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Nonlinear Dynamic Modeling and Analysis of Borehole Propagation for Directional Drilling

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Abstract

Boreholes with complex trajectories are drilled with the help of downhole rotary steerable systems. These robotic actuators, which are embedded in the drillstring, are used to steer the bit in the desired direction. This paper presents a dynamic non-smooth borehole propagation model for planar directional drilling. Essential nonlinearities, induced by the saturation of the bit tilt and by non-ideal (undergauged) stabilizers, are modeled using complementarity conditions, leading to a closed-form analytical description of the model in terms of a so-called delay complementarity system. The analytical form of the model allows for a comprehensive dynamic and parametric analysis. Firstly, (quasi-)stationary solutions generated by constant actuator forces are analyzed parametrically as a function of the actuation force. Secondly, an analysis of the local stability of these solutions shows the coexistence of multiple (stable and unstable) solutions and their dependency on key system parameters, such as the weight-on-bit and bit characteristics. Thirdly, a numerical simulation study shows the existence of steady-state oscillations, which are a consequence of the non-smooth characteristics of the bit tilt saturation and the stabilizers. Such limit cycles represent borehole rippling, which is the planar equivalent of the highly detrimental borehole spiraling observed in practice. The constructed model and the pursued analysis provide essential insights in the effects causing undesired borehole rippling. Herewith, the presented results can be used to support improved directional drilling system design and to form the basis for further work on automation techniques for the downhole robotic actuator to mitigate spiraled boreholes.

Keywords: Directional Drilling, Borehole Spiraling, Linear Complementarity Problems, Delay Complementarity Systems

2010 MSC: 37N15, 37N99, 34K07, 34K11

1. Introduction

Directional drilling is a technique to drill complex curved boreholes in the Earth's crust. In the context of resource 20 exploitation, applications of directional drilling include drilling multiple wells from the same rig to reduce costs and limit environmental impacts; drilling horizontal wells, which is crucial in the exploitation of shale oil and gas; and drilling relief wells to rescue out-of-control wells, as was applied to end the oil spillage in the New Horizon Oil 10 Spill in The Gulf of Mexico in 2010 [1].

Figure 1 shows a sketch of a directional drilling system. The drillstring is a hollow slender pipe that can be a few kilometers in length and is supported at the rig where the 30 rotary speed and the axial force (hook-load) are imposed. Most of the drillstring is in tension under its own weight, except for the bottom-hole assembly (BHA), which is in

compression to induce a sufficient weight on the bit. The BHA is usually about one hundred meters long and consists of drill collars, three to five stabilizers to center the BHA in the borehole, a bit to drill the rock formation, various logging tools and a rotary steerable system (RSS). The RSS is a downhole robotic actuator that steers the bit in the desired direction [3].

The work presented in this paper considers the family of tools called push-the-bit RSS. Such a RSS is located between the bit and the first stabilizer and uses a set of extendible pads to induce a lateral force on the side of the borehole and, thereby, on the BHA, which consequently steers the bit. For directional drilling applications, PDC (polycrystalline diamond compact) bits with diameters ranging from 6 inch (15.2 cm) to 17 inch (43.2 cm) are usually favored.

In practice, drilling with such directional drilling technology often results in self-excited instability-induced steady-state borehole oscillations, called borehole spiraling [4, 5, 6, 7]. An illustration of borehole spiraling is included in Figure 2, together with its two-dimensional equivalent called borehole rippling. The oscillation wavelength is re-

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Figure 1: Overview of a rotary directional drilling system [2].

Figure 2: Illustration of borehole spiraling and rippling.

lated to the distance between the bit and the first contact point of the BHA with the borehole walls, while the oscillation amplitude is of order $\mathcal{O}(10^{-3} \sim 10^{-2})$ m [8, 9, 10].

Borehole spiraling has negative effects on the drilling process and the borehole quality, since it (i) makes it harder to insert a casing after the borehole is drilled; (ii) decreases the drilling efficiency as evidenced by a lower rate-of-penetration and higher drilling costs due to increased drag forces; and (iii) reduces the accuracy of reaching a desired target position. By developing dynamic mathematical models for directional borehole propagation, we aim to gain insights in the borehole spiraling phenomenon. These insights can then be used to select the drill bit and design the BHA to mitigate borehole spiraling. Furthermore, these models can serve as a basis for the development of model-based controllers for the RSS, which have the objective to accurately track an intended (predefined) well path while mitigating borehole spiraling [11, 12, 13, 14, 15].

Figure 3: Nonlinearities in directional borehole propagation.

Many models [16, 17, 18, 19, 20, 21, 22, 23, 24, 8, 25, 26], mostly numerical, have been developed since the 1950's. An advanced mathematical model is developed in [27]. This model determines conditions leading to unstable oscillatory behavior, however, it is insufficient to predict steady-state borehole rippling and spiraling, since it does not account for essential nonlinearities. In [2, 28], an extension of this model is given which incorporates a saturation of the bit tilt as observed experimentally [29, 30]. The bit tilt is defined as the orientation difference between the bit and the borehole at the bit and is depicted in Figure 3 for the two-dimensional case. Such a saturation of the bit tilt occurs when the bit-gauge contacts a borehole wall. This nonlinearity prevents oscillations to grow unbounded and, consequently, this model captures borehole rippling and spiraling [10].

Another nonlinearity is induced by non-ideal stabilizers, which are defined as stabilizers with a smaller diameter than the borehole, see Figure 3. The clearance between the stabilizers and the borehole wall(s) is due to under-gauged stabilizers (by design), a hole enlargement caused by erosion of the mud and whirling of the bit/BHA. This nonlinearity is not taken into account in [2, 28].

