The initial positions of both the particles and the controls are fixed, i.e. \( X_{t=0} = X_0 \in \mathbb{R}^{dN} \) and \( U_{t=0} = U_0 \in \mathbb{R}^{dM} \). Define \( \mathcal{U}_{ad} = \{ U \in H^1([0, T], \mathbb{R}^{dM}) | U_{t=0} = U_0 \} \) with \( U_0 \) given as the admissible space of the controls. The initial condition for \( U \) ensures that all the \( U \in \mathcal{U}_{ad} \) are indeed admissible.

Let \( \mathcal{Y} = L^2([0, T], \mathbb{R}^{dN}) \), where \( L^2 \) denotes a Lebesgue space. The particle system considered is

\[
\frac{d}{dt} X = F(X, U)
\]

where \( F: \mathcal{X} \times \mathcal{U} \to \mathcal{Y} \) is defined by

\[
F_i(X, U)(t) = \frac{1}{N} \sum_{j=1}^{N} K(x_i^t - x_j^t) + \frac{1}{M} \sum_{l=1}^{M} \Phi(x_i^t - u_l^t)
\]

and \( K \) and \( \Phi : \mathbb{R}^d \to \mathbb{R}^d \) are the, respectively, particle-particle interaction force and the particle-control interaction force.

The following assumptions will be made:

- \( K \) is antisymmetric (odd).
- \( K \) and \( \Phi \) are bounded: \( \sup_{y \in \mathbb{R}^d} |K(y)| < \infty \) and \( \sup_{y \in \mathbb{R}^d} |\Phi(y)| < \infty \).
- \( K \) and \( \Phi \) are at least two times almost everywhere continuously differentiable.
- \( K, \Phi, \nabla K \) and \( \nabla \Phi \) are Lipschitz continuous.

Examples of explicit choices for \( K \) and \( \Phi \), based on literature, will be given in section 2.4. Note that since \( K \) is antisymmetric, it holds that \( K(-x) = -K(x) \) for \( x \in \mathbb{R}^d \) and hence \( K(0) = 0 \).

The aim of this paper is to guide the particles as close as possible to a certain position while minimizing the cost functional \( J(X, U) \), which will be defined in section 2.3. The goal is to find a solution to the optimization problem

\[
\min_{(X, U) \in (\mathcal{X} \times \mathcal{U}_{ad})} J(X, U) \text{ subject to the constraint } E(X, U) = 0 \text{ in } \mathcal{Y} \tag{OP}
\]

where \( E: \mathcal{X} \times \mathcal{U} \to \mathcal{Y} \) and \( E(X, U) = \frac{d}{dt} X - F(X, U) \). Note that \( E(X, U) = 0 \) in \( \mathcal{Y} \) means that \( \langle E(X, U), h \rangle_\mathcal{Y} = 0 \) \( \forall h \in \mathcal{Y} \). More information about the space \( L^2 \) and its corresponding inner product \( \langle \cdot, \cdot \rangle \) can be found in the next chapter and in Appendix A.I.

Some preliminary results about the spaces that are used throughout this report will be discussed in chapter 2. Furthermore, the cost functional will be defined and examples of interaction forces will be given, as was already stated before.

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Subsequently, it will be proven that the previously mentioned optimization problem actually has a solution in chapter 3.

After that, the first-order necessary conditions for optimality will be derived in chapter 4 using the method of Lagrange multipliers. These conditions will be rewritten to an initial value problem. It will be proven, with the help of Schäfer’s fixed point theorem, that there exists at least one solution to this initial value problem.

One of the results of the previous section is a system consisting of four first order differential equations and four variables. Chapter 5 will transform this system into a closed-loop control system of two first order differential equations depending only on the variables $X$ and $U$. Local existence of solutions to this system will be proven.

This new system is implemented in MATLAB to visualize the situation. The results of this implementation will be discussed in chapter 6.

Chapter 7 describes the main results of this paper and chapter 8 takes a critical look at these results. Furthermore, chapter 8 describes suggestions for future research in this field.
2 Preliminary results

This chapter begins with making some remarks on notation and discussing details about the spaces of the particles and the controls. Furthermore, a suitable cost functional will be defined and explained. Lastly, an example of the interaction forces will be given based on literature and proper constants will be chosen.

2.1 Notational remarks

Some remarks on notation are:

- A dot above a variable indicates a (weak) time derivative.
- I write \( (f, g) = \int_0^1 (f_t, g_t) dt \) for the \( L^2 \) inner product of \( f \) and \( g \) where \( f \) and \( g \) are real functions defined on the interval \([a, b] \subset \mathbb{R}\).
- \( C_c^\infty \) is the set of smooth, i.e. infinitely differentiable, functions with compact support.
- \( B(x; r) \) denotes the open ball with center \( x \) and radius \( r \).
- \( C^m([a, b]) \) with \( m \) nonnegative denotes the space of all functions that are \( m \) times continuously differentiable over the interval \([a, b]\) and \( C^m([a, b]) \) is equipped with the norm \( ||x||_{C^m} = \sum_{i=0}^{m} \sup_{t \in [a, b]} |x^{(m)}(t)| \). For \( m = 0 \), I will write \( C([a, b]) \) and \( ||x||_{C^0} = ||x||_{\sup} \).
- Spaces will be omitted when it should be clear from the context to avoid long notations.

2.2 Spaces

Information about the spaces that are used in this report can be found in Appendix A.1.

Another remark about the spaces used in this report is that for \( \mathbb{R}^d \) with \( d, n \geq 1 \) the 1–norm is used for all \( x, y \in \mathbb{R}^d \):

\[
||x - y||_{\mathbb{R}^d} = \sum_{i=1}^{n} |x^i - y^i|_{\mathbb{R}^d}
\]

For \( \mathbb{R}^d \), the 1–norm is also used. Note that, because \( \mathbb{R}^d \) is a finite dimensional vector space, any two norms on this space are equivalent [9], i.e. for any two norms \( || \cdot || \) and \( || \cdot ||' \) on \( \mathbb{R}^d \) there exist positive real constants \( a, b \) such that for all \( x \in \mathbb{R}^d \)

\[
a||x||' \leq ||x|| \leq b||x||'
\]

Below I will state and prove a lemma that implies that all functions in \( H^1([a, b]) \) are continuous:

**Lemma 2.1.** \( H^1([a, b]) \) is embedded in \( C([a, b]) \) with \( a, b \in \mathbb{R} \).

**Proof.** Let \([a, b]\) with \( a, b \in \mathbb{R} \) denote an interval. Furthermore, let \( y \in H^1([a, b]) \) and let \( t_1, t_2 \in [a, b] \).

Since \( y \in H^1([a, b]) \), its (weak) derivative \( \dot{y} \in L^2([a, b]) \). Hence, there exists a \( B > 0 \) such that \( \int_{a}^{b} |\dot{y}(s)|^2 ds \leq B \).

From the fundamental theorem of calculus for weak derivatives [5] I know that

\[
y(t_2) - y(t_1) = \int_{t_1}^{t_2} \dot{y}(s) ds
\]

Hence, for all \( \epsilon > 0 \) \( \exists \delta > 0 \) such that for all \( |t_2 - t_1| < \delta \) it holds that

\[
|y(t_2) - y(t_1)| \leq \int_{t_1}^{t_2} |\dot{y}(s)| ds
\]

\[
\leq \left( \int_{t_1}^{t_2} 1^2 ds \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} |\dot{y}(s)|^2 ds \right)^{\frac{1}{2}} \text{ Hölder's inequality (cf. Lemma 8.1)}
\]

\[
\leq \sqrt{|t_2 - t_1| \sqrt{B} < \sqrt{\delta B}}
\]

Choosing \( \delta = \frac{\epsilon}{\sqrt{B}} \) leads to \( |y(t_2) - y(t_1)| < \epsilon \). This implies that \( y \) is continuous on the interval \([a, b]\) and hence \( H^1([a, b]) \) is embedded in \( C([a, b]) \). \( \square \)
**Remark 2.1.** Since $H^1([a,b])$ is embedded in $C([a,b])$, $X$ and $U$ are embedded in $C([0,T])$. Hence, $F(X, U)$ is embedded in $C([0,T])$ for all $X \in X$ and $U \in U$, from which the conclusion can be drawn that $F: X \times U \to Y$ is continuous. In particular, since the particle system was defined as $\frac{d}{dt}X = F(X, U)$, this means that $X$ is continuously differentiable.

**Remark 2.2.** Note that there can also be functions in $H^1([a,b])$ with $a,b \in \mathbb{R}$ which aren’t differentiable in every point in $[a,b]$. An example will be given below.

**Example 2.1.** Let $a = 0$, $b = \hat{T}$, where $\hat{T} > 0$ and $\hat{T} \in \mathbb{R}$. Let $\phi \in C_c^\infty([0,\hat{T}])$.

Define the function

$$v(t) = \begin{cases} t^\frac{2}{T} & \text{if } t \in [0,\frac{\hat{T}}{2}] \\ 2 - \frac{2}{T}t & \text{if } t \in (\frac{\hat{T}}{2},\hat{T}] \end{cases}$$

The function is plotted for $\hat{T} = 10$:

![Figure 1: $v(t)$ for $\hat{T} = 10$.](image)

Clearly, $v$ is not differentiable in $t = \frac{\hat{T}}{2}$.

$$\int_0^\hat{T} v(t)\dot{\phi}(t)dt = \int_0^{\frac{\hat{T}}{2}} v(t)\dot{\phi}(t)dt + \int_{\frac{\hat{T}}{2}}^{\hat{T}} v(t)\dot{\phi}(t)dt$$

$$= \frac{2}{T} \int_0^{\frac{\hat{T}}{2}} t\dot{\phi}(t)dt + 2 \int_{\frac{\hat{T}}{2}}^{\hat{T}} (1 - \frac{t}{T})\dot{\phi}(t)dt$$

$$= -\int_0^{\frac{\hat{T}}{2}} \frac{2}{T}\phi(t)dt - \int_{\frac{\hat{T}}{2}}^{\hat{T}} \frac{2}{T}\phi(t)dt \text{ (Partial integration)}$$

$$= -\int_0^{\frac{\hat{T}}{2}} \left( \frac{2}{T} \mathbb{1}_{[0,\frac{\hat{T}}{2}]}(t) - \frac{2}{T} \mathbb{1}_{(\frac{\hat{T}}{2},\hat{T}]}(t) \right) \phi(t)dt$$

Let $\dot{v}(t) = \frac{2}{T} \mathbb{1}_{[0,\frac{\hat{T}}{2}]}(t) - \frac{2}{T} \mathbb{1}_{(\frac{\hat{T}}{2},\hat{T}]}(t)$ where the dot above $v$ indicates a weak derivative. Note that $\int_0^{\hat{T}} |v(t)|^2dt < \infty$ and $\int_0^{\hat{T}} |\dot{v}(t)|^2dt < \infty$.

From the calculation above it follows that $v \in H^1([0,\hat{T}])$.

### 2.3 Cost functional

Consider the following cost functional:

$$J(X, U) = \frac{1}{2N} \int_0^T \sum_{i=1}^N |x_i^N - \bar{x}|_2^2 dt + \frac{\lambda}{2M} \int_0^T \sum_{i=1}^M |\dot{x}_i^N|_2^2 dt$$

$$J: X \times U \to \mathbb{R} \quad \text{(CF)}$$
In the above notation, $\lambda$ is a real constant larger than 0. Furthermore, $\bar{x} \in \mathbb{R}^d$ denotes the desired position of $X$.

The first term of the cost functional (CF) measures the difference of the positions of the particles with the desired positions of the particles. The second term measures the control costs, i.e. the effort of the controls to steer the particles to $\bar{x}$. This second term is included because it is desirable that the controls are as efficient as possible. However, the minimization of the first term is the most important objective, since the ultimate aim of this paper is to bring the particles to $\bar{x}$, even if the controls need a high amount of kinetic energy to accomplish this task.

### 2.4 Examples of interaction forces

An example of a particle-particle interaction potential that would suit the particle system described in this paper is the Morse potential, which is defined by

$$V(r) = c_A e^{-\frac{r}{l_A}} + c_R e^{-\frac{r}{l_R}},$$

where $c_A$ and $c_R$ are the, respectively, attractive and repulsive strengths and $l_A$ and $l_R$ are the, respectively, attractive and repulsive length scales [6]. Defining $K(x) = -\nabla V(|x|)$ for $x \in \mathbb{R}^d$ results in the following example of a particle-particle interaction force

$$K(x - y) = -\nabla V(|x - y|) = \left(\frac{c_R}{l_R} e^{-\frac{|x-y|}{l_R}} - \frac{c_A}{l_A} e^{-\frac{|x-y|}{l_A}}\right) \frac{x - y}{|x - y|},$$

where I let $K(0) = 0$.

![Morse potential](image)

Figure 2: Morse potential $V(r)$ with $c_A = 0.2, l_A = 2, c_R = 0.8, l_R = 0.1$.

The chosen constants in Figure 2 illustrate the case of short range repulsion and long range attraction.

An example of a suitable particle-control interaction potential is $U : \mathbb{R}^d \to \mathbb{R}$ with $U(r) = c_R e^{-\frac{r}{l_R}}$, where $c_R$ is the repulsive strength and $l_R$ is the repulsive length scale. Defining $\Phi(x) = -\nabla U(|x|)$ for $x \in \mathbb{R}^d$ results in the following example of a particle-control interaction force

$$\Phi(x - u) = -\nabla U(|x - u|) = \frac{c_R}{l_R} e^{-\frac{|x-u|}{l_R}} \frac{x - u}{|x - u|},$$

where I let $\Phi(0) = 0$. 

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The chosen constants in Figure 3 illustrate the case of short range repulsion.

**Remark 2.3.** Note that the forces aren’t continuously differentiable in 0.
3 Existence of solutions to the optimization problem

First of all, recall that the following spaces were defined:

- \( \mathcal{U} = H^1([0, T], \mathbb{R}^{dM}) \)
- \( \mathcal{U}_{ad} = \{ U \in H^1([0, T], \mathbb{R}^{dM}) | U_{t=0} = U_0 \} \) and \( \mathcal{U}_{ad} \subset \mathcal{U} \)
- \( \mathcal{X} = H^1([0, T], \mathbb{R}^{dN}) \)
- \( \mathcal{Y} = L^2([0, T], \mathbb{R}^{dN}) \)

Furthermore, define \( \hat{\mathcal{Y}} := L^2([0, T], \mathbb{R}^{dM}) \).

Recall also that the goal of this paper is to solve the following optimization problem:

\[
\min_{(X,U) \in (\mathcal{X}, \mathcal{U}_{ad})} J(X, U) \quad \text{subject to the constraint } E(X, U) = 0 \quad \text{in } \mathcal{Y}
\]

In this chapter it will be shown that this optimization problem has a solution, i.e. the following theorem will be proven:

**Theorem 3.1.** The optimization problem (OP) has a solution.

Definitions that are used throughout this chapter are defined in Appendix A.II. Furthermore, the following lemmas will be used to prove Theorem 3.1:

**Lemma 3.1 ([12]).** All Hilbert spaces are reflexive Banach spaces.

**Lemma 3.2 ([12]).** Let \( Z \) be a reflexive Banach space. Then every closed and convex subset of \( Z \) is weakly closed.

**Lemma 3.3 ([12]).** Any weakly convergent sequence is bounded.

**Lemma 3.4 ([12]).** Every bounded sequence in a reflexive Banach space contains a weakly convergent subsequence.

**Lemma 3.5 ([12]).** Every convex and lower semicontinuous functional is weakly lower semicontinuous.

**Lemma 3.6.** Let \( \sigma \in [0, 1] \). If \( f : \mathbb{R}^m \to \mathbb{R}^n \) with \( m, n \geq 1 \) is a differentiable function, then

\[
f(y) = f(x) + \int_0^1 Df \left( (1 - \sigma)x + \sigma y \right) [y - x] d\sigma,
\]

where \( D \) indicates a Jacobian matrix.

**Proof.** Let \( f : \mathbb{R}^m \to \mathbb{R}^n \) with \( m, n \geq 1 \) be a differentiable function. Let \( g(\sigma) := f \left( (1 - \sigma)x + \sigma y \right) \), so \( g \) is also differentiable.

\[
g(1) - g(0) = \int_0^1 \frac{d}{d\sigma} g(\sigma) d\sigma \quad \text{(Fundamental theorem of calculus [8])}
\]

\[
= \int_0^1 Df \left( (1 - \sigma)x + \sigma y \right) [y - x] d\sigma
\]

Since \( g(1) = f(y) \) and \( g(0) = f(x) \), the result follows. \( \square \)

**Lemma 3.7.** Let \( K \) be Lipschitz continuous with Lipschitz constant \( L_K \), \( \Phi \) be Lipschitz continuous with Lipschitz constant \( L_\Phi \) and let \( U \in \mathcal{U}_{ad} \) be fixed. Then \( F(X, U) \) is Lipschitz continuous in \( X \).