The main contributions of this paper are (i) the development of a planar borehole propagation model that takes into account the bit tilt saturation and any number of non-ideal stabilizers; and (ii) a comprehensive analysis of the resulting model that shows the central role of the nonlinearities on the response of the model generated by constant RSS actuator forces. The derived model is an extension of the model proposed by [2]. The nonlinearities are modeled within a unified mathematical framework based on linear complementarity problems, which yields a closed-form model description. The resulting model is analyzed to investigate the influence of the imposed nonlinearities on the (steady-state) response generated by constant RSS actuator forces, an actuation technique used in practice. A numerical study shows the essential role of the nonlinearities in steady-state borehole oscillations representing so-called borehole rippling.

The paper is organized as follows. In Section 2, we define the model by introducing assumptions, on which we build the model, and the parameters and variables that define the model. Section 3 presents the different model components and their interaction with each other. The modeling of the non-ideal stabilizers and the bit tilt saturation is treated in Section 4. The total borehole propagation model is presented in Section 5. The length scales

Figure 4: Schematic overview of a moving n -stabilizer BHA.

associated to the system response to a step in RSS force are introduced in Section 6. The long range response generated by constant RSS actuation is described in Section 7 and its stability is assessed with methods presented in Section 8. Numerical simulations are presented in Section 9. The conclusions follow in Section 10.

2. Model Definition

The borehole trajectory is a result of the interaction between the BHA and the rock formation that is drilled by the bit. A set of hypotheses and assumptions are adopted upon which the mathematical model is built. These are mainly taken from [27] and commonly used [27, 2, 28, 31, 32, 33, 34]. Furthermore, we introduce in this section the parameters and variables that define the model.

The directional drilling process is averaged over several revolutions of the bit. As a consequence, the dynamic processes, generally occurring at a time scale of the order of the bit revolution time scale or less, are disregarded. Furthermore, we view the drilling process as being rate-independent for the typical range of bit angular velocity of order $\mathcal{O}(10 \sim 100 \text{ rev/min})$ used in practice. The time-averaging together with the assumed rate-independent nature of the cutting process imply that the independent variable in this system is the borehole length L rather than the time spent drilling. The variable L measures the increasing distance from the drill rig to the drill bit. Moreover, we only study the BHA and lump the rest of the drillstring into assumed known forces at the upper boundary of the BHA. The contribution of the upper part of the drillstring influences the axial force and axial torque at the bit, but does not significantly influence the lateral force and the moment experienced at the bit, which are responsible for the directional tendency. Finally, we formulate the model for homogeneous and isotropic rock formations.

The BHA is represented as a beam with uniformly distributed weight w and flexural stiffness EI . No contacts between the BHA and the borehole walls are assumed except for the bit, the pads of the RSS actuator and the stabilizers. A schematic overview of an n -stabilizer BHA is given in Figure 4. The stabilizers are numbered from 1 to n , starting with the stabilizer closest to the bit. The last stabilizer, i.e., stabilizer n , is chosen as an ideal stabilizer, which means that this stabilizer is always centered inside the borehole. The distance between the bit and the first stabilizer is defined as $\ell_{(1)}$, and the distance between stabilizer $i - 1$ and stabilizer i is defined as $\ell_{(i)}$, for $i = 2, 3, \dots, n$. The spatial coordinate s represents the arc length along the BHA, starting from the bit. The distance between the i -th stabilizer and the bit is defined as

$$s_i := \sum_{k=1}^i \ell_{(k)}, \text{ for } i = 1, 2, \dots, n.$$

The BHA is split into $n + 1$ segments starting from the bit. Segment $S_{(i)}$, for $i = 0, 1, \dots, n$, is defined as

$$S_{(i)} = \begin{cases} \{s \in [0, \Delta \ell_{(1)}]\} & \text{for } i = 0, \\ \{s \in [\Delta \ell_{(1)}, s_i]\} & \text{for } i = 1, \\ \{s \in [s_{i-1}, s_i]\} & \text{for } i = 2, 3, \dots, n. \end{cases}$$

The segments start and end at the RSS and at the stabilizers, which generally corresponds to shear force discontinuities in the BHA. A variable associated to a segment, e.g., the length $\ell_{(i)}$, has brackets around its index, in this case (i) . An index without brackets, e.g., in the variable s_i , refers, in general, to the bit for $i = 0$ and to stabilizer i for $i = 1, 2, \dots, n$.

Figure 5 depicts inclinations defined next. Inclinations are measured counterclockwise with respect to the downward vertical direction. Define $\theta_0(L)$ as the bit inclination,

evolving with the borehole length L . The borehole inclination at the bit is denoted as $\Theta_0(L)$. Furthermore, the spatially delayed borehole inclination $\Theta_i(L)$, corresponding to the borehole inclination at stabilizer i , is defined as

$$\Theta_i(L) := \Theta_0(L - s_i), \text{ for } i = 1, 2, \dots, n.$$

Chord $C_{(1)}(L)$, see Figure 4, is defined as the straight line segment between the bit and the center of the borehole at the first stabilizer and is a function of the increasing borehole length L . Chords $C_{(i)}(L)$, for $i = 2, 3, \dots, n$, are defined as straight line segments between the borehole center at locations of stabilizer $i - 1$ and i . The inclination of chord i is defined as

$$\langle \Theta \rangle_{(i)}(L) := \frac{1}{\ell_{(i)}} \int_{s_{i-1}}^{s_i} \Theta_0(L - s) ds. \quad (1)$$

This represents the average inclination of the borehole over the straight line segment $C_{(i)}(L)$.

For a variety of reasons that include bit/BHA whirling and the use of undergaged stabilizers, the borehole diameter can be larger than the diameter of the stabilizers and can vary over the borehole length and thus be a function of the borehole length L and the spatial coordinate s running over the BHA. Parameter $y_i^*(L)$, for $i = 1, 2, \dots, n - 1$, represents the clearance between stabilizer i and each borehole wall, when stabilizer i is centered inside the borehole. To limit complexity, we take y_i^* constant throughout the length of the borehole for all $i \in \{1, 2, \dots, n - 1\}$ and therefore drop the argument of y_i^* . Additional ideal stabilizers can be modeled by taking $y_j^* = 0$ for those $j \in \{1, 2, \dots, n - 1\}$.

Stabilizers are modeled as point supports of the BHA and, consequently, do not transmit any moment between the borehole walls and the BHA. The last stabilizer, assumed to be ideal, experiences the contact force F_n . Each other stabilizer experiences an upper wall contact force F_{u_i} and a lower wall contact force F_{ℓ_i} , for $i = 1, 2, \dots, n - 1$, when it contacts the upper or lower wall, respectively, see Figure 4. The total contact force experienced by stabilizer i is defined as

$$F_i := F_{\ell_i} - F_{u_i}, \text{ for } i = 1, 2, \dots, n - 1.$$

The bit experiences a lateral force F_0 , an axial force N_0 and a moment M_0 . An assumed known axial force N_n acts on the last stabilizer [27]. The RSS actuator, located at a distance $\Delta \ell_{(1)}$ from the bit with $\Delta \in (0, 1)$, applies a force F_{RSS} perpendicular to the borehole axis.

For modeling the BHA deformation inside the borehole, the Euler-Bernoulli beam theory [35] is adopted. In this respect, it is assumed that all lateral forces acting on the BHA are parallel. This assumption is motivated by the radius of curvature of the borehole being large with respect to the considered length of the BHA. This also allows us to take the direction of gravity (relative to the BHA axis) constant over the entire BHA length.

Figure 5: Interpretation of the bit, borehole and delayed borehole inclinations.

Figure 6: Components of the borehole propagation model and their interaction [2].

3. Model Components

The directional drilling model consists of three components [2]. Firstly, the bit kinematics describe the movement of the bit through the rock formation. Secondly, the bit-rock interface laws relate the forces and the moment acting on the bit to its penetration in the rock formation. Thirdly, the BHA model expresses the relation between forces and moments at the bit and other (external) loads on the BHA to the BHA deformation. The coupling between the model components leads to a model that describes the borehole propagation and the evolution of BHA deformation. Figure 6 depicts an overview of the model components and their interaction. In the following sections, the individual model components are treated in detail and combined to arrive at the borehole propagation model. Finally, scaled variables are introduced to write the model in a dimensionless form.

3.1. Bit Kinematics

The bit kinematics describe the movement of the bit through the rock formation. Two orthonormal reference frames are introduced. Frame i consists of vector $\mathbf{i}_1$, in the direction of the bit axis, and vector $\mathbf{i}_2$ orthonormal to $\mathbf{i}_1$ in a counterclockwise direction. Frame $\mathbf{I}$ consists of vector $\mathbf{I}_1$, in the longitudinal direction of the borehole at the bit, and vector $\mathbf{I}_2$ orthonormal to $\mathbf{I}_1$ in a counterclockwise direction. Figure 7 provides a graphical interpretation of the reference frames.

Notwithstanding its angular velocity around $\mathbf{i}_1$, the motion of the bit can be described by velocity $\mathbf{v}$ and spin

Figure 7: Orthonormal reference frames $\mathbf{i}$ and $\mathbf{I}$ defined at the bit.

velocity ω . Velocity $\mathbf{v}$ can be decomposed into $\mathbf{v}_1$, in the direction of $\mathbf{i}_1$, i.e., the direction of the bit axis, and $\mathbf{v}_2$, in the direction of $\mathbf{i}_2$, i.e., lateral to the bit axis. Spin velocity ω acts around the axis perpendicular to the plane of propagation.

Motion of the bit results in penetration into the rock. The penetration variables per revolution of the bit are defined as

$$\mathbf{d} := \frac{2\pi\mathbf{v}}{\Omega} \text{ and } \varphi := \frac{2\pi\omega}{\Omega}, \quad (2)$$

where Ω is the angular velocity of the bit around $\mathbf{i}_1$. The penetration per revolution of the bit is described by penetration vector $\mathbf{d}$ and angular penetration φ in (2). Angular penetration φ acts around the axis perpendicular to the plane of propagation.

Penetration vector $\mathbf{d}$ can be decomposed as $\mathbf{d} = d_1\mathbf{i}_1 + d_2\mathbf{i}_2$. By definition, the direction in which penetration vector $\mathbf{d}$ points is the direction in which the borehole propagates. Therefore, vector $\mathbf{d}$ is inclined by the borehole inclination Θ_0 on the vertical axis. Vector $d_1\mathbf{i}_1$ is defined as the component of $\mathbf{d}$ in the direction in which the bit is pointing. Therefore, the angle between $d_1\mathbf{i}_1$ and $\mathbf{d}$ can be expressed as $\Theta_0 - \theta_0$. The bit tilt is defined as

$$\psi := \theta_0 - \Theta_0, \quad (3)$$

which can be expressed in terms of the penetration variables as

$$\psi = -\arctan\left(\frac{d_2}{d_1}\right) \approx -\frac{d_2}{d_1} \quad (4)$$

for $d_1 \gg d_2$, in general true since drilling bits are designed to drill axially. In terms of penetration variables, the change of inclination of the bit with respect to the increasing borehole length can be written as

$$\frac{d\theta_0}{dL} \approx \frac{\varphi}{d_1}. \quad (5)$$

Equations (4) and (5) relate the bit motion to the penetration variables d_1, d_2 and φ . These two equations constitute the bit kinematics component of the model.

3.2. Bit-Rock Interaction

The forces and moment acting on the bit are related to the penetration variables by the bit-rock interface laws.

According to [36], the bit-rock interaction can be expressed as:

$$\begin{bmatrix} N_0 \\ F_0 \\ M_0 \end{bmatrix} = - \begin{bmatrix} G \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_3 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \varphi \end{bmatrix}, \quad (6)$$

where $G > 0$ is a measure of bit bluntness, N_0 and F_0 are the axial and lateral contact forces experienced by the bit, respectively, and M_0 is the contact moment acting on the bit. Coefficients $H_i > 0$, for $i = 1, 2, 3$, relate the forces and moment acting on the bit to the penetration variables measuring the amount of rock removed by the bit in one revolution. Generally, $H_1 \ll H_2$, as the bit is designed to drill in the axial direction. With these definitions, the active weight on the bit can be defined as $W_a := H_1 d_1 = -(N_0 + G)$. The active weight on bit is the part of the weight on the bit that is directly associated with the bit advancement into the rock.