**Proof.** Let \( X_1, X_2 \in \mathcal{X} \) and let

\[
(I) := \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N K(x_1^{1,i} - x_1^{1,j}) - \frac{1}{N} \sum_{j=1}^N K(x_2^{2,i} - x_2^{2,j})
\]

\[
(II) := \sum_{i=1}^N \frac{1}{M} \sum_{l=1}^M \Phi(x_1^{1,i} - u_l^1) - \frac{1}{M} \sum_{l=1}^M \Phi(x_2^{2,i} - u_l^1)
\]
By the triangle inequality
\[ |F(X_1,U)(t) - F(X_2,U)(t)|_{\mathbb{R}^{dN}} \leq (I) + (II) \]

The terms (I) and (II) will be evaluated separately. Since \( K \) is Lipschitz continuous with Lipschitz constant \( L_K \), it holds that
\[
(I) \leq \sum_{i=1}^{N} \frac{L_K}{N} \sum_{j=1}^{N} |(x_i^1 - x_i^2) - (x_i^2 - x_i^2)|
\]

Rewriting and using the triangle inequality yields:
\[
\leq 2L_K |X_i^1 - X_i^2|
\]

Since \( \Phi \) is Lipschitz continuous with Lipschitz constant \( L_\Phi \), it holds that
\[
(II) \leq L_\Phi |X_i^1 - X_i^2|
\]

Combining (I) and (II) yields
\[
|F(X_1,U)(t) - F(X_2,U)(t)|_{\mathbb{R}^{dN}} \leq (2L_K + L_\Phi) |X_i^1 - X_i^2|_{\mathbb{R}^{dN}} =: L_F |X_i^1 - X_i^2|_{\mathbb{R}^{dN}}
\]

Squaring both sides and taking the integral over both sides results in
\[
||F(X_1,U) - F(X_2,U)||_{L^2(Y)}^2 = \int_0^T |F(X_1,U)(t) - F(X_2,U)(t)|^2 dt
\]
\[
\leq L_F^2 \int_0^T |X_i^1 - X_i^2|^2 dt
\]
\[
\leq L_F^2 \left( \int_0^T |X_i^1 - X_i^2|^2 dt + \int_0^T |\dot{X}_i^1 - \dot{X}_i^2|^2 dt \right)
\]
\[
= L_F^2 ||X_1 - X_2||_{Y}^2
\]

Taking the square root on both sides gives the result. \( \Box \)

**Lemma 3.8.** If \( U \in \mathcal{U}_{ad} \) is fixed, then \( E(X, U) = 0 \) in \( \mathcal{Y} \) has a unique local solution \( X \in \mathcal{X} \).

**Proof.** Let \( U \in \mathcal{U}_{ad} \) be fixed. Recall that \( E(X, U) = \frac{d}{dt} X - F(X, U) = 0 \), which means that
\[
\frac{d}{dt} X(t) = F(X,U)(t), \quad X_{t=0} = X_0
\]

Since \( F \) is Lipschitz continuous in \( X \) (Lemma 3.7), \( F \) is continuous (see Remark 2.1) and bounded (since \( K \) and \( \Phi \) are bounded), there exists a unique local solution to (1) by Picard-Lindelöf (Lemma 8.4). \( \Box \)

**Lemma 3.9.** Let \( X \) be a solution of \( E(X, U) = 0 \) in \( \mathcal{Y} \). If \( K \) is Lipschitz continuous with Lipschitz constant \( L_K \) and \( C_\Phi := \sup_{y \in \mathbb{R}^d} \{ |\Phi(y)| + |D\Phi(y)| \} < \infty \), where \( D \) indicates a Jacobian matrix, then
\[
\max_{i \in \{1, \ldots, N\}, t \in [0,T]} |x_i^1| \leq \frac{\sqrt{e^{(4L_K+C_\Phi+1)T}} \left( NC_\Phi^2 T + \frac{N}{M} ||U||_{Y}^2 \right)}{C_X} =: C_X
\]

**Proof.** \( X \) is a solution of \( E(X, U) = 0 \) in \( \mathcal{Y} \) by Lemma 3.8. Therefore, I can make an a-priori estimate for \( X \):
\[
\frac{1}{2} \frac{d}{dt} ||x_i^1||^2 = (x_i^1, \frac{d}{dt} x_i^1) = \frac{1}{N} \sum_{j=1}^{N} \left( x_i^j, K(x_i^j - x_i^j) \right) + \frac{1}{M} \sum_{l=1}^{M} \left( x_i^j, \Phi(x_i^j - u_i^l) \right)
\]

From the fact that \( K \) is Lipschitz continuous and \( K(0) = 0 \) I can derive that
\[
\forall y \in \mathbb{R}^d : |K(y)| = |K(y) - K(0)| \leq L_K |y - 0| = L_K |y|
\]
Let \( C_\Phi := \sup_{y \in \mathbb{R}^d} \{ |\Phi(y)| + |D\Phi(y)| \} \). Since \( \Phi \) is Lipschitz continuous, its derivative is bounded (result of the mean value theorem [8]). In addition, \( \Phi \) is bounded and hence \( C_\Phi < \infty \). With the help of Lemma 3.6, the following estimate for the control function is obtained:

\[
\frac{1}{M} \sum_{i=1}^{M} \Phi(x_i^t - u_i^t) = \frac{1}{M} \sum_{i=1}^{M} \left( \Phi(x_i^t) - \int_0^1 D\Phi(x_i^t - \sigma u_i^t) u_i^t d\sigma \right) \\
\leq C_\Phi + C_\Phi \frac{1}{M} \sum_{i=1}^{M} |u_i^t| 
\]

Combining this information with the Cauchy-Schwarz inequality [8] results in:

\[
\frac{1}{2} \frac{d}{dt} |x_i^t|^2 \leq \frac{1}{N} \sum_{j=1}^{N} |x_i^t| |J_i| + |x_i^t| \Phi + |x_i^t| \Phi \frac{1}{M} \sum_{i=1}^{M} |u_i^t|
\]

Using Young’s inequality (cf. Lemma 8.2) and the triangle inequality:

\[
\leq \frac{1}{2} (3L_K + 1) |x_i^t|^2 + \frac{L_K}{2N} \sum_{j=1}^{N} |x_i^t|^2 + \frac{1}{2} C_\Phi^2 + \frac{1}{2} |x_i^t|^2 C_\Phi^2 + \frac{1}{2M} \sum_{i=1}^{M} |u_i^t|^2
\]

Multiplying the equation by 2 and summing up over all \( i \) gives:

\[
\frac{d}{dt} \sum_{i=1}^{N} |x_i^t|^2 \leq (4L_K + C_\Phi^2 + 1) \sum_{i=1}^{N} |x_i^t|^2 + N C_\Phi^2 + \frac{N}{M} \sum_{i=1}^{M} |u_i^t|^2
\]

Applying Gronwall’s inequality (Lemma 8.5) and using the fact that \( t \leq T \) for all \( t \in [0, T] \) results in

\[
\sum_{i=1}^{N} |x_i^t|^2 \leq e^{(4L_K + C_\Phi^2 + 1)T} \left( N C_\Phi^2 T + \frac{N}{M} \| |U| \|_2^2 \right) =: C_X^2
\]

Note that \( C_X \) doesn’t depend on the time \( t \). So for every element in the summation I can write:

\[
|x_i^t|^2 \leq C_X^2 \quad \forall i \in \{1, \ldots, N\} \forall t \in [0, T]
\]

Hence, \( \max_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} |x_i^t| \leq C_X. \)

The information to investigate the existence of solutions to the optimization problem \( \text{(OP)} \) is now complete and this will be done in the next section.

### 3.1 Existence of solutions

Theorem 3.1 holds true if the following six statements hold true [12]:

(A1) \( \mathcal{U}_{ad} \subset \mathcal{U} \) is a weakly closed subset of a reflexive Banach space \( \mathcal{U} \).

(A2) \( \mathcal{X} \) is a reflexive Banach space.

(A3) \( E(X, U) = 0 \) in \( \mathcal{Y} \) has a bounded solution operator \( A : \mathcal{U}_{ad} \rightarrow \mathcal{X} ; U \mapsto X \).

(A4) \( \mathcal{X} \times \mathcal{U} \rightarrow \mathcal{Y} ; (X, U) \mapsto E(X, U) \) is weakly continuous.

(A5) \( J(X, U) \) is weakly lower semicontinuous.

(A6) \( J(X, U) \) is coercive with respect to \( U \).

Before proving Theorem 3.1, it will be explained why the existence of minimizers for \( \text{(CF)} \) subject to the constraint \( E(X, U) = 0 \) in \( \mathcal{Y} \) indeed follows from the above conditions.

From (A3) it follows that I can define a reduced cost functional \( f(U) := J(A(U), U) : \mathcal{U}_{ad} \rightarrow \mathbb{R} \).

Let \( \{ U^{(n)} \}_{n \in \mathbb{N}} \subset \mathcal{U}_{ad} \) be a minimizing sequence for \( f \), i.e.

\[
\lim_{n \rightarrow \infty} f(U^{(n)}) = \inf_{U \in \mathcal{U}_{ad}} f(U)
\]
with \( E(A(U(n)), U(n)) = 0 \) for all \( n \in \mathbb{N} \). If \((U(n))_{n \in \mathbb{N}}\) is unbounded in \( U \), there should exist a subsequence \((U(n))_{n \in \mathbb{N}} \subseteq U \) that diverges to infinity [8]. Since \( J(X, U) \) is coercive in \( U \) because of (A6), \( f(U) \) is also coercive in \( U \). This would imply that \( f(U(n)) \to \infty (n \to \infty) \), which is a contradiction to the fact that \((U(n))_{n \in \mathbb{N}}\) is a subsequence of a minimizing sequence. So \((U(n))_{n \in \mathbb{N}}\) is bounded in \( U \) and because \( U \) is a reflexive Banach space (Lemma 3.1), there exists a subsequence \((U(n))_{n \in \mathbb{N}} \subseteq U_{ad} \) and a limit \( \tilde{U} \in U \) such that \( U_{nk} \to \tilde{U} (k \to \infty) \) in \( U \) (Lemma 3.4). From (A1) it is known that \( U_{ad} \subseteq U \) is weakly closed and hence \( \tilde{U} \in U_{ad} \). Since \((U(n))_{n \in \mathbb{N}}\) is bounded in \( U \) and, by (A3), \( A \) is a bounded solution operator, \((A(U(n)))_{n \in \mathbb{N}} \subseteq \mathcal{X} \) is also bounded.

According to (A2), \( \mathcal{X} \) is a reflexive Banach space and hence I can again apply Lemma 3.4 to conclude that there exists a \( \tilde{X} \in \mathcal{X} \) such that \( A(U(n)) \to \tilde{X} (k \to \infty) \) in \( \mathcal{X} \). As a consequence, by (A4),

\[
E\left(A(U(n)), U(n)\right) \to E(\tilde{X}, \tilde{U}) (k \to \infty) \text{ in } \mathcal{Y}
\]

Since all norms are continuous [9] and convex [4], all norms are weakly lower semicontinuous (Lemma 3.5). Combining this information with the above result that followed from (A4) implies

\[
||E(\tilde{X}, \tilde{U})||_\mathcal{Y} \leq \liminf_{k \to \infty} ||E\left(A(U(n)), U(n)\right)||_\mathcal{Y}
\]

where the right-hand side is 0. Therefore, \( E(\tilde{X}, \tilde{U}) = 0 \) in \( \mathcal{Y} \), which means that \( \tilde{X} = A(\tilde{U}) \). Consequently, \( A(U(n)) \to A(\tilde{U}) (k \to \infty) \) in \( \mathcal{X} \).

Hence, (A5) implies that \( f(U) \) is also weakly lower semicontinuous:

\[
f(\tilde{U}) \leq \liminf_{k \to \infty} f\left(U(n)\right)
\]

where the right-hand side is equal to \( \inf_{U \in U_{ad}} f(U) \), because \((U(n))_{n \in \mathbb{N}} \subseteq (U(n))_{n \in \mathbb{N}}\) is a minimizing sequence. Hence, the infimum is indeed attained and I can state that (OP) has a solution if the six statements are satisfied.

Now that it is clear why existence of a solution for (OP) follow from the six statements, Theorem 3.1 can be proven:

**Proof of Theorem 3.1.** It was already stated above that the theorem is indeed true if there are six statements that hold true. Each of these statements will be proven separately below:

(A1) \( U_{ad} \subseteq U \) is a weakly closed subset of a reflexive Banach space \( U \).

**Proof.** \( U \) is a Hilbert space and hence a reflexive Banach space (Lemma 3.1). I will first check that \( U_{ad} \subseteq U \) is convex. Let \( \sigma \in [0, 1] \) and let \( U_1, U_2 \in U_{ad} \). Since \( U_1, U_2 \in U_{ad} \), there exists constants \( M_1, M_2 \geq 0 \) such that \( ||U_1||_U \leq M_1 \) and \( ||U_2||_U \leq M_2 \). Therefore,

\[
||\sigma U_1 + (1 - \sigma) U_2||_U \leq \sigma ||U_1||_U + (1 - \sigma) ||U_2||_U
\]

\[
\leq \sigma M_1 + (1 - \sigma) M_2
\]

So \( \sigma U_1 + (1 - \sigma) U_2 \in U \). Also, the initial condition will be checked:

\[
\sigma U_1(0) + (1 - \sigma) U_2(0) = \sigma U_0 + (1 - \sigma) U_0 = U_0
\]

So \( \sigma U_1 + (1 - \sigma) U_2 \in U_{ad} \), which means that \( U_{ad} \subseteq U \) is convex.

Next, it will be shown that \( U_{ad} \subseteq U \) is closed. Let \((U(n))_{n \in \mathbb{N}} \subseteq U_{ad} \) be an arbitrary converging sequence with limit \( U \in U \). If \( U \in U_{ad} \), then \( U_{ad} \subseteq U \) is closed.

Since \( (U(n)) \in U_{ad} \) for every \( n \in \mathbb{N} \), \( U(n)(0) \) is bounded for every \( n \in \mathbb{N} \). So \( ||U(n)(0) - U_0||_{R^d} = 0 \). Furthermore, since \( H^1 \) is embedded in \( C \) (Lemma 2.1), there exists a \( C_0 \geq 0 \) such that \( ||y||_{C[0,T]} \leq C_0 ||y||_{H^1} \) for all \( y \in H^1 \). Hence, \( ||U - U(n)||_{C[0,T],R^d} \leq C_0 ||U - U(n)||_U \).

Since \((U(n))_{n \in \mathbb{N}}\) converges to \( U \in U \), it holds that \( ||U - U(n)||_U \to 0 \) for \( n \to \infty \). Using the above information and the triangle inequality:

\[
0 \leq ||U(0) - U_0||_{R^d} \leq ||U(0) - U(n)(0)||_{R^d} + ||U(n)(0) - U_0||_{R^d}
\]

\[
\leq ||U - U(n)||_{C[0,T],R^d} \leq C_0 ||U - U(n)||_U \to 0
\]
Hence, $U \in \mathcal{U}_{ad}$ and $\mathcal{U}_{ad} \subset \mathcal{U}$ is closed.

Consequently, $\mathcal{U}_{ad} \subset \mathcal{U}$ is weakly closed, since it is a closed and convex subset of $\mathcal{U}$ (Lemma 3.2).

(A2) $\mathcal{X}$ is a reflexive Banach space.

Proof. Since $\mathcal{X}$ is a Hilbert space, this follows from Lemma 3.1.

(A3) $E(X,U) = 0$ in $\mathcal{Y}$ has a bounded solution operator $A : \mathcal{U}_{ad} \to \mathcal{X} ; U \mapsto X$.

Proof. First of all, note that, by Lemma 3.8, $E(X,U) = 0$ in $\mathcal{Y}$ has a solution operator $A : \mathcal{U}_{ad} \to \mathcal{X} ; U \mapsto X$.

Furthermore, from Lemma 3.9 it is known that

$$\max_{i \in \{1, \ldots , N\}} \sup_{t \in [0,T]} |x^i_t| \leq \sqrt{e^{(4L_K + C^2 \hat{y})T} \left( NC^2 \hat{y} T + \frac{N}{M} ||U||^2 \right)} = C_X$$

where $L_K$ is the Lipschitz constant of $K$ and $C_\Phi = \sup_{y \in \mathbb{R}^d} \{|\Phi(y)| + |D\Phi(y)|\} < \infty$. Since $C_X$ doesn’t depend on $i$ or on the time $t$, it follows that

$$||X||^2_\mathcal{Y} = \int_0^T \sum_{i=1}^N |x^i_t|^2 dt \leq NTC^2_X$$

$K$ was assumed to be bounded, so there exists a $C_K \geq 0$ such that $\sup_{y \in \mathbb{R}^d} |K(y)| \leq C_K$. Furthermore, recall that the following estimate for the control function was obtained (see the proof of Lemma 3.9):

$$\frac{1}{M} \sum_{i=1}^M \Phi(x^i_t - u^i_t) \leq C_\Phi + C_\Phi \frac{1}{M} \sum_{i=1}^M |u^i_t|$$

Since $E(X,U) = 0$ in $\mathcal{Y}$, I get, with the triangle inequality:

$$|\dot{x}^i_t| = |F_i(X,U)(t)| = \left| \frac{1}{N} \sum_{j=1}^N K(x^j_t - x^i_t) + \frac{1}{M} \sum_{l=1}^M \Phi(x^l_t - u^l_t) \right| \leq C_K + C_\Phi + \frac{C_\Phi}{M} \sum_{l=1}^M |u^l_t|$$

(2)

From Young’s inequality (Lemma 8.2) it follows that for $a, b \geq 0$, $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$. Therefore,

$$(a + b)^2 = a^2 + 2ab + b^2 \leq 2a^2 + 2b^2$$

This will be used in (2):

$$|\dot{x}^i_t|^2 \leq 2(C_K + C_\Phi)^2 + 2C_\Phi^2 \left( \frac{1}{M} \sum_{l=1}^M |u^l_t| \right)^2$$

Since $f(x) = x^2$ is convex on $\mathbb{R}_+$, by Jensen’s inequality (cf. Lemma 8.3) I obtain

$$|\dot{x}^i_t|^2 \leq 2(C_K + C_\Phi)^2 + 2C_\Phi^2 \frac{1}{M} \sum_{l=1}^M |u^l_t|^2$$

Taking the summation and the integral on both sides results in

$$\int_0^T \sum_{i=1}^N |\dot{x}^i_t|^2 dt \leq 2TN(C_K + C_\Phi)^2 + 2C_\Phi^2 \frac{N}{M} \int_0^T \sum_{i=1}^M |u^i_t|^2 dt$$

So

$$||\dot{X}||^2_\mathcal{Y} \leq 2TN(C_K + C_\Phi)^2 + 2C_\Phi^2 \frac{N}{M} ||U||^2_\mathcal{Y}$$
Combining results yields
\[ ||X||^2_X = ||X||^2_Y + ||\dot{X}||^2_Y \]
\[ \leq NTC_X^2 + 2TN(C_K + C_\Phi)^2 + 2C_\Phi N \frac{||U||_Y^2}{M} \]
\[ =: B_1 + B_2||U||_Y^2 \]

Consequently,
\[ ||AU||_X = ||X||_X \leq \sqrt{B_1 + B_2||U||_Y^2} \]

So I can conclude that the solution operator A is indeed bounded.