Combining the kinematics (4) and (5) with the bit-rock interaction laws (6) leads to the following interface laws

$$F_0 = W_a \eta (\theta_0 - \Theta_0), \quad (7a)$$

$$M_0 = -W_a \epsilon \eta \ell_{(1)}^2 \frac{d\theta_0}{dL}, \quad (7b)$$

where

$$\eta := \frac{H_2}{H_1} \text{ and } \epsilon := \frac{H_3}{H_2 \ell_{(1)}^2}.$$

Parameters η and ϵ are dimensionless. The parameter η represents the lateral steering resistance of the bit and takes a value between 5 and 100 [36]. The combined parameter $\chi := \epsilon \eta$ represents the angular steering resistance of the bit and is generally one or two orders of magnitude smaller than η . The parameter χ taking a zero value implies that the bit is free to change its inclination inside the borehole. The small parameter ϵ represents the ratio of the angular steering resistance χ and the lateral steering resistance η of the bit. The interface law in (7a) states that the orientation difference between the bit and borehole, i.e., bit tilt, is proportional to the lateral force experienced by the bit. The interface law in (7b) states that the rate of orientation change of the bit is proportional to the moment experienced by the bit. These laws can be used for any n -stabilizer BHA.

3.3. BHA Model

The BHA deformation should conform to the already drilled borehole geometry. In particular, we are interested in the forces and moment that are experienced by the bit when fitting the BHA inside the borehole.

The BHA can be viewed as an elastic beam fixed at the bit, supported by all the stabilizers and subject to gravity and the RSS actuation force. We neglect the dimensions of the bit and, therefore, take the forces and moment acting on the bit equal to those acting on the bit-side end of the beam. Due to the averaging hypothesis introduced earlier,

a quasi-static model of the BHA is appropriate. However, this model is statically indeterminate, and, thus, the three static equilibrium equations are insufficient to determine the moment and forces at the bit. The axial force at the bit N_0 can directly be found from the static axial equilibrium equation, resulting in

$$N_0 = N_n + w\ell \cos \langle \Theta \rangle_{(1)}. \quad (8)$$

The axial force N_0 incorporates the imposed hook-load at the rig via N_n and thus can be seen as a control parameter. Recall the definition of the active weight on the bit $W_a := -(N_0 + G)$ and note that the axial force N_0 is embedded in this parameter. Therefore, the axial force N_0 does not show up explicitly in further equations, but is implicitly present in W_a . The cosine term in (8) represents the effect of gravity that acts on the BHA when the BHA is aligned with its reference configuration along the chord $C_{(1)}$ with inclination $\langle \Theta \rangle_{(1)}$. To obtain expressions for the lateral force F_0 and moment M_0 at the bit, the BHA deformation inside the borehole should be considered.

The variables $\phi(s, L)$ and $y(s, L)$ are defined as the relative inclination and deflection of the BHA with respect to the undeformed BHA, respectively, and are functions of the spatial coordinate s and the borehole length L . The internal moment of the beam is defined as $M(s, L)$, also a function of s and L . For notational convenience, we drop the variable L in the argument of these variables. The inclination of a beam subjected to lateral loads and moments is given by the Euler-Bernoulli beam theory, which consists of the three static equilibrium equations and

$$\frac{d\phi(s)}{ds} = \frac{M(s)}{EI}, \quad (9)$$

where EI is the flexural stiffness of the BHA. The differential equation in (9) is subjected to a set of (mathematical) constraints. First of all, the BHA deflection $y(s)$ is a function in the twice differentiable class C^2 . This means that both the deflection and the relative inclination $\phi(s)$, as well as the internal moment $M(s)$ are continuous functions over the complete BHA length. Secondly, a boundary condition for the BHA inclination at the bit is applied. Furthermore, the bit and the last stabilizer are constrained in the center of the borehole. Finally, unilateral constraints ensuring that non-ideal stabilizers conform to the borehole geometry are imposed (see Section 4.1). The beam equation is solved in Appendix A, where we obtain analytical expressions for scaled versions of F_0 , M_0 and the BHA deflection inside the borehole for any number of n stabilizers. Hereby, we generalize previous works [2, 10, 27] which only considered a small number of stabilizers.

By combining the lateral force F_0 at the bit and the moment M_0 at the bit with the relations found in the previous model components, i.e., (7a), which relates F_0 to the bit tilt, and (7b), which relates M_0 to the rate of change of the bit inclination, the borehole propagation model is formulated. This model governs the evolution of the bit inclination over the increasing length of the borehole.

3.4. Scaling of Variables

Scaling is now used to express the model in a dimensionless form. Such a form is useful to identify the key model parameters. To this extent, the distance $\ell_{(1)}$ between the bit and the first stabilizer is adopted as the characteristic length. This results in the definition of the following dimensionless quantities: the length of the segment running between the bit and the first stabilizer $\lambda_{(1)} := \ell_{(1)}/\ell_{(1)} = 1$, the length of segments running between two consecutive stabilizers $\lambda_{(i)} := \ell_{(i)}/\ell_{(1)}$, for $i = 2, 3, \dots, n$, the borehole length $\xi := L/\ell_{(1)}$, the arc length along the BHA starting from the bit $\tilde{s} := s/\ell_{(1)}$, the stabilizer positions $\tilde{s}_i := s_i/\ell_{(1)}$, for $i = 1, 2, \dots, n$, and the nominal clearances $\tilde{y}_i^* := y_i^*/\ell_{(1)}$ between stabilizer i and the borehole walls, for $i = 1, 2, \dots, n-1$. Using these scaling definitions, the scaled BHA segments can be written as

$$\tilde{S}_{(i)} = \begin{cases} \{\tilde{s} \in [0, \Delta]\} & \text{for } i = 0, \\ \{\tilde{s} \in [\Delta, \tilde{s}_i]\} & \text{for } i = 1, \\ \{\tilde{s} \in [\tilde{s}_{i-1}, \tilde{s}_i]\} & \text{for } i = 2, 3, \dots, n. \end{cases} \quad (10)$$