(A4) \( X \times U \to \mathcal{Y}; (X, U) \mapsto E(X, U) \) is weakly continuous.

Proof. Assume \( (X_k, U_k) \to (X, U) \) \((k \to \infty)\) in \( X \times U \). If this implies that \( E(X_k, U_k) \to E(X, U) \) \((k \to \infty)\) in \( \mathcal{Y} \), the statement is indeed true.

Note that \( (X_k, U_k) \to (X, U) \) \((k \to \infty)\) in \( X \times U \) means that
- \( \langle \dot{X}_k, f_1 \rangle \to \langle \dot{X}, f_1 \rangle \) \((k \to \infty)\) for all \( f_1 \in \mathcal{Y} \)
- \( \langle X_k, f_2 \rangle \to \langle X, f_2 \rangle \) \((k \to \infty)\) for all \( f_2 \in \mathcal{Y} \)
- \( \langle \dot{U}_k, f_3 \rangle \to \langle \dot{U}, f_3 \rangle \) \((k \to \infty)\) for all \( f_3 \in \dot{\mathcal{Y}} \)
- \( \langle U_k, f_4 \rangle \to \langle U, f_4 \rangle \) \((k \to \infty)\) for all \( f_4 \in \dot{\mathcal{Y}} \)

\( E(X_k, U_k) \to E(X, U) \) \((k \to \infty)\) in \( \mathcal{Y} \) means that \( \langle E(X_k, U_k), h \rangle \to \langle E(X, U), h \rangle \) \((k \to \infty)\) for all \( h \in \mathcal{Y} \). Let \( h \in \mathcal{Y} \). Then
\[ \langle E(X_k, U_k), h \rangle = \langle \dot{X}_k, h \rangle - \langle F(X_k, U_k), h \rangle \]

From the assumptions, it already follows that \( \langle \dot{X}_k, h \rangle \to \langle \dot{X}, h \rangle \) \((k \to \infty)\) for all \( h \in \mathcal{Y} \).

Proving convergence of the other term requires a bit more work and will be done below. For all \( i \in \{1, \ldots, N\} \) and for all \( k \in \mathbb{N} \) it holds that
\[ (F_i(X_k, U_k), h^i) = \int_0^T \left( \frac{1}{N} \sum_{j=1}^N K(x^i_k(t) - x^j_k(t)) + \frac{1}{M} \sum_{l=1}^M \Phi(x^i_k(t) - u^l_k(t)), h^i(t) \right) dt \]

To prove that \( \langle F(X_k, U_k), h \rangle \to \langle F(X, U), h \rangle \) \((k \to \infty)\), I will start by showing that for all \( i \in \{1, \ldots, N\} \)
\[ \lim_{k \to \infty} \frac{1}{N} \sum_{j=1}^N K(x^i_k(t) - x^j_k(t)) = \frac{1}{N} \sum_{j=1}^N K(x^i(t) - x^j(t)) \text{ for almost every } t \in [0, T] \quad (3) \]

and
\[ \lim_{k \to \infty} \frac{1}{M} \sum_{l=1}^M \Phi(x^i_k(t) - u^l_k(t)) = \frac{1}{M} \sum_{l=1}^M \Phi(x^i(t) - u^l(t)) \text{ for almost every } t \in [0, T] \quad (4) \]

It is known that \( X_k \to X \) \((k \to \infty)\) in \( X \). Therefore, every subsequence of \( (X_k)_{k \in \mathbb{N}} \) converges weakly to \( X \) in \( X \) [9]. Let \( (X_{k_n})_{n \in \mathbb{N}} \) be an arbitrary subsequence of \( (X_k)_{k \in \mathbb{N}} \). Then \( X_{k_n} \to X \) \((n \to \infty)\) in \( X \).

From the Rellich-Kondrachov theorem, it follows that \( H^1([0, T]) \) is compactly embedded in \( C([0, T]) \) [7]. Therefore, any uniformly bounded sequence in \( H^1([0, T]) \) has a subsequence that converges in \( C([0, T]) \) [2].

\( (X_{k_n})_{n \in \mathbb{N}} \) is weakly convergent in \( X \), so it is uniformly bounded in \( X \) by Lemma 3.3. Hence, there exists a \( X \) such that \( X_{k_n} \to X \) as \( n \to \infty \) in \( C([0, T]) \). Convergence in \( C([0, T]) \) is uniform convergence [9], so \( X_{k_n} \to X \) as \( n \to \infty \) uniformly. Since \( X_{k_n} \to X \) as \( n \to \infty \) in \( X \), it holds that \( X = X \). Therefore, \( (X_{k_n})_{n \in \mathbb{N}} \) has a subsequence converging uniformly to \( X \). Since \( (X_{k_n})_{n \in \mathbb{N}} \) was an arbitrary subsequence of \( (X_k)_{k \in \mathbb{N}} \), every subsequence of \( (X_k)_{k \in \mathbb{N}} \) has a subsequence converging uniformly to \( X \). Hence, \( (X_k)_{k \in \mathbb{N}} \) converges uniformly (and hence also pointwise) to \( X \). In the same manner, it can be shown that \( (U_k)_{k \in \mathbb{N}} \) converges uniformly (and hence also pointwise) to \( U \). Because \( K \) and \( \Phi \) are continuous, I indeed obtain (3) and (4).
Next, it must be investigated if for all $i \in \{1, \ldots, N\}$ the limit can be taken inside the integral, i.e.

$$\lim_{k \to \infty} (F_i(X_k, U_k), h^i) = \int_0^T \lim_{k \to \infty} g_k^i(t) dt$$

where, for all $k \in \mathbb{N}$, $g_k^i : [0, T] \to \mathbb{R}$ is defined by

$$g_k^i(t) := \left( \frac{1}{N} \sum_{j=1}^N K(x_j^i(t) - x_k^j(t)) + \frac{1}{M} \sum_{l=1}^M \Phi(x_k^l(t) - u_k^l(t)), h^i(t) \right)$$

It was already shown above that for all $i \in \{1, \ldots, N\}$ $g_k^i(t) \to g^i(t)$ ($k \to \infty$) for almost every $t \in [0, T]$. Furthermore, from the assumptions made in the Introduction, it is known that $K$ and $\Phi$ are bounded and hence there exists a $C \geq 0$ such that, with the help of the Cauchy-Schwarz inequality [8],

$$|g_k^i(t)| \leq C|h^i(t)|$$

for all $i \in \{1, \ldots, N\}, k \in \mathbb{N}$.

Taking the integral on both sides results in

$$\sup_{k \in \mathbb{N}} \int_0^T |g_k^i(t)| dt \leq C \int_0^T |h^i(t)| dt$$

for all $i \in \{1, \ldots, N\}$.

With Hölder’s inequality (Lemma 8.1), it follows that for all $i \in \{1, \ldots, N\}$

$$\int_0^T |h^i(t)| dt \leq \sqrt{T}||h^i||_{L^2((0,T),\mathbb{R}^dN)} =: \sqrt{T}C_h$$

Hence,

$$\sup_{k \in \mathbb{N}} \int_0^T |g_k^i(t)| dt \leq C\sqrt{T}C_h < \infty$$

for all $i \in \{1, \ldots, N\}$.

Therefore, Lebesgue’s dominated convergence theorem (Lemma 8.6) can be applied to conclude that

$$\lim_{k \to \infty} \int_0^T g_k(t) dt = \int_0^T \lim_{k \to \infty} g_k(t) dt = \int_0^T g(t) dt$$

So, the limit can indeed be taken inside the integral and I can conclude that $\langle F(X_k, U_k), h \rangle \to \langle F(X, U), h \rangle$ ($k \to \infty$) for all $h \in \mathcal{Y}$. Hence, $E(X_k, U_k) \to E(X, U)$ ($k \to \infty$) in $\mathcal{Y}$.

\(\square\)

(A5) $J(X, U)$ is weakly lower semicontinuous.

Proof. $J(X, U)$ is a continuous function, since all norms are continuous [9]. Therefore it follows that $J(X, U)$ is also lower semicontinuous.

Convexity of $J(X, U)$ will be shown in order to be able to apply Lemma 3.5.

Let $n \geq 1$. Let $g : \mathbb{R}^n \to \mathbb{R}_+$ with $g(x) = |x|_{\mathbb{R}^n}$ and $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Since $f(x)$ is convex and non-decreasing on $\mathbb{R}_+$, and all norms are convex, it follows that $f(g(x)) = |x|_{\mathbb{R}^n}^2$ is convex [4]. Furthermore, it is known that a convex function after a linear transformation is still convex [4]. Using this information, it follows that $|X(t) - \bar{x}|_{\mathbb{R}^dN}^2$ and $|\dot{X}(t)|_{\mathbb{R}^dN}^2$ are convex and hence convexity of $J(X, U)$ can be shown.

Let $\sigma \in [0,1]$, $X_1, X_2 \in \mathcal{X}$ and $U_1, U_2 \in \mathcal{U}$.

In the derivation below, the spaces are omitted to avoid long notations. However, this should cause no confusion.

$$J(\sigma X_1 + (1 - \sigma)X_2, \sigma U_1 + (1 - \sigma)U_2)$$

$$= \frac{1}{2N} \int_0^T \sigma |X_1(t) - \bar{x}|^2 + (1 - \sigma)|X_2(t) - \bar{x}|^2 dt + \frac{\lambda}{2M} \int_0^T \sigma |\dot{U}_1(t) + (1 - \sigma)\dot{U}_2(t)|^2 dt$$

$$\leq \frac{1}{2N} \int_0^T \sigma |X_1(t) - \bar{x}|^2 + (1 - \sigma)|X_2(t) - \bar{x}|^2 dt + \frac{\lambda}{2M} \int_0^T \sigma |\dot{U}_1(t)|^2 + (1 - \sigma)|\dot{U}_2(t)|^2 dt$$

$$= \sigma J(X_1, U_1) + (1 - \sigma)J(X_2, U_2)$$

So $J(X, U)$ is also convex and consequently, $J(X, U)$ is weakly lower semicontinuous (Lemma 3.5).

\(\square\)
(A6) \(J(X, U)\) is coercive with respect to \(U\).

Proof. If \(J\) is coercive with respect to \(U\) it must hold that \(J(X, U) \to \infty\) as \(\|U\|_U \to \infty\).

By the fundamental theorem of calculus [8] and the triangle inequality, it is known that

\[
|U(t)| = |U(0)| + \int_0^t |\dot{U}(s)|ds
\]

Using Hölder’s inequality (Lemma 8.1):

\[
\leq |U(0)| + \sqrt{T} \left( \int_0^t |\dot{U}(s)|^2ds \right)^{\frac{1}{2}}
\]

Squaring both sides:

\[
|U(t)|^2 \leq \left( |U(0)| + \sqrt{T} \left( \int_0^t |\dot{U}(s)|^2ds \right)^{\frac{1}{2}} \right)^2
\]

From Young’s inequality (Lemma 8.2) it follows that for \(a, b \geq 0\), \((a + b)^2 \leq 2a^2 + 2b^2\). Therefore

\[
|U(t)|^2 \leq 2|U(0)|^2 + 2t \int_0^t |\dot{U}(s)|^2ds
\]

Integrating both sides from 0 to \(T\):

\[
\int_0^T |U(t)|^2dt \leq 2T|U(0)|^2 + 2t \int_0^T |\dot{U}(s)|^2dsdt
\]

\[
\leq 2T|U(0)|^2 + 2T \int_0^T |\dot{U}(s)|^2dsdt
\]

\[
= 2T|U(0)|^2 + T^2 \int_0^T |\dot{U}(t)|^2dt
\]

This results in the following estimate:

\[
\|U\|^2_U = \int_0^T |U(t)|^2dt + \int_0^T |\dot{U}(t)|^2dt
\]

\[
\leq 2T|U(0)|^2 + (T^2 + 1) \int_0^T |\dot{U}(t)|^2dt
\]

Since the initial condition for \(U\) is fixed, because \(U \in U_{ad}\), it holds that if \(\|U\|_U \to \infty\), then \(\int_0^T \sum_{l=1}^M |\dot{u}_l|^2_{R^d}dt \to \infty\).

Since \(\frac{1}{2N} \int_0^T \sum_{i=1}^N |x_i - \bar{x}|^2_{R^d}dt \geq 0\), it follows that

\[
J(X, U) = \frac{1}{2N} \int_0^T \sum_{i=1}^N |x_i - \bar{x}|^2_{R^d}dt + \frac{\lambda}{2M} \int_0^T \sum_{i=1}^M |\dot{u}_i|^2_{R^d}dt \to \infty \text{ as } \|U\|_U \to \infty
\]

Hence, all the conditions are satisfied and (OP) indeed has a solution.
4 First-order necessary conditions for optimality

This chapter consists of two parts. In the first part (section 4.1), the first-order necessary conditions for optimality are derived, as the title already suggests. The first-order necessary conditions for optimality consist of the state equation, the control equation and the adjoint equation. These equations will be derived with the help of the method of Lagrange multipliers, which will be explained in that section.

In the second part (section 4.2), the first-order necessary conditions for optimality will be rewritten to an initial value problem with initial conditions for \( t = 0 \) only. Thereafter, Schäfer’s fixed point theorem is used to verify the existence of solutions of that initial value problem. Concepts that are used in this chapter are defined in Appendix A.III.

4.1 Derivation first-order necessary conditions for optimality

The goal of this paper is to minimize the cost functional \( \mathbf{CF} \) subject to the constraint \( E(X, U) = 0 \) in \( \mathcal{Y} \), with \( \mathcal{Y} = L^2([0, T], \mathbb{R}^{dX}) \), which was already stated before. To accomplish this, the method of Lagrange multipliers [11] can be used. This method is used for finding stationary points of a function subject to a constraint, which makes it suitable for the optimization problem (OP). Note that this method only gives a necessary condition for optimality, so the stationary points may be saddle points or (local or global) minimizers or maximizers.

Let \( Q = (q^1, \ldots, q^N) \in \mathcal{X} \) be the Lagrange multipliers (also called adjoint variables). For the method of Lagrange multipliers, the Gâteaux derivatives of the Lagrangian \( \mathcal{L} : \mathcal{X} \times \mathcal{U}_{ad} \times \mathcal{X} \to \mathbb{R} \) defined by

\[
\mathcal{L}(X, U, Q) = J(X, U) + \langle E(X, U), Q \rangle
\]

with respect to \( X, U \) and \( Q \) will be computed. If \( X, U \) and \( Q \) are saddle points of \( \mathcal{L} \), the Gâteaux derivatives of the Lagrangian with respect to \( X, U \) and \( Q \) are 0. Therefore, the Gâteaux derivatives of the Lagrangian with respect to \( X, U \) and \( Q \) will be set equal to 0.

The equations that follow from these computations are called, respectively, the adjoint equation, the control equation and the state equation. Together they form the first-order necessary conditions for optimality, which are stated in the theorem below:

**Theorem 4.1.** The first-order necessary conditions for optimality are:

\[
\frac{d}{dt} x^i_t = \frac{1}{N} \sum_{j=1}^{N} K(x^i_t - x^j_t) + \frac{1}{M} \sum_{l=1}^{M} \Phi(x^i_t - u^j_t) \text{ and } x^i_{t=0} = x_0 \quad \text{(SE)}
\]

\[
\frac{d^2}{dt^2} u^j_t = \frac{1}{N} \sum_{i=1}^{N} \nabla \Phi(x^i_t - u^j_t) q^j_i \text{ and } u^j_{t=0} = u^j_0, \dot{u}^j_T = 0 \quad \text{(CE)}
\]

\[
\frac{d}{dt} q^j_t = -\frac{1}{N} \sum_{i=1}^{N} \nabla K(x^i_t - x^j_t)(q^i_t - q^j_t) - \frac{1}{M} \sum_{l=1}^{M} \nabla \Phi(x^i_t - u^j_t) q^j_l + \frac{1}{N} (x^i_t - \bar{x}) \text{ and } q^j_T = 0 \quad \text{(AE)}
\]

The state, control and adjoint equation will be derived to obtain the first-order necessary conditions for optimality. In these derivations, the inner product corresponding to \( L^2([a, b]) \) is used quite frequently. Recall that this inner product was defined as \( \langle f, g \rangle = \int_a^b (f_t, g_t) dt \) for all real functions \( f, g \in L^2([a, b]) \).

4.1.1 State equation

**Lemma 4.1.** The state equation is

\[
\frac{d}{dt} x^i_t = \frac{1}{N} \sum_{j=1}^{N} K(x^i_t - x^j_t) + \frac{1}{M} \sum_{l=1}^{M} \Phi(x^i_t - u^j_t) \quad x^i_{t=0} = x_0
\]
Proof. The Gâteaux derivative with respect to $Q$ will be computed. Since the cost functional doesn't depend on $Q$, its Gâteaux derivative will be 0.

$$D_Q \mathcal{L}(X,U,Q)[h_Q] = D_Q J(X,U)[h_Q] + D_Q \langle E(X,U),Q \rangle[h_Q]$$

$$= \lim_{\tau \to 0} \frac{(E(X,U),Q + \tau h_Q) - \langle E(X,U),Q \rangle}{\tau} = \langle E(X,U),h_Q \rangle$$

Since it has to hold that $D_Q \mathcal{L}(X,U,Q)[h_Q] = 0$ for all $h_Q \in X$, it must hold that $E(X,U) = 0$. This results in:

$$\frac{d}{dt} x_i^t = \frac{1}{N} \sum_{j=1}^{N} K(x_i^t - x_j^t) + \frac{1}{M} \sum_{l=1}^{M} \Phi(x_i^t - u_i^t)$$

Remark 4.1. The state equation is just the constraint.

4.1.2 Control equation

Lemma 4.2. The control equation is

$$\frac{d^2}{dt^2} u_l^t = \frac{1}{\lambda} \sum_{i=1}^{N} \nabla \Phi(x_i^t - u_l^t) q_i^t$$

$$u_{i,0}^t = u_0^t, \quad u_T^t = 0$$

Proof. The Gâteaux derivative with respect to $U$ will be computed below.

$$D_U \mathcal{L}(X,U,Q)[h_U] = D_U J(X,U)[h_U] + D_U \langle E(X,U),Q \rangle[h_U]$$

First, the Gâteaux derivative of the cost functional (CF) will be computed. Note that only the second term of the cost functional depends on $U$ and hence the first term doesn’t contribute to the Gâteaux derivative with respect to $U$.

Let $\mathcal{Y} = L^2 ([0,T], \mathbb{R}^{dM})$.

$$D_U J(X,U)[h_U] = D_U \frac{\lambda}{2M} \| \dot{U} \|_2^2 [h_U]$$

$$= \frac{\lambda}{2M} \lim_{\tau \to 0} \frac{\langle \dot{U} + \tau \dot{h}_U, \dot{U} + \tau \dot{h}_U \rangle - \langle \dot{U}, \dot{U} \rangle}{\tau}$$

$$= \frac{\lambda}{M} \int_0^T \langle \dot{U}, \dot{h}_U \rangle dt$$

This integral can be rewritten with the help of partial integration:

$$= \frac{\lambda}{M} (\langle \dot{U}, h_U \rangle \bigg|_0^T - \int_0^T (\dot{U}, h_U) dt)$$

Assume $h_U(0) = 0$:

$$= \frac{\lambda}{M} (\langle \dot{U}, h_U \rangle - \langle \dot{U}, h_U \rangle)$$

Now the second part of the Gâteaux derivative will be computed. For simplification I will write

$$\langle E(X,U),Q \rangle = \langle X, Q \rangle - \langle F(X,U),Q \rangle$$

$$= \int_0^T \sum_{i=1}^{N} (x_i^t, q_i^t) dt - \int_0^T \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} \left( K(x_i^t - x_j^t), q_i^t \right) dt - \int_0^T \sum_{i=1}^{N} \frac{1}{M} \sum_{l=1}^{M} \left( \Phi(x_i^t - u_i^t), q_i^t \right) dt$$

$$=: f_1(X,Q) - f_2(X,Q) - f_3(X,U,Q)$$
Note that \( f_1 \) and \( f_2 \) don’t depend on \( U \) and hence, only the Gâteaux derivative of \( f_3 \) has to be considered here:

\[
D_U((E(X, U), Q))[h_U] = D_U \left( - \int_0^T \sum_{i=1}^N \frac{1}{M} \sum_{l=1}^M (\Phi(x^i_l - u^i_l), q^i_l) dt \right) [h_U]
\]
\[
= \lim_{\tau \to 0} \int_0^T \sum_{i=1}^N \frac{1}{M} \sum_{l=1}^M \left( (\Phi(x^i_l - u^i_l), q^i_l) - (\Phi(x^i_l - u^i_l - \tau h^i_l), q^i_l) \right) dt
\]

Using Lemma 3.6:

\[
= \lim_{\tau \to 0} \int_0^T \sum_{i=1}^N \frac{1}{M} \sum_{l=1}^M \left( \int_0^1 D\Phi((1 - \sigma)(x^i_l - u^i_l) + \sigma(x^i_l - u^i_l - \tau h^i_l)) [h^i_l] d\sigma, q^i_l \right) dt
\]
\[
= \int_0^T \sum_{i=1}^N \frac{1}{M} \sum_{l=1}^M \left( \int_0^1 D\Phi(x^i_l - u^i_l) [h^i_l] d\sigma, q^i_l \right) dt
\]

The inner integral from 0 to 1 doesn’t contribute to the expression, because the expression in the integral doesn’t depend on \( \sigma \). Furthermore, the inner product can be rewritten:

\[
= \int_0^T \sum_{i=1}^N \frac{1}{M} \sum_{l=1}^M \left( h^i_l, (D\Phi(x^i_l - u^i_l))^T q^i_l \right) dt
\]

Note that the transpose of the derivative is the gradient. Also, since \( h^i_U \) doesn’t depend on \( i \), the summation over \( i \) can be taken inside the inner product:

\[
= \frac{1}{M} \int_0^T \sum_{l=1}^M \left( h^l_U, (D\Phi(x^l - u^l)) q^l \right) dt
\]
\[
= \frac{1}{M} \langle \Psi(X, U, Q), h_U \rangle , \text{ where } \Psi^l(X, U, Q) = \sum_{i=1}^N \nabla \Phi(x^i - u^i) q^i
\]

Summarizing these two results gives the Gâteaux derivative with respect to \( U \):

\[
D_U \mathcal{L}(X, U, Q)[h_Q] = \frac{1}{M} \left( (\lambda \hat{U}_T, h_U) + \langle \Psi(X, U, Q) - \lambda \hat{U}, h_U \rangle \right)
\]

Since it has to hold that \( D_U \mathcal{L}(X, U, Q)[h_U] = 0 \) for all \( h_U \in \mathcal{U} \) such that \( h_U(0) = 0 \), it must hold that \( \hat{U}_T = 0 \) and \( \Psi(X, U, Q) - \lambda \hat{U} = 0 \).

This results in:

\[
\frac{d^2}{dt^2} u^i_l = \frac{1}{\lambda} \sum_{i=1}^N \nabla \Phi(x^i_l - u^i_l) q^i_l
\]
\[
\hat{u}^i_l = 0
\]

\[\square\]

4.1.3 Adjoint equation

**Lemma 4.3.** The adjoint equation is

\[
\frac{d}{dt} q^i_T = -\frac{1}{N} \sum_{j=1}^N \nabla K(x^i_l - x^j_l) (q^i_T - q^j_T) - \frac{1}{M} \sum_{l=1}^M \nabla \Phi(x^i_l - u^i_l) q^i_T + \frac{1}{N} (x^i - \bar{x}) q^i_T = 0
\]

**Proof.** The Gâteaux derivative with respect to \( X \) will be computed:

\[
D_X \mathcal{L}(X, U, Q)[h_X] = D_X J(X, U)[h_X] + D_X (E(X, U), Q)[h_X]
\]
Again, this Gâteaux derivative will be computed in two parts, starting with the Gâteaux derivative of the cost functional (CF). Note that only the first term of the cost functional depends on \( X \).

\[
D_X J(X,U)[h_X] = D_X \left( \frac{1}{2N} \int_0^T \sum_{i=1}^N |x_i^t - \bar{x}|^2 dt + \frac{1}{N} \int_0^T \sum_{i=1}^N 2\tau(x_i^t - \bar{x}, h_X^i) + \tau^2(h_X^i, h_X^i)dt \right)
= \frac{1}{2N} \lim_{\tau \to 0} \int_0^T \sum_{i=1}^N 2\tau(x_i^t - \bar{x}, h_X^i) \frac{dt}{\tau} \quad \langle X_t - \bar{x}, h_X \rangle
\]

For the second part of the Gâteaux derivative, again the notation of \( f_1, f_2, \) and \( f_3 \) is used. Recall that:

\[
\langle E(X,U), Q \rangle = \langle \dot{X}, Q \rangle - \langle F(X,U), Q \rangle
= \int_0^T \sum_{i=1}^N (\dot{x}_i^t, q_i^t)dt - \int_0^T \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N \left( K(x_i^t - x_j^t), q_i^t \right)dt - \int_0^T \sum_{i=1}^N \frac{1}{M} \sum_{l=1}^M \left( \Phi(x_i^t - u_l^t), q_i^t \right)dt
=: f_1(X,U) - f_2(X,U) - f_3(X,U, Q)
\]

\( f_1, f_2 \) and \( f_3 \) depend on \( X \) and hence each of their Gâteaux derivatives will be calculated, starting with the Gâteaux derivative of \( f_1 \):

\[
D_X f_1(X,U)[h_X] = \lim_{\tau \to 0} \int_0^T \sum_{i=1}^N \frac{\left( (x_i^{t+} + \tau h_X^i)^t - (x_i^t, q_i^t) \right)dt}{\tau}
= \int_0^T \langle \dot{X}, Q_t \rangle dt
\]

Using partial integration I can write:

\[
= \left. (Q_t, h_X) \right|_0^T - \int_0^T \langle \dot{Q}_t, h_X \rangle dt
\]

Assume \( h_X(0) = 0 \):

\[
= \langle Q_T, h_X \rangle - \langle \dot{Q}_T, h_X \rangle
\]

Secondly, the Gâteaux derivative of \( f_2 \) is considered:

\[
D_X f_2(X,U)[h_X] = \lim_{\tau \to 0} \int_0^T \sum_{i=1}^N \frac{\left( K(x_i^{t+} + \tau h_X^i - x_i^t, q_i^t) - (K(x_i^t, q_i^t)) \right)dt}{\tau}
= \int_0^T \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N \left( \int_0^1 DK(x_i^t - x_j^t)[h_X^j - h_X^i]d\sigma, q_i^t \right)dt
\]

Again it is used that the inner integral has no contribution and that the gradient is the transpose of the derivative. Also, the inner product can be split into two inner products because of linearity. Hence, the inner product will be rewritten to:

\[
= \int_0^T \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N \left( h_X^i, \nabla K(x_i^t - x_j^t)q_i^t \right) \left( h_X^j, \nabla K(x_i^t - x_j^t)q_i^t \right) dt
- \frac{1}{N} \int_0^T \sum_{i=1}^N \left( h_X^i, \sum_{j=1}^N \nabla K(x_i^t - x_j^t)q_j^t \right) dt
\]

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Interchanging the summations in the second term gives:

\[
= \frac{1}{N} \int_0^T \sum_{i=1}^N \left( h_X^i \sum_{j=1}^N \nabla K(x^i - x^j)q^i_j \right) dt - \frac{1}{N} \int_0^T \sum_{i=1}^N \left( h_X^i \sum_{j=1}^N \nabla K(x^i - x^j)q^i_j \right) dt
\]

= \langle \Psi_1(X, Q), h_X \rangle,

where \( \Psi_1 = \frac{1}{N} \sum_{j=1}^N \left( \nabla K(x^i - x^j)q^i_j - \nabla K(x^j - x^i)q^j_i \right) = \frac{1}{N} \sum_{j=1}^N \nabla K(x^i - x^j)(q^i_j - q^j_i) \)

Rewriting \( \Psi_1 \) is justified since \( K \) is antisymmetric and hence \( K(-x) = -K(x) \) for \( x \in \mathbb{R}^d \). Therefore, its gradient \( \nabla K \) is symmetric: \( \nabla K(-x) = \nabla K(x) \) for \( x \in \mathbb{R}^d \).

Lastly, the Gâteaux derivative of \( f_3 \) will be investigated:

\[
D_X f_3(X)[h_X] = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^T \sum_{i=1}^N \sum_{l=1}^M \left( \Phi(x^i_l - u^i_l), q^i_l \right) - \left( \Phi(x^i_l - u^i_l), q^i_l \right) dt
\]

\[
= \int_0^T \sum_{i=1}^N \sum_{l=1}^M \left( \int_0^1 D\Phi(x^i_l - u^i_l)[h_X]d\sigma, q^i_l \right) dt \quad (\text{Lemma 3.6})
\]

Once more, the fact that the transpose of the derivative is the gradient and the fact that the inner integral has no contribution can be used:

\[
= \int_0^T \sum_{i=1}^N \sum_{l=1}^M \nabla \Phi(x^i_l - u^i_l)q^i_l dt
\]

\[
= \langle \Psi_2(X, U, Q), h_X \rangle,
\]

where \( \Psi_2(X, U, Q) = \frac{1}{M} \sum_{i=1}^M \nabla \Phi(x^i - u^i)q^i \)

So the Gâteaux derivative with respect to \( X \) becomes:

\[
D_X\mathcal{L}(X, U, Q)[h_X] = (Q_T, h_X) - (\dot{Q} + \Psi_1 + \Psi_2 - \frac{1}{N}(X_t - \bar{x}), h_X)
\]

Since it has to hold that \( D_X\mathcal{L}(X, U, Q)[h_X] = 0 \) for all \( h_X \in X \) such that \( h_X(0) = 0 \), it must hold that \( Q_T = 0 \) and \( \dot{Q} = -\Psi_1(X, Q) - \Psi_2(X, U, Q) + \frac{1}{N}(X_t - \bar{x}) \).

This results in:

\[
\frac{d}{dt} q^i_t = -\frac{1}{N} \sum_{j=1}^N \nabla K(x^i_t - x^j_t)(q^i_t - q^j_t) - \frac{1}{M} \sum_{l=1}^M \nabla \Phi(x^i_l - u^i_l)q^i_l + \frac{1}{N}(x^i_t - \bar{x}) \quad q^i_T = 0
\]

The results of the three lemmas combined give the first-order necessary conditions for optimality:

\[
\text{Proof of Theorem 4.1.} \quad \text{This follows from lemmas 4.1, 4.2 and 4.3.}
\]

### 4.2 Existence of solutions

To verify the existence of solutions, the first-order necessary conditions for optimality will be rewritten. The control equation (CE) can be rewritten into a coupled system consisting of two first-order differential equations:

\[
\frac{d}{dt} u^i_t = v^i_t \quad \text{u}^i_{t=0} = u^i_0
\]

\[
\frac{d}{dt} v^i_t = \frac{1}{\lambda} \sum_{i=1}^N \nabla \Phi(x^i_t - u^i_t)q^i, \quad v^i_T = 0
\]

where \( V = (v^1, ..., v^M) \in U \).
Remark 4.2. $V$ is the velocity of the controls $U$.

Substituting $P_t := Q_{T-t}$ into the first-order necessary conditions for optimality and $W_t := V_{T-t}$ into the above coupled system, where $P = (p^1, ..., p^N) \in \mathcal{X}$, $W = (w^1, ..., w^M) \in \mathcal{U}$, gives rise to the following equations:

$$
\frac{d}{dt} x^i_t = \frac{1}{N} \sum_{j=1}^{N} K(x^j_t - x^i_t) + \frac{1}{M} \sum_{l=1}^{M} \Phi(x^i_t - u^i_l) \quad x^i_{t=0} = x^i_0
$$

$$
\frac{d}{dt} u^i_t = w^i_{T-t} \quad u^i_{t=0} = u^i_0
$$

$$
\frac{d}{dt} p^i_t = \frac{1}{N} \sum_{j=1}^{N} \nabla K(x^j_{T-t} - x^j_t)(p^j_t - p^i_t) + \frac{1}{M} \sum_{l=1}^{M} \nabla \Phi(x^j_{T-t} - u^j_{T-t})p^j_l - \frac{1}{N} (x^j_{T-t} - x^j_t) p^i_{t=0} = 0
$$

$$
\frac{d}{dt} w^i_t = -\frac{1}{\lambda} \sum_{j=1}^{N} \nabla \Phi(x^j_{T-t} - u^j_{T-t})p^j_l \quad w^i_{t=0} = 0
$$

Let $Y = (X, U, P, W)$ and let $G$ be the vector of all the functions on the right-hand side, so I can rewrite the system of equations into the initial value problem

$$
\frac{d}{dt} Y(t) = G(Y)(t) \quad Y(0) = Y_0 \tag{IVP}
$$

where $G : \mathcal{M} \rightarrow \mathcal{M}$, with $\mathcal{M} = (C([0, T], \mathbb{R}^k), || \cdot ||_{\text{sup}})$ and $k = d(2M + 2N)$. Note that this choice of $\mathcal{M}$ is allowed since $H^1([0, T])$ is embedded in $C([0, T])$ (Lemma 2.1).

Integrating both sides of (IVP) with respect to $t$ gives $Y(t) = Y_0 + \int_0^t G(Y)(s)ds$.

Define the operator $\Gamma : \mathcal{M} \rightarrow \mathcal{M}$ as $\Gamma(Y)(t) := Y_0 + \int_0^t G(Y)(s)ds$.

Theorem 4.2. Let $G : \mathcal{M} \rightarrow \mathcal{M}$. There exists at least one solution to the initial value problem (IVP).

To see that the above theorem holds true, the following theorem is used:

Theorem 4.3 (Schäfer’s fixed point theorem [7]). Let $Z$ be a Banach space and let $H : Z \rightarrow Z$ be a compact mapping with the property that there exists $r > 0$ such that $\{ z \in Z ; \sigma H(z) = z \text{ for some } 0 \leq \sigma \leq 1 \} \subset B(0; r)$. Then $H$ has at least one fixed point in $\overline{B}(0; r)$.

To use the above theorem, three properties (defined in lemmas 4.4, 4.5 and 4.10) need to be satisfied. It will be proven that this is indeed the case in the rest of this chapter.