The characteristic force

$$F^* := \frac{3EI}{\ell_{(1)}^2} \quad (11)$$

is introduced next. This leads to the definition of the following dimensionless quantities: lateral force at the bit $\tilde{F}_0 := F_0/F^*$, axial force at the bit $\tilde{N}_0 := N_0/F^*$, moment at the bit $\tilde{M}_0 := M_0/M^*$ with $M^* := 3EI/\ell_{(1)} = F^*\ell_{(1)}$, active weight on the bit $\Pi := W_a/F^*$, distributed BHA weight $\tilde{w} := w\ell_{(1)}/F^*$, RSS actuator force $\tilde{F}_{\text{RSS}} := F_{\text{RSS}}/F^*$, measure of bit bluntness $\tilde{G} := G/F^*$, contact force experienced by the n^{th} stabilizer $\tilde{F}_n := F_n/F^*$ and contact forces experienced by the non-ideal stabilizers $\tilde{F}_{\alpha_i} := F_{\alpha_i}/F^*$, for $\alpha \in \{u, \ell\}$ and for $i = 1, 2, \dots, n-1$, and the total contact forces $\tilde{F}_i := F_i/F^*$, for $i = 1, 2, \dots, n-1$. For convenience, the following notation for the total dimensionless BHA length is introduced $\lambda := \sum_{i=1}^n \lambda_{(i)} = \tilde{s}_n$.

4. Modeling the Nonlinearities

The derivations of the BHA model are treated in Appendix A, where expressions for the BHA deflection are given. In those expressions, the external forces due to the unilateral contact of the non-ideal stabilizers with the borehole walls enter as (unknown) variables. Here, we formulate a linear complementarity problem (LCP) that deals with the unilateral contact at any number of non-ideal stabilizers. Furthermore, we derive a similar linear complementarity problem to deal with the saturation of the bit tilt. Finally, we combine both LCPs to arrive at a single LCP formulation.

Figure 8: Illustration of $\tilde{y}_i$, defined as the BHA deflection at stabilizer i with respect to chord $C_{(1)}$, and δ_i , defined as the BHA deflection at stabilizer i with respect to the borehole center.

4.1. Non-Ideal Stabilizers

Define the deflection at stabilizer i with respect to the undeformed reference configuration of the BHA, which is aligned with chord $C_{(1)}$, as

$$\tilde{y}_i := \tilde{y}_{(i+1)}(\tilde{s}_i), \text{ for } i = 1, 2, \dots, n-1, \quad (12)$$

where $\tilde{y}_{(i)}(\tilde{s})$ is the BHA deflection of segment i given in (A.5). The BHA deflection at stabilizer i with respect to the borehole center is defined as

$$\begin{aligned} \delta_i &:= \tilde{y}_i - \sum_{k=2}^i \left(\lambda_{(k)} \left(\langle \Theta \rangle_{(1)} - \langle \Theta \rangle_{(k)} \right) \right) \\ &= \sum_{k=1}^i \left(\zeta_{k+n}(\tilde{s}_i) \tilde{F}_k \right) + \sum_{k=i+1}^{n-1} \left(\zeta_{i+n}(\tilde{s}_k) \tilde{F}_k \right) + \tilde{q}_i, \end{aligned} \quad (13)$$

where

$$\begin{aligned} \tilde{q}_i &= \sum_{k=1}^i \left(\lambda_{(k)} \left(\langle \Theta \rangle_{(k)} - \theta_0 \right) \right) + \sum_{k=1}^n \left(\zeta_k(\tilde{s}_i) \left(\langle \Theta \rangle_{(k)} - \theta_0 \right) \right) \\ &\quad + \zeta_{2n}(\tilde{s}_i) \tilde{F}_{\text{RSS}} + \zeta_{2n+1}(\tilde{s}_i) \sin \langle \Theta \rangle_{(1)}. \end{aligned}$$

The inclination terms $\langle \Theta \rangle_{(k)}$ and θ_0 are present because the BHA has to conform to the borehole geometry. Contact forces $\tilde{F}_k$ experienced by the non-ideal stabilizers result from the BHA being constrained inside the borehole. The RSS actuation force enters via $\tilde{F}_{\text{RSS}}$, while gravity acting on the BHA is represented by the sine term. The term $\sum_{k=2}^i \left(\lambda_{(k)} \left(\langle \Theta \rangle_{(1)} - \langle \Theta \rangle_{(k)} \right) \right)$ in (13) represents the distance between the borehole center at stabilizer i and the reference configuration of the BHA, see Figure 8. This term relates the BHA deflection $\tilde{y}_i$, which is measured with respect to chord $C_{(1)}$, to the BHA deflection δ_i , which is measured with respect to the borehole center. Figure 8 illustrates the defined variables $\tilde{y}_i$ and δ_i .

Signorini's contact law [37] is employed for the unilateral contact between the non-ideal stabilizers and the (upper

Figure 9: Gap variables γ_{ℓ_i} and γ_{u_i} are defined for each non-ideal stabilizer. In this illustration, stabilizer j is perfectly centered inside the borehole: $\gamma_{\ell_j} = \gamma_{u_j} = \tilde{y}^*$.

and lower) borehole walls. This idealized unilateral contact model can be described by:

$$\begin{aligned} \tilde{F}_{\ell_i} &\geq 0, & \gamma_{\ell_i} &:= \tilde{y}_i^* + \delta_i \geq 0, & \tilde{F}_{\ell_i} \gamma_{\ell_i} &= 0, \\ \tilde{F}_{u_i} &\geq 0, & \gamma_{u_i} &:= \tilde{y}_i^* - \delta_i \geq 0, & \tilde{F}_{u_i} \gamma_{u_i} &= 0. \end{aligned} \quad (14)$$

The gap variables γ_{ℓ_i} and γ_{u_i} measure the gap between stabilizer i and the lower and upper borehole wall, respectively. These gap variables are depicted in Figure 9. The relations (14) follow from considering that the lower wall contact of stabilizer i , for $i = 1, 2, \dots, n-1$, i.e., the gap $\gamma_{\ell_i} = 0$. On the other hand, when there is a positive gap $\gamma_{\ell_i} > 0$, for $i = 1, 2, \dots, n-1$, the contact force $\tilde{F}_{\ell_i} = 0$. The same reasoning can be applied for upper wall contact.

Next, we use expression (13) for δ_i to formulate expressions for the gap variables γ_{ℓ_i} and γ_{u_i} , for $i = 1, 2, \dots, n-1$, defined in (14). We collect the gap variables γ_{ℓ_i} , for $i = 1, 2, \dots, n-1$, into the column vector γ_{ℓ} defined as

$$\gamma_{\ell} := [\gamma_{\ell_1} \quad \dots \quad \gamma_{\ell_i} \quad \dots \quad \gamma_{\ell_{n-1}}]^T.$$

In the same way, we collect the gap variables γ_{u_i} , contact forces $\tilde{F}_{\ell_i}$, $\tilde{F}_{u_i}$, the variables $\tilde{q}_i$ and $\tilde{y}_i^*$, for $i = 1, 2, \dots, n-1$, into the column vectors γ_u , $\tilde{F}_{\ell}$, $\tilde{F}_u$, $\tilde{q}$ and $\tilde{y}^*$, respectively. Having all these definitions, using (13), the column vectors of gap variables can be written as

$$\begin{aligned} \gamma_{\ell} &= K^{-1} \left(\tilde{F}_{\ell} - \tilde{F}_u \right) + \tilde{q} + \tilde{y}^*, \\ \gamma_u &= -K^{-1} \left(\tilde{F}_{\ell} - \tilde{F}_u \right) - \tilde{q} + \tilde{y}^*, \end{aligned} \quad (15)$$

where

$$K^{-1} = \begin{bmatrix} \zeta_{n+1}(1) & \dots & \zeta_{n+1}(\tilde{s}_i) & \dots & \zeta_{n+1}(\tilde{s}_{n-1}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \zeta_{n+1}(\tilde{s}_i) & \dots & \zeta_{n+i}(\tilde{s}_i) & \dots & \zeta_{n+i}(\tilde{s}_{n-1}) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \zeta_{n+1}(\tilde{s}_{n-1}) & \dots & \zeta_{n+i}(\tilde{s}_{n-1}) & \dots & \zeta_{2n-1}(\tilde{s}_{n-1}) \end{bmatrix}$$

with $\zeta_i(\cdot)$ defined in (A.6). The matrices K^{-1} and K are always positive definite (and hence symmetric) for a linear Euler-Bernoulli beam and have the interpretation of a flexibility and stiffness matrix [35], respectively. In our case, the relation (A.7) is critical to arrive at (A.5), which, in essence, generates the matrix K^{-1} and brings its symmetry property to light, see Appendix A.

A linear complementarity problem is formulated using the complementarity relations (14) and the linear equations (15):

$$\underbrace{\begin{bmatrix} \gamma_\ell \\ \gamma_u \end{bmatrix}}_w = \underbrace{K^{-1} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}}_M \underbrace{\begin{bmatrix} \tilde{F}_\ell \\ \tilde{F}_u \end{bmatrix}}_z + \underbrace{\begin{bmatrix} \tilde{q} + \tilde{y}^* \\ -\tilde{q} + \tilde{y}^* \end{bmatrix}}_q, \quad (16)$$

$$0 \leq w \perp z \geq 0.$$

Solving this LCP returns the contact forces $\tilde{F}_i := \tilde{F}_{\ell_i} - \tilde{F}_{u_i}$ and the gap variables γ_{u_i} and γ_{ℓ_i} , for $i = 1, 2, \dots, n-1$.

Remark 1. An LCP, as given in (16), returns unique solutions for any column vector q if and only if the matrix M is a P -matrix [38], which is a matrix with all principal minors positive. The matrix M in (16) is not a P -matrix, since it is singular, which implies that there exists a column vector q for which this LCP has no or multiple solutions. Even though M is not a P -matrix, this also implies that still for some q a unique solution can exist. Proving that solutions of (16) exist and are unique is left for future work.

Remark 2. The BHA deflection with respect to the borehole center, δ_i , is measured perpendicular to the chord $C_{(1)}$, while the nominal clearance $\tilde{y}_i^*$ is defined perpendicular to the borehole axis at stabilizer i , for $i = 1, 2, \dots, n-1$. As a consequence, if there is an orientation difference between chord $C_{(1)}$ and the borehole axis at stabilizer i , then non-ideal stabilizer i does not contact the borehole walls for $\delta_i = \pm \tilde{y}_i^*$, but for a slightly different $\tilde{y}_i^*$. However, as the BHA length is generally small compared to the radius of curvature of the borehole, it is assumed that this orientation difference is negligibly small and, therefore, this correction is not taken into account.

4.2. Saturation of the Bit Tilt

Next, we account for the bit tilt saturation using a similar approach based on linear complementarity problems. First, we introduce dimensionless equivalents of the interface laws (7):

$$\tilde{F}_0 = \eta \Pi (\theta_0 - \Theta_0), \quad (17a)$$

$$\tilde{M}_0 = -\epsilon \eta \Pi \frac{d\theta_0}{d\xi}. \quad (17b)$$

As can be seen, the active weight on the bit Π does not appear by itself in the equations, but is always multiplied by the lateral steering resistance η . The particular choice for the characteristic force F^* in (11) implies that the dimensionless group $\eta \Pi$ is of order $\mathcal{O}(0.1 \sim 1)$. The small parameter ϵ is typically of order $\mathcal{O}(10^{-4} \sim 10^{-2})$. The parameter group $\eta \Pi$ can be interpreted as a pseudo-stiffness contrasting the flexural stiffness of the BHA with the penetration stiffness of the rock formation. In other words, if the bit is tilted, then the drillstring tends to deform back

Figure 10: Gap variables ψ_ℓ and ψ_u measure the gap between the bit tilt ψ and the saturation boundaries $\pm\psi^*$, respectively.

to its undeformed shape due to the flexural stiffness of the drillstring, however, this is counteracted by the rock formation not allowing the bit to change its orientation.