### 4.2.1 Banach space

Lemma 4.4. $\mathcal{M} = (C([0, T], \mathbb{R}^k), || \cdot ||_{\text{sup}})$ with $k = d(2M + 2N)$ is a Banach space.

Proof. The proof can be found in [2].

Remark 4.3 ([2]). In fact, for all nonnegative $m$ and $[a, b] \subset \mathbb{R}$, the space $C^m([a, b])$ equipped with the norm $||x||_{C^m} = \sum_{i=0}^{m} \sup_{t \in [a, b]} |x^{(i)}(t)|$ is a Banach space.

### 4.2.2 Compact mapping

Lemma 4.5. $\Gamma : \mathcal{M} \rightarrow \mathcal{M}$ is a compact mapping.

To prove the above lemma, the following information is used:

Lemma 4.6 (Generalized embedding theorem [2]). Let $\Omega \subset \mathbb{R}$ be an open and bounded interval and let $k_1, k_2 \geq 0$ with $k_1 > k_2$. Then the embedding $\text{Id} : (C^{k_1}(\Omega, \mathbb{R}), || \cdot ||_{C^{k_1}}) \rightarrow (C^{k_2}(\Omega, \mathbb{R}), || \cdot ||_{C^{k_2}})$ is compact.

Corollary 4.6.1 (Generalized embedding theorem in multiple dimensions). Let $\Omega \subset \mathbb{R}$ be an open and bounded interval. Then the embedding $\text{Id} : (C^1(\Omega, \mathbb{R}^n), || \cdot ||_{C^1}) \rightarrow (C(\Omega, \mathbb{R}^n), || \cdot ||_{\text{sup}})$ with $n \geq 1$ is compact.
Proof. Let $Ω = (0, T)$. Assume $A ⊂ C²((Ω, R^n))$ is bounded. Then $∃B ≥ 0$ such that $∀x ∈ A$ it holds that $||x||_{C¹} ≤ B$. This can be used to derive the fact that $||x^i||_{C¹} ≤ B$ for all $i = 1, ..., n$.

$$||x^i||_{C¹} = \sup_{t \in [0, T]} |x^i(t)|_R + \sup_{t \in [0, T]} |x^i(t)|_R \\
≤ \sup_{t \in [0, T]} \sum_{i=1}^n |x^i(t)|_R + \sup_{t \in [0, T]} \sum_{i=1}^n |x^i(t)|_R$$

Recall that the 1-norm is used in this report, as was stated in section 2.2:

$$= \sup_{t \in [0, T]} |x(t)|_R + \sup_{t \in [0, T]} |x(t)|_R \\
= ||x||_{C¹} ≤ B \quad ∀i = 1, ..., n$$

Hence, $A = \{x^i ∈ C²((Ω, R), x ∈ A) ⊂ C²((Ω, R))$ is bounded.

Since $Id : C²((Ω, R)) → C(Ω, R)$ is compact (Lemma 4.6), $Id(A_1) = A_1 ⊂ C(Ω, R)$ is totally bounded.

Thus, $∀ε > 0 \exists \{y^1_1, ..., y^k_1\} ⊂ A_1$ such that $A_1 \subset \cup_{i=1}^{k} B(y_i^1; ε') \quad ∀i = 1, ..., n$.

Since $A_1$ is totally bounded, $∀x_i ∈ A_1$ there exists a $l^i_{x_i}$ such that $x^i ∈ B(y^i_1, ε)'$ with $l^i_{x_i} ∈ \{1, ..., k_i\} \quad ∀i = 1, ..., n$.

$$\sup_{t \in [0, T]} |x^i(t) - y^i_{l^i_{x_i}}(t)|_R < ε'$$

$$\sum_{i=1}^n \sup_{t \in [0, T]} |x^i(t) - y^i_{l^i_{x_i}}(t)|_R < nε'$$

Note that the supremum of the sum is always less or equal than the sum of the supremum. Let $x_h = (y^1_{l^1_{x_1}}, ..., y^n_{l^n_{x_n}})$ and use the above inequality to arrive at the following inequality:

$$||x - x_h||_{L^\sup} = \sup_{t \in [0, T]} \sum_{i=1}^n |x^i(t) - y^i_{l^i_{x_i}}(t)|_R ≤ \sum_{i=1}^n \sup_{t \in [0, T]} |x^i(t) - y^i_{l^i_{x_i}}(t)|_R < nε'$$

Since $x$ was arbitrary, $∀x ∈ A$ there exists a $x_h$ such that $x ∈ B(x_h; nε')$.

Choosing $ε = nε'$ results in $x ∈ B(x_h; ε)$.

There are finitely many, namely $k = k_1 · k_2 · ... · k_n$, $x_h$.

This implies that $∀ε > 0 \exists \{x_1, ..., x_k\} ∈ A$ such that $A \subset \cup_{l=1}^{k} B(x_l; ε)$.

Therefore, $Id(A) = A ⊂ C(Ω, R^n)$ is totally bounded.

Hence, the embedding $Id : (C²((Ω, R^n), ||·||_{C¹}) → (C(Ω, R^n), ||·||_{L^\sup})$ is compact. □

**Lemma 4.7** ([1]). Local Lipschitz continuity implies continuity.

**Lemma 4.8.** Let $K, Φ, ∇K$ and $∇Φ$ be Lipschitz continuous with Lipschitz constants $L_K, L_Φ, L_{∇K}$ and $L_{∇Φ}$ respectively. Then $G : M → M$ is locally Lipschitz continuous. In particular, $G : M → M$ is continuous as a mapping from $M$ to $M$.

Proof. Let $Y^1, Y^2 ∈ M$. Recall that the 1-norm, defined in section 2.2, is used in this report. Furthermore, spaces will be omitted here to shorten the notation. However, this should not be confusing for the reader.

I will start by making estimates for the four different components of $G$. After that, these results will be combined to show the local Lipschitz continuity of $G$.

Let

$$\textbf{(I)} := \sum_{i=1}^N \sum_{j=1}^N \frac{1}{N} K(x^1_{i,j} - x^1_{i,j}) - \frac{1}{N} \sum_{j=1}^N K(x^2_{i,j} - x^2_{i,j})$$

$$\textbf{(II)} := \sum_{i=1}^N \frac{1}{M} \sum_{l=1}^M Φ(x^1_{i,j} - u^1_{i,j}) - \frac{1}{M} \sum_{l=1}^M Φ(x^2_{i,j} - u^2_{i,j})$$

For the first component of $G$ it holds that, by the triangle inequality:

$$|G_1(Y^1)(t) - G_1(Y^2)(t)| = \sum_{i=1}^N |G^1_i(Y^1)(t) - G^1_i(Y^2)(t)| \\
≤ \textbf{(I)} + \textbf{(II)}$$
The terms (I) and (II) will be evaluated separately. The first term was already evaluated in the proof of Lemma 3.7, where it was shown that

\[(I) \leq 2L_K |X^1_t - X^2_t|\]

By the Lipschitz continuity of \(\Phi\), for the second term it holds that

\[(II) \leq \frac{N}{M} \sum_{i=1}^{N} \frac{1}{M} \sum_{j=1}^{M} L_{\Phi}(\Phi(x^1_{t,i} - u^1_{t,i}) - (x^2_{t,i} - u^2_{t,i}))\]

Again, the triangle inequality is used:

\[\leq L_{\Phi}|X^1_t - X^2_t| + L_{\Phi} \frac{N}{M}|U^1_t - U^2_t|\]

Combining the two terms yields

\[|G_1(Y^1)(t) - G_1(Y^2)(t)| \leq (2L_K + L_{\Phi} \frac{N}{M})||Y^1 - Y^2||_{\text{sup}} =: C_1||Y^1 - Y^2||_{\text{sup}}\]

It can easily be seen that the following estimate holds for the second component of \(G\):

\[|G_2(Y^1)(t) - G_2(Y^2)(t)| = \sum_{i=1}^{M} \left|G^2_i(Y^1)(t) - G^2_i(Y^2)(t)\right|\]

\[\leq \left||Y^1 - Y^2\right||_{\text{sup}} =: C_2||Y^1 - Y^2||_{\text{sup}}\]

The estimate for the third component is less trivial and will be made below: Let

\[(I) := \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} \nabla K(x^1_{t,i} - x^1_{t,j})(p^1_{t,i} - p^1_{t,j}) - \frac{1}{N} \sum_{j=1}^{N} \nabla K(x^2_{t,i} - x^2_{t,j})(p^2_{t,i} - p^2_{t,j})\]

\[(II) := \sum_{i=1}^{N} \left|\frac{1}{M} \sum_{l=1}^{M} \nabla \Phi(x^1_{t,i} - u^1_{t,i})p^1_{t,i} - \frac{1}{M} \sum_{l=1}^{M} \nabla \Phi(x^2_{t,i} - u^2_{t,i})p^2_{t,i}\right|\]

\[(III) := \sum_{i=1}^{N} \left|\frac{1}{N}(x^1_{t,i} - \bar{x}) - \frac{1}{N}(x^2_{t,i} - \bar{x})\right|\]

Using the above notation, for the third component of \(G\) it holds that, by the triangle inequality:

\[|G_3(Y^1)(t) - G_3(Y^2)(t)| = \sum_{i=1}^{N} \left|G^3_i(Y^1)(t) - G^3_i(Y^2)(t)\right|\]

\[\leq (I) + (II) + (III)\]

First, (I) will be rewritten:

\[\text{(I)} = \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} \left(\nabla K(x^1_{t,i} - x^1_{t,j})(p^1_{t,i} - p^1_{t,j}) - \frac{1}{N} \nabla K(x^2_{t,i} - x^2_{t,j})(p^2_{t,i} - p^2_{t,j})\right)\]

The triangle inequality is used to receive an expression that allows the use of Lipschitz continuity:

\[\leq \sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} \left(\left|\nabla K(x^1_{t,i} - x^1_{t,j}) - \nabla K(x^2_{t,i} - x^2_{t,j})\right| |p^1_{t,i} - p^1_{t,j}| + |\nabla K(x^1_{t,i} - x^1_{t,j})| |p^2_{t,i} - p^2_{t,j}| + |\nabla K(x^2_{t,i} - x^2_{t,j})| |p^1_{t,i} - p^1_{t,j}| + |\nabla K(x^2_{t,i} - x^2_{t,j})| |p^2_{t,i} - p^2_{t,j}|\right)\]
Since $K$ is Lipschitz continuous, due to the mean value theorem [8] it holds that its gradient is bounded: $|\nabla K(y)| \leq L_K$ for $y \in \mathbb{R}^d$. Substituting this into the above expression and using Lipschitz continuity of $\nabla K$ results in:

$$\sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} \left( L_{\nabla K} \left( x_{T-t}^{1,i} - x_{T-t}^{1,j} \right) - (x_{T-t}^{2,i} - x_{T-t}^{2,j}) \right) ||Y^2||_{\text{sup}} + L_K |p_t^{1,i} - p_t^{1,j}|
$$

$$+ L_{\nabla K} \left( x_{T-t}^{1,i} - x_{T-t}^{1,j} \right) - (x_{T-t}^{2,i} - x_{T-t}^{2,j}) ||Y^1||_{\text{sup}} + L_K |p_t^{1,j} - p_t^{2,j}|$$

Again, the triangle inequality is used:

$$\sum_{i=1}^{N} \frac{1}{N} \sum_{j=1}^{N} \left( L_{\nabla K} \left( x_{T-t}^{1,i} - x_{T-t}^{1,j} \right) - (x_{T-t}^{2,i} - x_{T-t}^{2,j}) \right) ||Y^2||_{\text{sup}} + L_K |p_t^{1,i} - p_t^{1,j}|
$$

$$+ \sum_{i=1}^{N} L_K |p_t^{1,i} - p_t^{2,i}| + \sum_{j=1}^{N} L_K |p_t^{1,j} - p_t^{2,j}|$$

Next, (II) will be rewritten:

$$(\text{II}) = \sum_{i=1}^{N} \frac{1}{M} \sum_{l=1}^{M} \left( \nabla \Phi(x_{T-t}^{1,i} - u_{T-t}^{1,l})(p_t^{i,l} - p_t^{2,i}) - \nabla \Phi(x_{T-t}^{2,i} - u_{T-t}^{2,l})p_t^{2,i} \right)$$

The triangle inequality is used to receive an expression that allows the use of Lipschitz continuity:

$$\sum_{i=1}^{N} \frac{1}{M} \sum_{l=1}^{M} \left( \nabla \Phi(x_{T-t}^{1,i} - u_{T-t}^{1,l}) - \nabla \Phi(x_{T-t}^{2,i} - u_{T-t}^{2,l}) \right) ||\nabla \Phi||_{\text{sup}} + \frac{1}{M} ||\nabla \Phi||_{\text{sup}}$$

Since $\Phi$ is Lipschitz continuous, due to the mean value theorem [8] it holds that its gradient is bounded: $|\nabla \Phi(y)| \leq L_\Phi$ for $y \in \mathbb{R}^d$. Substituting this into the above expression and using Lipschitz continuity of $\nabla \Phi$ results in:

$$\sum_{i=1}^{N} \frac{1}{M} \sum_{l=1}^{M} \left( \nabla \Phi(x_{T-t}^{1,i} - u_{T-t}^{1,l}) - \nabla \Phi(x_{T-t}^{2,i} - u_{T-t}^{2,l}) \right) ||\nabla \Phi||_{\text{sup}} + L_\Phi |p_t^{1,i} - p_t^{2,i}|$$

Again, the triangle inequality is used:

$$\sum_{i=1}^{N} \frac{1}{M} \sum_{l=1}^{M} \left( \nabla \Phi(x_{T-t}^{1,i} - u_{T-t}^{1,l}) - \nabla \Phi(x_{T-t}^{2,i} - u_{T-t}^{2,l}) \right) ||\nabla \Phi||_{\text{sup}} + L_\Phi |p_t^{1,i} - p_t^{2,i}|$$

Lastly, I will rewrite the third term:

$$(\text{III}) = \sum_{i=1}^{N} \frac{1}{N} \left( ||x_{T-t}^{1,i} - x_{T-t}^{2,i}|| \right)$$

Combining the three terms yields

$$|G_3(Y^1)(t) - G_3(Y^2)(t)| \leq \left( 2(||Y^1||_{\text{sup}} + ||Y^2||_{\text{sup}} L_{\nabla K} + ||Y^2||_{\text{sup}} L_{\nabla \Phi} + \frac{N}{M} ||Y^2||_{\text{sup}} L_{\nabla \Phi} + 2L_K 
$$

$$+ L_\Phi + \frac{1}{N} \right) ||Y^1 - Y^2||_{\text{sup}}
$$

$$=: C_3 ||Y^1 - Y^2||_{\text{sup}}$$

Finally, the estimate for the fourth component of $G$ will be derived:

$$|G_4(Y^1)(t) - G_4(Y^2)(t)| = \sum_{i=1}^{M} \left( G_4^1(Y^1)(t) - G_4^1(Y^2)(t) \right)$$

$$= \sum_{i=1}^{M} \left( \frac{1}{\lambda} \sum_{l=1}^{N} \nabla \Phi(x_{T-t}^{1,i} - u_{T-t}^{1,l})p_t^{1,l} + \frac{1}{\lambda} \sum_{l=1}^{N} \nabla \Phi(x_{T-t}^{2,i} - u_{T-t}^{2,l})p_t^{2,l} \right) - \sum_{i=1}^{N} \nabla \Phi(x_{T-t}^{2,i} - u_{T-t}^{2,l})p_t^{2,l}$$

$$= \frac{1}{\lambda} \sum_{i=1}^{M} \sum_{l=1}^{N} \nabla \Phi(x_{T-t}^{1,i} - u_{T-t}^{1,l})(p_t^{1,i} - p_t^{2,i}) + \frac{1}{\lambda} \sum_{i=1}^{N} \nabla \Phi(x_{T-t}^{2,i} - u_{T-t}^{2,l})p_t^{2,i}$$
Using the triangle inequality and the Lipschitz continuity of \( \nabla \Phi \):

\[
\begin{align*}
&\leq \frac{1}{\lambda} \sum_{i=1}^{M} \sum_{i=1}^{N} \left| \nabla \Phi(y_{T,t}^{1,i} - u_{T,t}^{1,i}) \right| |p_{t}^{1,i} - p_{t}^{2,i}| \\
&\quad + \frac{1}{\lambda} \sum_{i=1}^{M} \sum_{i=1}^{N} L_{\nabla \Phi} \left| (x_{T,t}^{1,i} - u_{T,t}^{1,i}) - (x_{T,t}^{2,i} - u_{T,t}^{2,i}) \right| |p_{t}^{2,i}|
\end{align*}
\]

Again, I use the fact that \( |\nabla \Phi(y)| \leq L_{\Phi} \) for \( y \in \mathbb{R}^{d} \). Then I get, with the triangle inequality:

\[
\begin{align*}
&\leq \frac{1}{\lambda} (ML_{\Phi} + \|I2\|_{\text{sup}} ML_{\nabla \Phi} + \|I2\|_{\text{sup}} NL_{\nabla \Phi}) \|Y^{1} - Y^{2}\|_{\text{sup}} \\
&= C_{4} \|Y^{1} - Y^{2}\|_{\text{sup}}
\end{align*}
\]

As can be seen above, \( C_{3} \) and \( C_{4} \) depend on the norms of \( Y^{1} \) and \( Y^{2} \). Therefore, I will get that \( G \) is locally Lipschitz continuous instead of Lipschitz continuous. It will be shown below that it is indeed the case that \( G \) is locally Lipschitz continuous:

\[
\|G(Y^{1}) - G(Y^{2})\|_{\text{sup}} = \sum_{i=1}^{4} \|G_{i}(Y^{1}) - G_{i}(Y^{2})\|_{\text{sup}}
\]

Since \( C_{1}, C_{2}, C_{3} \) and \( C_{4} \) don’t depend on the time \( t \):

\[
\begin{align*}
&\leq (C_{1} + C_{2} + C_{3} + C_{4}) \|Y^{1} - Y^{2}\|_{\text{sup}} \\
&=: L_{G} \|Y^{1} - Y^{2}\|_{\text{sup}}
\end{align*}
\]

Note that \( L_{G} > 0 \) and \( L_{G} \) depends on the supremum norms of \( Y^{1} \) and \( Y^{2} \). From the above calculations it follows that there exists a \( r > 0 \) such that \( \forall Y^{2} \in \mathcal{M}, Y^{1} \in B(Y^{2}; r) \) there exists a \( L_{G} = L_{G}(Y^{2}) \) such that

\[
\|G(Y^{1}) - G(Y^{2})\|_{\text{sup}} \leq L_{G} \|Y^{1} - Y^{2}\|_{\text{sup}}
\]

Hence, \( G : \mathcal{M} \to \mathcal{M} \) is locally Lipschitz continuous. In particular, \( G : \mathcal{M} \to \mathcal{M} \) is continuous as a mapping from \( \mathcal{M} \) to \( \mathcal{M} \) (Lemma 4.7).

Let \( \mathcal{M}_{1} := \{C^{1}([0, T], \mathbb{R}^{k}), \| \cdot \|_{C^{1}}\} \), where \( \|x\|_{C^{1}} = \sup_{t \in [0, T]} |\dot{x}(t)|_{\mathbb{R}^{k}} + \sup_{t \in [0, T]} |x(t)|_{\mathbb{R}^{k}} \) and \( k = d(2M + 2N) \).

**Lemma 4.9.** Let \( G : \mathcal{M} \to \mathcal{M} \) be locally Lipschitz continuous with local Lipschitz constant \( L_{G} \). Then \( \Lambda : \mathcal{M} \to \mathcal{M} \) defined by \((\Lambda Y)(t) := Y_{0} + \int_{0}^{t} G(Y)(s)ds \) is locally Lipschitz continuous. In particular, \( \Lambda \) is continuous as a mapping from \( \mathcal{M} \) to \( \mathcal{M}_{1} \).

**Proof.** Let \( Y^{1}, Y^{2} \in \mathcal{M} \).

Since \( G : \mathcal{M} \to \mathcal{M} \) is locally Lipschitz continuous (Lemma 4.8), there exists a \( r > 0 \) such that \( \forall Y^{2} \in \mathcal{M}, Y^{1} \in B(Y^{2}; r) \) there exists a \( L_{G} = L_{G}(Y^{2}) \) such that

\[
\|G(Y^{1}) - G(Y^{2})\|_{\text{sup}} = \sup_{t \in [0, T]} |G(Y^{1})(t) - G(Y^{2})(t)| \leq L_{G} \|Y^{1} - Y^{2}\|_{\text{sup}}
\]

This will be used to show local Lipschitz continuity of \( \Lambda \):

\[
\begin{align*}
&\|\Lambda Y^{1} - \Lambda Y^{2}\|_{C^{1}} = \sup_{t \in [0, T]} |\Lambda Y^{1}(t) - \Lambda Y^{2}(t)| + \sup_{t \in [0, T]} |\Lambda Y^{1}(t) - \Lambda Y^{2}(t)| \\
&\leq \sup_{t \in [0, T]} |G(Y^{1})(t) - G(Y^{2})(t)| + \sup_{t \in [0, T]} \int_{0}^{t} |G(Y^{1})(s) - G(Y^{2})(s)|ds
\end{align*}
\]

Using the local Lipschitz continuity of \( G \):

\[
\begin{align*}
&\leq (1 + T)L_{G} \|Y^{1} - Y^{2}\|_{\text{sup}} \\
&=: L_{\Lambda} \|Y^{1} - Y^{2}\|_{\text{sup}}
\end{align*}
\]
Note that $L_A > 0$ and $L_A$ depends on the supremum norms of $Y^1$ and $Y^2$, because $L_G$ depends on the supremum norms of $Y^1$ and $Y^2$. From the above calculations it follows that there exists a $r > 0$ such that
\[
\forall Y^2 \in M, Y^1 \in B(Y^2; r) \text{ there exists a } L_A = L_A(Y^2) \text{ such that }
\]
\[
||Y^1 - Y^2||_{C^1} \leq L_A||Y^1 - Y^2||_{sup}
\]
Hence, $\Lambda : M \to M_1$ is locally Lipschitz continuous. In particular, $\Lambda$ is continuous as a mapping from $M$ to $M_1$ (Lemma 4.7).

The information needed for proving Lemma 4.5 is now complete and the proof can be written down:

**Proof of Lemma 4.5.** Let $\Omega = (0, T), k_1 = 1, k_2 = 0$. Then the embedding $Id : M_1 \to M$ is compact (Lemma 4.6).

First of all, note that $\Lambda : M \to M_1$ is continuous (Lemma 4.9) and that $\Lambda(A) \subset M_1$ is bounded for $A \subset M$ bounded. Since $Id : M_1 \to M$ is compact, $Id(\Lambda(A)) \subset M$ is totally bounded for all $\Lambda(A) \subset M_1$ bounded. Since $\Lambda(A) \subset M_1$ is bounded for $A \subset M$ bounded, $Id \circ \Lambda(A) \subset M$ is totally bounded for all $A \subset M$ bounded. Thus $Id \circ \Lambda : M \to M$ is a compact mapping. Recall that $\Gamma : M \to M$ was defined as $\Gamma(Y)(t) := Y_0 + \int_0^t G(Y(s))ds$.

Hence, $\Gamma = Id \circ \Lambda$ and consequently, $\Gamma : M \to M$ is a compact mapping.

### 4.2.3 A-priori estimates

**Lemma 4.10.** Let $C_\Phi := \sup_{y \in \mathbb{R}^d} \{|\Phi(y)| + |D\Phi(y)|\} < \infty$, where $D$ indicates a Jacobian matrix, $K$ be Lipschitz continuous with Lipschitz constant $L_K$ and $\Phi$ be Lipschitz continuous with Lipschitz constant $L_\Phi$. Then $\exists r > 0$ such that
\[
\{Y \in M : \sigma \Gamma(Y) = Y \text{ for some } 0 \leq \sigma \leq 1\} \subset B(0; r)
\]

**Proof.** Let $Y \in M$ and $\sigma \in [0, 1]$ arbitrary such that $\sigma \Gamma(Y) = Y$. Therefore, $Y(t) = \sigma Y_0 + \int_0^t G(Y(s))ds$, where $G : M \to M$. Due to the fundamental theorem of calculus, it holds that $\frac{d}{dt}Y(t) = \sigma G(Y(t))$ and that $Y$ is continuously differentiable [8]. In addition, $Y(0) = \sigma Y_0$. Hence, I can assume that there exists a solution to
\[
\frac{d}{dt}Y(t) = \sigma G(Y(t))
\]
and make an a-priori estimate for $Y$.

The condition $\exists r > 0$ such that $\{Y \in M : \sigma \Gamma(Y) = Y \text{ for some } 0 \leq \sigma \leq 1\} \subset B(0; r)$ is satisfied if $\exists r > 0$ such that $||Y||_M < r$, which is the result of the a-priori estimates for $X, P, W$ and $U$.

The a-priori estimate for $X$ was already stated in Lemma 3.9:
\[
\max_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} |x_i^t| \leq \sqrt{\frac{e^{(4L_K+C_\Phi^2+1)T} (NC_\Phi^2 T + N M ||U||_2^2)}{2}} = C_X
\]

The a-priori estimate for $P$ can now be made. This estimate depends on the a-priori estimate for $X$.
\[
\frac{1}{2} \frac{d}{dt} |p_i|^2 = \frac{1}{N} \sum_{j=1}^{N} \left( p_j \nabla K(x_j^t - x_i^t) (p_i - p_j) \right) + \frac{1}{M} \sum_{l=1}^{M} \left( p_{l_1} \nabla \Phi(x_{l_1}^t - u_{l_1}^t) p_{l_1} \right) - \frac{1}{N} (p_i, x_i^t - \bar{x})
\]
Since $\max_{i \in \{1, \ldots, N\}} |x_i^t| \leq C_X$, it follows from the triangle inequality that $\max_{i \in \{1, \ldots, N\}} |x_i^t - \bar{x}| \leq 2C_X$.

Furthermore, I know that if $K$ and $\Phi$ are Lipschitz continuous, their gradients are bounded (result of the mean value theorem [8]), i.e. $|\nabla K(y)| \leq L_K$ and $|\nabla \Phi(y)| \leq L_\Phi$ for $y \in \mathbb{R}^d$. Combining this information with the Cauchy-Schwarz inequality [8] results in:
\[
\frac{1}{2} \frac{d}{dt} |p_i|^2 \leq \frac{1}{N} \sum_{j=1}^{N} |p_j| L_K |p_i - p_j| + \frac{1}{M} \sum_{l=1}^{M} |p_{l_1}|^2 L_\Phi + |p_i|^2 \frac{2}{N} C_X
\]
Using Young’s inequality (cf. Lemma 8.2) and the triangle inequality:

\[ \leq L_K |p_i|^2 + L_K \frac{1}{2} |p_i|^2 + \frac{1}{N} \sum_{j=1}^{N} L_K \frac{1}{2} |p_j|^2 + L_\Phi |p_i|^2 + \frac{1}{2} |p_i|^2 + \frac{2}{N^2} C_X^2 \]

Multiplying the equation by 2 and summing up over all \( i \) gives:

\[ \frac{d}{dt} \sum_{i=1}^{N} |p_i|^2 \leq (4L_K + 2L_\Phi + 1) \sum_{i=1}^{N} |p_i|^2 + \frac{4}{N} C_X^2 \]

Hence, the following estimate can be established:

\[ \frac{d}{dt} (e^{-(4L_K+2L_\Phi+1)t} \sum_{i=1}^{N} |p_i|^2) = -(4L_K + 2L_\Phi + 1)e^{-(4L_K+2L_\Phi+1)t} \sum_{i=1}^{N} |p_i|^2 + e^{-(4L_K+2L_\Phi+1)t} \frac{d}{dt} \sum_{i=1}^{N} |p_i|^2 \]

\[ \leq \frac{4C_X^2}{N} e^{-(4L_K+2L_\Phi+1)t} \]

Integrating both sides and rewriting:

\[ \sum_{i=1}^{N} |p_i|^2 \leq \sum_{i=1}^{N} |p_0|^2 e^{(4L_K+2L_\Phi+1)t} + \frac{4C_X^2}{N} \left( \frac{e^{(4L_K+2L_\Phi+1)t}}{4L_K + L_\Phi + 1} - \frac{1}{4L_K + L_\Phi + 1} \right) \]

Recall that \( p_0 = 0 \). Eliminating the last term (since that one is negative) and using the fact that \( t \leq T \) for all \( t \in [0, T] \) gives:

\[ \sum_{i=1}^{N} |p_i|^2 \leq \frac{4C_X^2}{N} e^{(4L_K+2L_\Phi+1)T} \frac{e^{(4L_K+2L_\Phi+1)T}}{4L_K + L_\Phi + 1} =: C_\Phi^2 \forall t \in [0, T] \]

So for every element in the summation it holds:

\[ |p_i|^2 \leq C_\Phi^2 \forall i \in \{1, \ldots, N\} \forall t \in [0, T] \]

Hence, \( \max_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} |p_i| \leq C_\Phi \).

The a-priori estimate for \( W \), which depends on the a-priori estimate for \( P \), will be determined below:

\[ \frac{1}{2} \frac{d}{dt} |w_i|^2 = -\frac{1}{\lambda} \sum_{i=1}^{N} \left( w_i^t, \nabla \Phi (x_{T-t} - u_{T-t}^0) p_i^t \right) \]

Again, the Lipschitz continuity of \( \Phi \) is used to see that the gradient of \( \Phi \) is bounded. Combining this with the Cauchy-Schwarz inequality [8]:

\[ \leq \frac{1}{\lambda} \sum_{i=1}^{N} |w_i^t| |L_\Phi| |p_i^t| \]

\[ \leq \frac{1}{\lambda} \sum_{i=1}^{N} L_\Phi \frac{1}{2} |w_i^t|^2 + \frac{1}{\lambda} \sum_{i=1}^{N} L_\Phi \frac{1}{2} |p_i^t|^2 \] (Young’s inequality (Lemma 8.2))

Using the a-priori estimate for \( P \) and rewriting gives:

\[ \frac{d}{dt} |w_i|^2 \leq \frac{1}{\lambda} NL_\Phi |w_i|^2 + \frac{1}{\lambda} NL_\Phi C_\Phi^2 \]

Hence, the following estimate can be established:

\[ \frac{d}{dt} (e^{\frac{1}{\lambda} NL_\Phi t} |w_i|^2) = -\frac{1}{\lambda} NL_\Phi e^{\frac{1}{\lambda} NL_\Phi t} |w_i|^2 + e^{\frac{1}{\lambda} NL_\Phi t} \frac{d}{dt} |w_i|^2 \]

\[ \leq e^{\frac{1}{\lambda} NL_\Phi t} \frac{1}{\lambda} NL_\Phi C_\Phi^2 \]
Integrating both sides and rewriting:
\[ |w^t|^2 \leq |w^0|^2 e^{\frac{1}{2}NLs} + C^2_P e^{\frac{1}{2}NLs} - C^2_P \]

Recall that \( w^0 = 0 \). Eliminating the last term (since that one is negative) and using the fact that \( t \leq T \) for all \( t \in [0,T] \) gives:
\[ |w^t|^2 \leq C^2_P e^{\frac{1}{2}NLs} \]

Hence, \( \max_{t \in [1,...,T]} \sup_{t \in [0,T]} |w^t| \leq C_W \).

Lastly, the a-priori estimate for \( U \), which depends on the a-priori estimate for \( W \), will be calculated:
\[
\frac{1}{2} \frac{d}{dt} |u^t|^2 = \langle u^t, u^t_{t-1} \rangle \\
\leq \frac{1}{2} |u^t|^2 + \frac{1}{2} |u^t_{t-1}|^2
\]
(Cauchy-Schwarz inequality [8] and Young’s inequality (Lemma 8.2))

Multiplying the equation by 2 and using the a-priori estimate for \( W \) gives:
\[
\frac{d}{dt} |u^t|^2 \leq |u^t|^2 + C^2_W
\]

Hence, the following estimate can be established:
\[
\frac{d}{dt} e^{-t} |u^t|^2 = e^{-t} \frac{d}{dt} |u^t|^2 - e^{-t} |u^t|^2 \\
\leq e^{-t} C^2_W
\]

Integrating both sides and rewriting:
\[
|u^t|^2 \leq |u^0|^2 e^t + e^t C^2_W = |u^0|^2 e^t + C^2_W
\]

Again, using the fact that \( t \leq T \) for all \( t \in [0,T] \):
\[
|u^t|^2 \leq e^T (|u^0|^2 + C^2_W) =: C^2_U \quad \forall t \in [0,T]
\]

Hence, \( \max_{t \in [1,...,T]} \sup_{t \in [0,T]} |u^t| \leq C_U \).

Note that \( C_X, C_P, C_W \) and \( C_U \) don’t depend on the time \( t \). Combining these a-priori estimates gives:
\[
\sup_{t \in [0,T]} |Y^t|_{\mathbb{R}^K} = \sup_{t \in [0,T]} \left( \sum_{i=1}^{N} |x^i_t|_{\mathbb{R}^d} + \sum_{i=1}^{N} |p^i_t|_{\mathbb{R}^d} + \sum_{l=1}^{M} |w^l_t|_{\mathbb{R}^d} + \sum_{l=1}^{M} |u^l_t|_{\mathbb{R}^d} \right) \\
\leq NC_X + NC_P + MC_W + MC_U \\
< NC_X + NC_P + MC_W + MC_U + 1 =: r
\]

The information to prove Theorem 4.2 is now complete.

**Proof of Theorem 4.2.** Since all the conditions of Schäfer’s fixed point theorem are met (Lemma 4.4, Lemma 4.5 and Lemma 4.10), I can apply the theorem to conclude that \( \Gamma \) has at least one fixed point in \( B(0;r) \) where \( r = NC_X + NC_P + MC_W + MC_U + 1 \) (Theorem 4.3).

Since \( \Gamma : \mathcal{M} \to \mathcal{M} \) was defined as \( \Gamma(Y)(t) = Y_0 + \int_0^t G(Y)(s)ds \), it follows that there is at least one \( \tilde{Y} \in B(0;r) \) such that \( \Gamma(\tilde{Y})(t) = \tilde{Y}(t) \forall t \in [0,T] \).

Therefore, \( \tilde{Y}(t) = Y_0 + \int_0^t G(\tilde{Y})(s)ds \). Since \( G \) is continuous, the fundamental theorem of calculus can be applied to conclude that, when taking the derivative on both sides, \( \frac{d}{dt} \tilde{Y}(t) = G(\tilde{Y})(t) \) and that \( \tilde{Y} \) is continuously differentiable [8]. At \( t = 0 \), it holds that \( \tilde{Y}(0) = Y_0 \). This means that there exists at least one solution to the initial value problem (IVP).
5 Instantaneous control

One of the results of the previous chapter are the first-order necessary conditions for optimality, consisting of the state equation (SE), the control equation (CE) and the adjoint equation (AE). This system was transformed into a system consisting of four first order differential equations and four variables, while we’re only interested in the behaviour of $X$ and $U$. Therefore, $W$ and $P$ will be eliminated and the number of variables will thus be reduced to two, which gives rise to a system that is less computationally expensive to solve. Furthermore, the system derived in the previous chapter depends on future times, which means that this is an open-loop control system and that there is no feedback system involved, in contrast to closed-loop control systems, which are systems that do rely on feedback. In a closed-loop control system, the controls do receive feedback about the positions of the particles and can adjust their behavior accordingly. Since it is desirable that the controls are as efficient as possible in bringing the particles to their desired position, closed-loop control systems are preferred over open loop control systems. Therefore, the system will be transformed into a closed-loop control system.