According to (17a), the lateral force at the bit $\tilde{F}_0$ is proportional to the bit tilt ψ . However, laboratory experiments show that the bit tilt ψ saturates [29, 30]. In fact, the lateral force model at the bit is set-valued when the bit tilt saturates. Saturation of the bit tilt implies that $|\psi| \leq \psi^*$, where ψ^* is the saturation boundary. Therefore, the interface law (17a) is only valid for $|\psi| < \psi^*$. For $|\psi| = \psi^*$, the lateral force at the bit $\tilde{F}_0$ is not anymore proportional to the bit tilt ψ . The bit tilt saturation boundary ψ^* depends on the bit-gauge length and profile and is typically about 1° or less [2].

To model this saturation of the bit tilt, the following variables are introduced. The bit tilt measured from the lower saturation boundary is defined as

$$\psi_\ell := \psi + \psi^*, \quad (18)$$

and from the upper boundary

$$\psi_u := -\psi + \psi^*. \quad (19)$$

These variables are depicted in Figure 10. Similar to γ_{ℓ_i} and γ_{u_i} , the variables ψ_ℓ and ψ_u can be interpreted as gap variables. The following set-valued force law, see [37, 39], is introduced to characterize the bit tilt saturation:

$$\begin{aligned} \hat{F}_\ell &:= -\tilde{F}_0 + \eta \Pi \psi \geq 0, & \psi_\ell &\geq 0, & \hat{F}_\ell \psi_\ell &= 0, \\ \hat{F}_u &:= \tilde{F}_0 - \eta \Pi \psi \geq 0, & \psi_u &\geq 0, & \hat{F}_u \psi_u &= 0. \end{aligned} \quad (20)$$

The force variables $\hat{F}_\ell$ and $\hat{F}_u$ are zero in the linear part $|\psi| \leq \psi^*$. The variable $\hat{F}_\ell$ can take any positive value when $\psi = -\psi^*$, while $\hat{F}_u$ can take any positive value when $\psi = \psi^*$. Figure 11 provides a graphical interpretation of the force variables $\hat{F}_\ell$ and $\hat{F}_u$.

From (20), we can observe the relation

$$\hat{F} := \hat{F}_\ell - \hat{F}_u = -2\tilde{F}_0 + 2\eta \Pi \psi, \quad (21)$$

which, by using (18) and (19), can be rewritten as

$$\begin{aligned} \psi_\ell &= \frac{1}{2\eta \Pi} (\hat{F}_\ell - \hat{F}_u) + \frac{\tilde{F}_0}{\eta \Pi} + \psi^*, \\ \psi_u &= \frac{1}{2\eta \Pi} (\hat{F}_u - \hat{F}_\ell) - \frac{\tilde{F}_0}{\eta \Pi} + \psi^*. \end{aligned} \quad (22)$$

Figure 11: Graphical interpretation of the set-valued force law (20).

A linear complementarity problem is formulated using the complementarity relations (20) and the linear equations (22):

$$\underbrace{\begin{bmatrix} \psi_\ell \\ \psi_u \end{bmatrix}}_w = \underbrace{\frac{1}{2\eta\Pi} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_M \underbrace{\begin{bmatrix} \hat{F}_\ell \\ \hat{F}_u \end{bmatrix}}_z + \underbrace{\begin{bmatrix} \frac{\tilde{F}_0}{\eta\Pi} + \psi^* \\ -\frac{\tilde{F}_0}{\eta\Pi} + \psi^* \end{bmatrix}}_q, \quad (23)$$

$$0 \leq w \perp z \geq 0.$$

Here, the lateral force at the bit $\tilde{F}_0$ comes from the BHA model given in (A.8). For the calculation of the lateral force at the bit $\tilde{F}_0$, the contact forces experienced by the non-ideal stabilizers are necessary. This implies a unidirectional coupling between the LCP in (16), which should be solved first to obtain the contact forces, and the LCP in (23), which should be solved in the second step.

4.3. Combining both Nonlinearities

The LCPs in (16) and (23) can be combined to arrive at a single LCP. To do so, first we write the lateral force at the bit, given by (A.8), more concisely as

$$\tilde{F}_0 = \sum_{i=1}^n a_i \left(\langle \Theta \rangle_{(i)} - \theta_0 \right) + \sum_{i=1}^{n-1} \tilde{b}_i \tilde{F}_i + c_1 \tilde{F}_{\text{RSS}} + c_2 \sin \langle \Theta \rangle_{(1)} \quad (24)$$

with coefficients

$$a_i = \frac{\lambda_i}{\lambda^3}, \quad (25a)$$

$$\tilde{b}_i = -\frac{2\lambda^3 - 3\lambda \tilde{s}_i^2 + \tilde{s}_i^3}{2\lambda^3}, \quad (25b)$$

$$c_1 = -\frac{\Delta^3 - 3\lambda \Delta^2 + 2\lambda^3}{2\lambda^3}, \quad (25c)$$

$$c_2 = \frac{5}{8} \tilde{w} \lambda. \quad (25d)$$

Next, we write the term $\sum_{k=1}^{n-1} \tilde{b}_k \tilde{F}_k$ in (24) as $\tilde{b} \tilde{F}$, where $\tilde{b}$ is a row vector containing the individual coefficients $\tilde{b}_k$, defined as

$$\tilde{b} := [\tilde{b}_1, \dots, \tilde{b}_{n-1}],$$

and $\tilde{F}$ is a column vector containing the individual contact forces $\tilde{F}_k$ experienced by the non-ideal stabilizers defined as

$$\tilde{F} := [\tilde{F}_1 \quad \dots \quad \tilde{F}_{n-1}]^\top. \quad (26)$$