In order to obtain this new system, (CF) will be discretized, its derivative will be calculated and the roots of this derivative will be computed. These roots will be used to receive the following closed-loop control system consisting of two variables:

$$\frac{d}{dt} x_i^k = \frac{1}{N} \sum_{j=1}^{N} K(x_i^k - x_i^j) + \frac{1}{M} \sum_{l=1}^{M} \Phi(x_i^k - u_{k+1}^l) \quad x_i^{k,0} = x_i^k$$  \hspace{1cm} (SE2)

$$\frac{d}{dt} u_i^k = \frac{\gamma}{MN} \sum_{j=1}^{N} \nabla \Phi(x_i^k - u_{l+1}^k)(x_i^k - \bar{x}) \quad u_i^{k,0} = u_i^k$$  \hspace{1cm} (CE2)

**Remark 5.1.** (SE2) is equal to (SE).

In this chapter, let $Y_k$ denote the position of $Y$ at time step $k = 1, ..., H$, where $H$ is the total number of time steps. Furthermore, the difference between two time steps, denoted by $\Delta t^k$, is assumed to be very small.

5.1 Temporal discretization

To derive (SE2) and (CE2), it is necessary to discretize (CF), which depends on the discretized equations for (SE) and (CE).

One potential choice of first-order temporal discretization for the state equation (SE) and the control equation (CE) results in:

$$\frac{x_i^{k+1} - x_i^k}{\Delta t^k} = \frac{1}{N} \sum_{j=1}^{N} K(x_i^k - x_i^j) + \frac{1}{M} \sum_{l=1}^{M} \Phi(x_i^k - u_{k+1}^l) \quad x_i^{k,1} = x_i^0$$

$$\frac{u_i^{k+1} - u_i^k}{\Delta t^k} = w_i^{k+1} \quad u_i^{k,1} = u_i^0$$

where $k \in \{1, ..., H - 1\}$.

The cost functional (CF) will be approximated by a discretized cost functional below:

$$J(X, U) = \frac{1}{2N} \int_0^T \sum_{i=1}^{N} |x_i^k - \bar{x}_i|^2_{R^d} dt + \frac{\lambda}{2M} \int_0^T \sum_{l=1}^{M} |u_{k+1}^l|^2_{R^d} dt$$

$$= \sum_{k=0}^{H-1} \left( \frac{1}{2N} \int_{t_k}^{t_{k+1}} \sum_{i=1}^{N} |x_i^k - \bar{x}_i|^2_{R^d} dt + \frac{\lambda}{2M} \int_{t_k}^{t_{k+1}} \sum_{l=1}^{M} |u_{k+1}^l|^2_{R^d} dt \right)$$

$$\approx \sum_{k=0}^{H-1} \Delta t^k \dot{j}^k(W_{k+1})$$

where

$$\dot{j}^k(W_{k+1}) = \frac{1}{2N} \sum_{i=1}^{N} |x_{k+1}^i - \bar{x}_i|^2_{R^d} + \frac{\lambda}{2M} \sum_{l=1}^{M} |w_{k+1}^l|^2_{R^d}$$
Note that this approximation is justified since the difference between two time steps is assumed to be very small and the derivative (the velocity) of \( U \) is equal to \( W \).

Substitution of the discretized state and control equation in \( \hat{J}^k(W_{k+1}) \) yields

\[
\hat{J}^k(W_{k+1}) = \frac{1}{2N} \sum_{i=1}^{N} |x^i_k + \frac{\Delta t}{N} \sum_{j=1}^{N} K(x^i_k - x^i_j) + \frac{\Delta t}{M} \sum_{l=1}^{M} \Phi (x^i_k - u^i_l - \Delta t k w^i_{k+1}) - \bar{x}|^2_{\mathbb{R}^d} + \frac{\lambda}{2M} \sum_{i=1}^{M} |w^i_{k+1}|^2_{\mathbb{R}^d}
\]

which is the discretized cost functional.

### 5.2 Closed-loop control system

If \( w^i_{k+1} \) is a minimizer of \( \hat{J}^k \), then \( \frac{\partial \hat{J}^k(w^i_{k+1})}{\partial w^i_{k+1}} = 0 \). Therefore, the stationary points of \( \hat{J}^k \) will be computed in this section.

**Remark 5.2.** *In principle, the stationary points of \( \hat{J}^k \) can be minimizers, maximizers or saddle points. If the second derivative of \( \hat{J}^k \) in \( w^i_{k+1} \) is strictly larger than 0, then \( w^i_{k+1} \) is a local minimizer of \( \hat{J}^k \) (second derivative test).*

The following lemma is used in the search for these stationary points:

**Lemma 5.1** (Neumann series [1]). *Let \( I \) denote the identity matrix. If, in a Banach space, \( ||S|| < 1 \), then \( I - S \) is bijective and hence \( [I - S]^{-1} \) exists.*

The cost functional \( \hat{J}^k \) computed in the previous section is differentiable and its derivative is equal to

\[
\frac{\partial \hat{J}^k(w^i_{k+1})}{\partial w^i_{k+1}} = - \frac{(\Delta t)^2}{MN} \sum_{i=1}^{N} \nabla \Phi (x^i_k - u^i_k - (\Delta t) w^i_{k+1}) (x^i_{k+1} - \bar{x}) + \frac{\lambda}{M} w^i_{k+1}
\]

Taylor’s theorem gives

\[
\nabla \Phi (x^i_k - u^i_k - (\Delta t) w^i_{k+1}) = \nabla \Phi (x^i_k - u^i_k) - \nabla^2 \Phi (x^i_k - u^i_k) (\Delta t) w^i_{k+1} + O ((\Delta t)^2) \quad \text{for } \Delta t^2 \to 0
\]

Applying the result of Taylor’s theorem to the derivative of the cost functional \( \hat{J}^k \) results in the following expression:

\[
\frac{\partial \hat{J}^k(w^i_{k+1})}{\partial w^i_{k+1}} = - \frac{(\Delta t)^2}{MN} \sum_{i=1}^{N} \nabla \Phi (x^i_k - u^i_k) (x^i_{k+1} - \bar{x}) + \frac{\Delta t}{MN} \sum_{i=1}^{N} \nabla^2 \Phi (x^i_k - u^i_k) (x^i_{k+1} - \bar{x}) w^i_{k+1}
\]

\[+ \frac{\lambda}{M} w^i_{k+1} + O ((\Delta t)^4) \quad \text{for } \Delta t^2 \to 0
\]

Let \( I \) denote the identity matrix. Setting the above expression equal to 0 gives rise to the following equation:

\[
\left( \frac{(\Delta t)^2}{MN} \sum_{i=1}^{N} \nabla^2 \Phi (x^i_k - u^i_k) (x^i_{k+1} - \bar{x}) + \frac{\lambda}{M} I \right) w^i_{k+1} = \frac{(\Delta t)^2}{MN} \sum_{i=1}^{N} \nabla \Phi (x^i_k - u^i_k) (x^i_{k+1} - \bar{x}) + O ((\Delta t)^4) \quad \text{for } \Delta t^2 \to 0
\]

Let

\[
A := \frac{(\Delta t)^2}{MN} \sum_{i=1}^{N} \nabla^2 \Phi (x^i_k - u^i_k) (x^i_{k+1} - \bar{x}) + \frac{\lambda}{M} I
\]

\[= \frac{\lambda}{M} \left( I - \frac{(\Delta t)^2}{\lambda} \sum_{i=1}^{N} \nabla^2 \Phi (x^i_k - u^i_k) (x^i_{k+1} - \bar{x}) \right) =: \frac{\lambda}{M} (I - B)
\]

where \( \frac{(\Delta t)^2}{\lambda} = \mathcal{O}(\gamma) \) for \( \Delta t \to 0 \), with \( \gamma > 0 \).

According to Lemma 5.1, if \( ||B|| < 1 \) then \( I - B \) is invertible and hence \( A \) is invertible.
It was already assumed that \( \nabla \Phi \) is Lipschitz continuous, so, by the mean value theorem [8], its gradient is bounded. Thus, there exists a \( C_{\Phi} \geq 0 \) such that \( |\nabla^2 \Phi(x^i_k - u^i_k)| \leq C_{\Phi} \). Furthermore, I know that there exists a \( C_{\Phi, K} \geq 0 \) such that \( |x^i_{k+1} - \bar{x}| \leq |x^i_k - \bar{x}| + \Delta t^k C_{\Phi, K} \). Hence, \( \Delta t^k \) can be chosen sufficiently small such that \( |B| < 1 \) is satisfied. So \( A \) is invertible and I can write

\[
w^i_{k+1} = \frac{(\Delta t^k)^2}{2N} \left[ 1 - \frac{2(\Delta t^k)^2}{\lambda N} \sum_{i=1}^N \nabla^2 \Phi(x^i_k - u^i_k)(x^i_{k+1} - \bar{x}) \right]^{-1} \sum_{i=1}^N \nabla \Phi(x^i_k - u^i_k)(x^i_{k+1} - \bar{x}) + O((\Delta t^k)^4)
\]

\[
= \frac{1}{N} \left[ \Delta t^k \sum_{i=1}^N \nabla^2 \Phi(x^i_k - u^i_k)(x^i_{k+1} - \bar{x}) + \frac{\lambda}{(\Delta t^k)^2} \sum_{i=1}^N \nabla \Phi(x^i_k - u^i_k)(x^i_{k+1} - \bar{x}) + O((\Delta t^k)^4) \right] \quad \text{ for } \Delta t^k \to 0
\]

which are the stationary points of \( J^k \).

Substituting the equation for \( w^i_{k+1} \) into the discretized control equation and letting \( \Delta t^k \to 0 \) results in the following closed-loop control system:

\[
d \frac{dx^i_t}{dt} = \frac{1}{N} \sum_{j=1}^N K(x^i_t - x^j_t) + \frac{1}{M} \sum_{l=1}^M \Phi(x^i_t - u^i_l)
\]

\[
d \frac{du^i_t}{dt} = \frac{\gamma}{MN} \sum_{i=1}^N \nabla \Phi(x^i_t - u^i_l)(x^i_t - \bar{x})
\]

\[
x^i_{t=0} = x^i_0
\]

\[
u^i_{t=0} = u^i_0
\]

I will end this chapter by proving local existence of solutions to the closed-loop control system:

**Lemma 5.2.** Let \( K, \Phi \) and \( \nabla \Phi \) be Lipschitz continuous with Lipschitz constants \( L_K, L_\Phi \) and \( L_{\nabla \Phi} \) respectively and let \( C_{\Phi} := \sup_{y \in \mathbb{R}^d} |\Phi(y)| + |D \Phi(y)| < \infty \). Then there exists a local solution to the closed-loop control system consisting of \((\text{SE2})\) and \((\text{CE2})\).

**Proof.** The closed-loop control system can be rewritten to the following initial value problem:

\[
\frac{d}{dt} Z(t) = R(Z)(t) \quad Z(0) = Z_0
\]

where \( Z = (X, U) \) and \( R : C([-T], \mathbb{R}^{d(N+M)}) \to C([0, T], \mathbb{R}^{d(N+M)}) \). Note that \( K, \Phi \) and \( \nabla \Phi \) are all Lipschitz continuous and bounded and \( \max_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} |x^i_t| \leq C_x < \infty \) (see Lemma 3.9). Hence, \( R \) is bounded. I already know that (see the proof of Lemma 4.8)

\[
|R_1(Z^1)(t) - R_1(Z^2)(t)| \leq (2L_K + L_\Phi + L_{\nabla \Phi} \frac{N}{M}) ||Z^1 - Z^2||_{\sup}
\]

\[
=: C_1 ||Z^1 - Z^2||_{\sup}
\]

because \((\text{SE2})\) is equal to \((\text{SE})\) (and hence the first component of \( R \) is equal to the first component of \( G \), defined in section 4.2).

Using the Lipschitz continuity of \( \nabla \Phi \), the boundedness of \( \nabla \Phi \) and the fact that \( \max_{i \in \{1, \ldots, N\}} \sup_{t \in [0, T]} |x^i_t| \leq C_x \), the following estimate can be obtained for the second component of \( R \):

\[
|R_2(Z^1)(t) - R_2(Z^2)(t)| \leq \frac{\gamma L_{\nabla \Phi}}{MN} (2C_X N + 2C_X + 1) ||Z^1 - Z^2||_{\sup}
\]

\[
=: C_2 ||Z^1 - Z^2||_{\sup}
\]

Since \( C_1 \) and \( C_2 \) don’t depend on the time \( t \),

\[
||RZ^1 - RZ^2||_{\sup} \leq L_R ||Z^1 - Z^2||_{\sup}
\]

where \( L_R = C_1 + C_2 \). Therefore, \( R \) is Lipschitz continuous in \( Z \). Consequently, local existence of solutions to the closed-loop control system follows from Picard-Lindelöf (cf. Lemma 8.4).

**Remark 5.3.** The closed-loop control system is well-posed.
6 Visualization

The equations (SE2) and (CE2), derived in the previous chapter, are implemented in MATLAB to visualize the situation. For the implementation, \( d = 2 \) (dimension), \( N = 100, M = 3, T = 500, \lambda = 0.01, \gamma = 10, \bar{x} = (0, 0) \in \mathbb{R}^2 \) and the number of time steps \( H = 1000 \) are used, unless stated otherwise. For \( K \) and \( \Phi \), the examples described in section 2.4 are used together with the constants that were chosen in that section, unless stated differently.

Let \( A := \frac{C}{|x-u|^n} e^{-\frac{|x-u|}{\tau^n}}, B := -\frac{C}{|x-u|^n} e^{-\frac{|x-u|}{\tau^n}}. \)

The gradient of \( \Phi \) is

\[
\nabla \Phi = \begin{bmatrix}
A \frac{1}{|x-u|^3} \left( x_1 - u_1 \right)^2 + B \frac{1}{|x-u|^2} \left( x_1 - u_1 \right)^2 - A \left( x_1 - u_1 \right) \left( x_2 - u_2 \right) \frac{1}{|x-u|^3} \\
-A \left( x_1 - u_1 \right) \left( x_2 - u_2 \right) \frac{1}{|x-u|^2} + B \frac{1}{|x-u|^2} \left( x_2 - u_2 \right)^2 - A \left( x_2 - u_2 \right) \frac{1}{|x-u|^3} \end{bmatrix}
\]

where \( x = (x_1, x_2) \in \mathbb{R}^2 \) and \( u = (u_1, u_2) \in \mathbb{R}^2 \). In the examples in this chapter, the red dots represent the positions of the controls \( U \) and the blue dots represent the positions of the particles \( X \).

Furthermore, the time-dependent cost functional is used to calculate the costs at each time step \( k \):

\[
J_k(X, U) = \frac{\Delta t}{2N} \sum_{i=1}^{N} |x_i^k - \bar{x}|^2 + \frac{\lambda \Delta t}{2M} \sum_{l=1}^{M} |\dot{u}_l^k|^2
\]

6.1 Basic example

An example of the output of the MATLAB program is given below.

![Graphs showing the basic example](image-url)
From the above figures it can be seen that the controls are able to steer the particles to a position close to \( \bar{x} \) over time. However, they don’t stop immediately when the particles arrive in a circle around \( \bar{x} \). It would make more sense if the controls would stop moving a few time steps earlier, for example at \( H = 850 \), but the controls never stop exerting forces on the particles, even though the distance between them is relatively large. This results in a cost functional that is slightly increasing in the end:

6.2 Cutting off the particle-control interaction potentials

As was already stated in the previous section, the controls keep exerting forces on the particles, even when the distance between them is relatively large and the particles are already centered around \( \bar{x} \). It would be more realistic if the particle-control interaction potentials are cut off at a certain range. For example, suppose that the controls are herding dogs that need to steer a flock of sheep (the particles) to a certain location. It would make sense that the sheep aren’t afraid of the herding dogs if the distance between them is relatively large. The same constants as before are implemented, only now, if the distance between a particle and a control is larger than 1.5, the control doesn’t exert any forces on the particle anymore.

An example of the output of the MATLAB program where the potentials are cut off, is presented below. Note that the starting positions of the particles and the controls differ for each run, so the begin positions are different from the begin positions described in the previous example.
Note that this time, in the end, the particles are centered around $\bar{x}$ instead of centered around a point close to $\bar{x}$. Apparently, the best strategy for the controls is again to stick together and approach the particles from a specific angle. The cost functional is now constantly decreasing over time on the whole time interval, as pictured below:
A disadvantage of this method is that it can be the case that, if the distance between the starting positions of the controls and the starting positions of the particles is relatively large, the controls aren’t able to steer the particles to $\bar{x}$ in the given time interval. An example can be seen below.

Two of the controls are initially too far removed from the group of particles to exert forces on them. Hence the group of particles must be steered to $\bar{x}$ by only one control, which is apparently not possible in the given time interval. The following cost functional corresponds to this run:

Figure 6: Value of the cost functional over time.
This cost functional is decreasing strongly in about the first 100 time steps. Thereafter, a slight increase can be seen, which can be explained by the fact that the positions of the particles aren’t changing much after $H = 100$, but the steering control still has to use a lot of energy.

In the rest of this chapter, the particle-control interaction potentials are cut off if the distance between a particle and a control is larger than 1.5.

### 6.3 Split initial conditions for the particles

In the examples in this section, the starting positions of the particles are divided over four different areas. The other constants remain the same for the example that is presented below:
The cost functional is constantly decreasing over time:

![Figure 7: Value of the cost functional over time.](image)

In the above example, the particles are immediately clustering together and they don’t really avoid the controls. A possible reason for this are the constants that were chosen for the interaction forces. Therefore, I will consider a more interesting example below where the particle-control interaction potentials are stronger and the urge of the particles to cluster together is less strong when the distance between them is relatively large. For the particle-particle interaction force, the attractive length scale $l_A$ is now 1 instead of 2 and for the particle-control interaction force, the repulsive strength $c_R$ is now 0.2 instead of 0.02. The other constants remain the same.
The controls are now moving around the cluster of particles instead of going right through them. The controls are focusing on bringing the cluster of particles in the right top corner to the desired position. The other clusters of particles are then also approaching \( \bar{x} \), because they are attracted by the particles that are steered to \( \bar{x} \) by the controls. The positions of the particles and the controls aren’t changing any more after time step \( H = 500 \).

![Figure 8: Value of the cost functional over time.](image)

The above figure shows that the cost functional is constantly decreasing over time, except for the first few time steps. The slight increase in the cost functional in the first few time steps can be explained by the fact that the controls have to use quite some energy to move around the group of particles in the right top corner and start steering them in the desired direction.

### 6.4 A different cost functional

In this section, I will consider a variant of the cost functional (CF), namely

\[
J^2_2(X, U) = \frac{1}{2N} \int_0^T \sum_{i=1}^N \min\{|x_i^t - \bar{x}_1|^2, |x_i^t - \bar{x}_2|^2\} dt + \frac{\lambda}{2M} \int_0^T \sum_{l=1}^M |u_l^t|^2 dt
\]

where \( J_2 : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R} \). The time-dependent cost functional corresponding to \( J_2 \) is

\[
J^k_2(X, U) = \frac{\Delta t^k}{2N} \sum_{i=1}^N \min\{|x_i^k - \bar{x}_1|^2, |x_i^k - \bar{x}_2|^2\} + \frac{\lambda \Delta t^k}{2M} \sum_{l=1}^M |u_l^k|^2
\]

Instead of bringing the whole group of particles to one location, the controls should now choose between bringing a particle to \( \bar{x}_1 \) or to \( \bar{x}_2 \), depending on whether the particle is closer to \( \bar{x}_1 \) or to \( \bar{x}_2 \). The following system is implemented in MATLAB:

\[
\begin{align*}
\frac{dx_i^t}{dt} &= \frac{1}{N} \sum_{j=1}^N K(x_i^t - x_j^t) + \frac{1}{M} \sum_{l=1}^M \Phi(x_i^t - u_l^t) \quad \text{for} \quad t = 0, \ldots, k-1 \\
\frac{du_l^t}{dt} &= \frac{\gamma}{MN} \sum_{i=1}^N \nabla \Phi(x_i^t - u_l^t) \min\{|x_i^t - \bar{x}_1, x_i^t - \bar{x}_2\} \quad \text{for} \quad t = 0, \ldots, k-1
\end{align*}
\]

\( x_i^{t=0} = x_i^0 \)

\( u_l^{t=0} = u_l^0 \)

In this section, \( \bar{x}_1 = (1, 1) \in \mathbb{R}^2 \) and \( \bar{x}_2 = (-1, -1) \in \mathbb{R}^2 \). The constants used for generating the next example are the same as in the previous example (except for the initial conditions).
As can be seen in the pictures above, the controls can indeed successfully bring the particles to their desired positions. However, the cost functional is increasing in the beginning:

This can possibly be explained by the fact that the controls have to use a relatively large amount of energy in the beginning to push the particles in the right direction. Once the particles are further away from the controls (around $H = 50$ and further) and hence closer to their desired positions, the cost functional is decreasing over time.
However, if I again reduce the particle-control interaction potential, i.e. letting $c_R = 0.02$ instead of $c_R = 0.2$, it can happen that the two groups of particles are clustering together somewhere between $\bar{x}_1$ and $\bar{x}_2$.

Since the two groups of particles are clustering together, and hence moving away from their desired positions, from about $H = 700$ and further, the cost functional is increasing from about $H = 700$ and further, which is shown in the figure below.

Figure 10: Value of the cost functional over time.
In the last example that will be discussed in this report, the constants remain the same as in the previous example (except for the initial conditions). All the starting positions of the particles are now centered around the origin. So the particles are not longer divided in two groups initially. There are now two possibilities for the controls that would seem logical: they could either try to split the group of particles and bring around half of them to $\bar{x}_1$ and the other ones to $\bar{x}_2$ or they could bring the whole group of particles to either $\bar{x}_1$ or $\bar{x}_2$.

As can be seen above, the controls chose the second strategy. The particles immediately cluster together and hence it is hard for the controls to split them into two separate groups. Apparently this is a good strategy, since the cost functional is constantly decreasing over time:

![Graph showing the value of the cost functional over time.](image)

**Figure 11:** Value of the cost functional over time.
7 Conclusion

In this paper, an interacting particle system was proposed of which the behaviour has been investigated. The aim of this paper was to solve the following optimization problem

$$\min_{(X,U) \in \{X,U\}_{ad}} J(X,U) \text{ subject to the constraint } E(X,U) = 0 \text{ in } Y$$

(\text{OP})

First, it has been proven that this optimization problem indeed has a solution. Thereafter, the first-order necessary conditions for optimality were derived with the help of the method of Lagrange multipliers. These conditions have been rewritten to an initial value problem. Existence of solutions to this initial value problem has been proven. Furthermore, the first-order necessary conditions for optimality have been transformed into a closed-loop control system, of which local existence of solutions has been proven, that describes the behaviour of the particles and the controls. Examples of potentials were given and these were used for the implementation of the closed-loop control system in MATLAB. The output that followed from this implementation made it clear that the controls indeed can succeed in bringing the particles to a certain position while minimizing the time-dependent cost functional if the particle-control interaction potential is set equal to 0 if the distance between a control and a particle is too large. This was tested for different situations. Also, a variant of the cost functional has been investigated, where the controls had to choose between bringing a particle to $\bar{x}_1$ or to $\bar{x}_2$, depending on whether the particle is more close to $\bar{x}_1$ or to $\bar{x}_2$. Again, the controls could succeed in bringing the particles to the desired positions for certain choices of constants while minimizing this variant of the time-dependent cost functional. The results of this paper can be used to investigate and model several types of behaviours in the real world (e.g. crowd dynamics or collective animal behaviour).
8 Discussion

As was already stated before in section 6.1, the controls don’t always stop exerting forces the moment that the particles arrive in a position centered around \( \bar{x} \), which could result in a time-dependent cost functional that is slightly increasing in the last time steps. Cutting off the particle-control interaction potentials when the distance between a particle and a control is relatively large, solved this issue. However, if the particle-control interaction potential is set equal to 0 if the distance between a control and a particle is too large, the controls don’t always succeed in bringing the particles to the desired destination in the given time interval. Also, the time-dependent cost functional is increasing in the first few time steps in certain situations. For future research, it could be interesting to investigate if this behaviour is also present for other choices of constants, potentials, initial conditions or cost functionals. In particular, the influence of different choices of potentials could be a fascinating subject for future research, especially since the potentials chosen in section 2.4 are highly sensitive to the choice of constants, as for example the runs described in section 6.4 point out. In addition, more variations of the cost functional could be investigated. On top of this, it might be interesting to explore the situation in which the number of particles approaches infinity.
Appendices

Appendix A: Prior knowledge

This appendix starts with defining the spaces that are used for the particles and the controls. In addition, some remarks about those spaces are made. Lastly, the other concepts that are used in this report are defined.

Appendix A.I: Spaces

**Definition 8.1** (Normed space [9]). A normed space $(Z, \|\cdot\|_Z)$ is a vector space with a metric defined by a norm.

**Definition 8.2** (Inner product space [9]). An inner product space is a normed vector space with an inner product defined on it.

**Definition 8.3** (Complete space [9]). A space $Z$ is complete if every Cauchy sequence in $Z$ converges.

**Definition 8.4** (Lebesgue space [7]). Let $\Omega$ be an interval. Given any integer $1 \leq p \leq \infty$, let the Lebesgue space $L^p(\Omega)$ be the set of all measurable functions $f : \Omega \rightarrow [-\infty, \infty]$ that satisfy

$$\int_{\Omega} |f(x)|^p dx < \infty \text{ if } 1 \leq p < \infty$$

$$\inf\{C \geq 0; |f| \leq C \text{ almost everywhere in } \Omega\} < \infty \text{ if } p = \infty$$

$L^p(\Omega)$ is equipped with the norm

$$\|f\|_{L^p} = \left(\int_{\Omega} |f(x)|^p dx\right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty$$

$$\|f\|_{L^\infty} = \inf\{C \geq 0; |f| \leq C \text{ almost everywhere in } \Omega\} \text{ if } p = \infty$$

**Definition 8.5** (Sobolev space [7]). Let $\Omega$ be an interval and let $m \geq 1$ be an integer. A function $v \in L^2(\Omega)$ is in the Sobolev space $H^m(\Omega)$ if for each multi-index $\alpha$, where $1 \leq |\alpha| \leq m$, there exists a (weak) partial derivative $\partial^\alpha v \in L^2(\Omega)$ such that

$$\int_{\Omega} (\partial^\alpha v)\phi dx = (-1)^{|\alpha|} \int_{\Omega} v(\partial^\alpha \phi) dx \forall \phi \in C^\infty_c(\Omega)$$

$H^m(\Omega)$ is equipped with the norm

$$\|v\|_{H^m} = \left(\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha v\|_{L^2}^2\right)^{\frac{1}{2}}$$

**Definition 8.6** (Banach space [9]). A normed space is a Banach space if it is complete.

**Definition 8.7** (Hilbert space [9]). An inner product space is a Hilbert space if it is complete.

Remark 8.1. 1. All Hilbert spaces are Banach spaces [9].

2. The Lebesgue space $L^2$ is a Hilbert space [9].

3. All Sobolev spaces $H^m$ where $m \geq 1$ are Hilbert spaces [7].

Appendix A.II: Preliminaries chapter 3

**Definition 8.8** (Reflexivity [12]). Let $Z$ be a Banach space and let $Z^{**}$ be the bidual space of $Z$. $Z$ is reflexive if $Z \rightarrow Z^{**}$ is surjective.

**Definition 8.9** (Convex set [9]). Let $\sigma \in [0, 1]$. A subset $S$ of a vector space is convex if for any $s_1, s_2 \in S$

$$\sigma s_1 + (1 - \sigma)s_2 \in S$$

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Definition 8.10 (Convex functional [12]). Let $Z$ be a Banach space and let $\sigma \in [0, 1]$. A functional $f : Z \to \mathbb{R}$ is convex if for any $z_1, z_2 \in Z$

$$f(\sigma z_1 + (1 - \sigma)z_2) \leq \sigma f(z_1) + (1 - \sigma)f(z_2)$$

Definition 8.11 (Weak convergence [12]). Let $Z$ be a real Banach space. A sequence $\{z_k\} \subset Z$ is weakly convergent and we write $z_k \rightharpoonup z$, if there exists a $z \in Z$ such that for $k \to \infty \forall \phi \in Z^*$

$$\langle \phi, z_k \rangle_{Z^*, Z} \to \langle \phi, z \rangle_{Z^*, Z}$$

where $Z^*$ indicates the dual space of $Z$.

Remark 8.2 ([12]). Weak convergence $z_k \rightharpoonup z$ in a Hilbert space $Z$ is equivalent to $\forall f \in Z$ 

$$\langle f, z_k \rangle_Z \to \langle f, z \rangle_Z$$

Definition 8.12 (Weak continuity [12]). Let $Z$ and $\hat{Z}$ be Banach spaces. A mapping $H : Z \to \hat{Z}$ is weakly continuous if 

$$z_k \rightharpoonup z \text{ in } Z \implies H(z_k) \rightharpoonup H(z) \text{ in } \hat{Z}$$

Definition 8.13 ((Weak) lower semicontinuity [12]). A functional $f : Z \to \mathbb{R}$ on a Banach space is sequentially (weakly) lower semicontinuous if for $z \in Z$

$$(z_k \rightharpoonup z) \implies f(z) \leq \liminf_{k \to \infty} f(z_k)$$

Definition 8.14 (Coercivity [12]). A functional $f : Z \to \mathbb{R}$ on a Banach space is coercive if for $z \in Z$

$$f(z) \to \infty \text{ for } ||z|| \to \infty$$

Appendix A.III: Preliminaries chapter 4

Definition 8.15 (Gâteaux derivative [7]). Let $Z$ and $\hat{Z}$ be Banach spaces. The Gâteaux derivative of a function $f : Z \to \hat{Z}$ at $z \in Z$ in the direction $h_z \in Z$ is 

$$D_zf(z)[h_z] = \lim_{\tau \to 0} \frac{f(z + \tau h_z) - f(z)}{\tau} \in \hat{Z}$$

Definition 8.16 (Totally bounded [3]). A set $S$ is totally bounded if $\forall \epsilon > 0 \exists s_1, ..., s_n \in S$ such that $S \subset \bigcup_{i=1}^{n} B(s_i; \epsilon)$.

Definition 8.17 (Compact mapping [3]). Let $Z$ and $\hat{Z}$ be Banach spaces. A mapping $H : Z \to \hat{Z}$ is compact if for all bounded sets $A \subset Z$, it holds that $H(A) \subset \hat{Z}$ is totally bounded.

Definition 8.18 (Lipschitz continuity [1]). Let $Z$ and $\hat{Z}$ be Banach spaces. A mapping $H : Z \to \hat{Z}$ is Lipschitz continuous if $\forall z_1, z_2 \in Z$

$$||Hz_1 - Hz_2||_{\hat{Z}} \leq L_H ||z_1 - z_2||_Z$$

where $L_H > 0$ is called the Lipschitz constant.

Definition 8.19 (Local Lipschitz continuity [1]). Let $Z$ and $\hat{Z}$ be Banach spaces. A mapping $H : Z \to \hat{Z}$ is locally Lipschitz continuous if $\forall z \in Z$ there exists a $r > 0$ such that the restriction of $H$ to $B(z; r)$ is Lipschitz continuous.
Appendix B: Inequalities

In this appendix, some basic inequalities that are used throughout the report are stated.

**Lemma 8.1** (Hölder’s inequality for integrals [7]). Let $\Omega$ be a subset of $\mathbb{R}^n$ and let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $f, g : \Omega \rightarrow [\infty, \infty]$ are two measurable functions that satisfy $\int_{\Omega} |f(x)|^p dx < \infty$ and $\int_{\Omega} |g(x)|^q dx < \infty$. Then

$$\int_{\Omega} |f(x)g(x)| dx \leq \left( \int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |g(x)|^q dx \right)^{\frac{1}{q}}$$

**Lemma 8.2** (Young’s inequality [2]). Let $a, b \geq 0$ and let $p, q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

**Lemma 8.3** (Jensen’s inequality [8]). If $f$ is a real and convex function on a domain $\Omega$ and $x_1, ..., x_n$ are in $\Omega$, then

$$f \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \leq \frac{1}{n} \sum_{i=1}^{n} f(x_i)$$
Appendix C: Ordinary differential equations

Here, two important theories about ordinary differential equations will be stated.

**Lemma 8.4** (Picard-Lindelöf [10]). Let \( k \geq 1, t_0 \in \mathbb{R}, x_0 \in \mathcal{Z} \) and \( \mathcal{Z} \) and \( \hat{\mathcal{Z}} \) Banach spaces. Let \( f : \mathbb{R} \times \mathcal{Z} \to \hat{\mathcal{Z}} \) be continuous and bounded on some region

\[
Q = \{ (t, x) : |t - t_0| \leq a, ||x - x_0||_{\mathcal{Z}} \leq b \} \quad (a, b > 0)
\]

Consider the initial value problem

\[
x'(t) = f(t, x(t)) \quad x(t_0) = x_0
\]

If \( f \) is Lipschitz continuous with respect to \( x \) on \( Q \), then there exists a \( \delta > 0 \) such that the initial value problem has a unique solution on \([t_0 - \delta, t_0 + \delta]\).

**Lemma 8.5** (Gronwall’s inequality [13]). Let \( \alpha, \beta \) and \( y \) be real and continuous functions on an interval \([a, b]\) with \( a < b \). If

\[
y(t) \leq \alpha(t) + \int_a^t \beta(s)y(s)ds \quad \text{for} \quad t \in [a, b]
\]

and \( \beta \) is nonnegative and \( \alpha \) is nondecreasing on \([a, b]\), then

\[
y(t) \leq \alpha(t) \exp \left( \int_a^t \beta(s)ds \right) \quad \text{for} \quad t \in [a, b]
\]
Appendix D: Measure theory

In this last appendix, an important statement about measure theory will be written down.

**Lemma 8.6** (Lebesgue’s dominated convergence theorem [2]). Let \( f_j, f : S \rightarrow Y \) be \( \mu \)-measurable and let \( g \in L^1(\mu; \mathbb{R}) \). If

\[
|f_j| \leq g \text{ \( \mu \)-almost everywhere for all } j \in \mathbb{N} \text{ and } \\
f_j \rightarrow f \text{ \( \mu \)-almost everywhere as } j \rightarrow \infty
\]

then \( f_j, f \in L^1(\mu; Y) \) and

\[
f_j \rightarrow f \text{ in } L^1(\mu; Y) \text{ as } j \rightarrow \infty
\]
References