By substituting (24) for $\tilde{F}_0$ in (21) (or the LCP in (23)), we arrive at the combined LCP

$$\underbrace{\begin{bmatrix} \gamma_\ell \\ \gamma_u \\ \psi_\ell \\ \psi_u \end{bmatrix}}_w = \underbrace{\begin{bmatrix} K^{-1} & -K^{-1} & 0 & 0 \\ -K^{-1} & K^{-1} & 0 & 0 \\ \frac{\tilde{b}}{\eta\Pi} & -\frac{\tilde{b}}{\eta\Pi} & \frac{1}{2\eta\Pi} & -\frac{1}{2\eta\Pi} \\ -\frac{\tilde{b}}{\eta\Pi} & \frac{\tilde{b}}{\eta\Pi} & -\frac{1}{2\eta\Pi} & \frac{1}{2\eta\Pi} \end{bmatrix}}_M \underbrace{\begin{bmatrix} \tilde{F}_\ell \\ \tilde{F}_u \\ \hat{F}_\ell \\ \hat{F}_u \end{bmatrix}}_z + \underbrace{\begin{bmatrix} \tilde{q} + \tilde{y}^* \\ -\tilde{q} + \tilde{y}^* \\ \hat{q} + \psi^* \\ -\hat{q} + \psi^* \end{bmatrix}}_q, \quad 0 \leq w \perp z \geq 0, \quad (27)$$

where

$$\tilde{q} = \sum_{i=1}^n \frac{a_i}{\eta\Pi} \left(\langle \Theta \rangle_{(i)} - \theta_0 \right) + \frac{c_1}{\eta\Pi} \tilde{F}_{\text{RSS}} + \frac{c_2}{\eta\Pi} \sin \langle \Theta \rangle_{(1)}.$$

This LCP returns both the contact forces at the non-ideal stabilizers $\tilde{F}_\ell$ and $\tilde{F}_u$, together with their gap variables γ_ℓ and γ_u , as well as the bit tilt gap variables ψ_ℓ and ψ_u and the force variables $\hat{F}_\ell$ and $\hat{F}_u$. Once solved, the bit tilt can be calculated from the definition (18):

$$\psi = \psi_\ell - \psi^*. \quad (28)$$

Using relation (21), the lateral force at the bit can be retrieved from

$$\tilde{F}_0 = \frac{\hat{F}_u - \hat{F}_\ell}{2} + \eta\Pi\psi. \quad (29)$$

The borehole inclination at the bit is simply given by

$$\Theta_0 = \theta_0 - \psi, \quad (30)$$

which is deduced from the definition of the bit tilt (3).

5. Borehole Propagation Model

The total borehole propagation model is formulated next. With such a model, we can study the evolution of bit inclination θ_0 (and hence also the borehole inclination which gives the borehole shape) with the borehole length ξ . The dynamics of θ_0 can be found by combining the moment at the bit $\tilde{M}_0$ derived in the interface laws in (17b) and in the BHA component in (A.9), which results in the following differential equation:

$$\theta'_0 := \frac{d\theta_0(\xi)}{d\xi} = -\frac{\tilde{M}_0}{\epsilon\eta\Pi} = -\sum_{i=1}^n a_i \left(\langle \Theta \rangle_{(i)} - \theta_0 \right) + \sum_{i=1}^{n-1} \tilde{b}_i \tilde{F}_i + c_1 \tilde{F}_{\text{RSS}} + c_2 \sin \langle \Theta \rangle_{(1)} \quad (31)$$

with coefficients

$$a_i = \frac{\lambda_{(i)}}{\epsilon\eta\Pi\lambda^2}, \quad (32a)$$

$$b_i = -\frac{\tilde{s}_i}{2\epsilon\eta\Pi\lambda^2}(\lambda - \tilde{s}_i)(2\lambda - \tilde{s}_i), \quad (32b)$$

$$c_1 = -\frac{\Delta}{2\epsilon\eta\Pi\lambda^2}(\lambda - \Delta)(2\lambda - \Delta), \quad (32c)^{450}$$

$$c_2 = \frac{1}{8\epsilon\eta\Pi}\tilde{w}\lambda^2. \quad (32d)$$

This differential equation is expressed in terms of average borehole inclinations $\langle\Theta\rangle_{(i)}$, for $i = 1, 2, \dots, n$, which, therefore, must be considered as additional state variables. The dynamical behavior of $\langle\Theta\rangle_{(i)}$, for $i = 1, 2, \dots, n$, for an increasing borehole length is given by

$$\langle\Theta\rangle'_{(i)} := \frac{d\langle\Theta\rangle_{(i)}(\xi)}{d\xi} = \frac{\Theta_{i-1} - \Theta_i}{\lambda_{(i)}}, \quad (33)^{460}$$

410 which is the ξ -derivative of $\langle\Theta\rangle_{(i)}$ defined in (1). The output of the model is the borehole inclination $\Theta_0(\xi)$, which is calculated via (30).

The differential equations (31) and (33) as well as the output equation (30) are all subject to the combined LCP (27). This LCP returns both the contact forces experienced by the non-ideal stabilizers $\tilde{F}_i := \tilde{F}_{\ell_i} - \tilde{F}_{u_i}$, for $i = 1, 2, \dots, n-1$, as well as the bit tilt ψ .⁴⁷⁰

420 The borehole propagation model consists of the set of delay differential equations (31) and (33), and the output equation (30), all subject to the LCP in (27). This model can be considered as a delay complementary system [40]. The delay nature of this model stems from the delayed influence of the borehole geometry on the stabilizers, which have to conform to the borehole that has been drilled before. Linear complementarity problems are used to model the unilateral contact of non-ideal stabilizers and the saturation of the bit tilt.⁴⁸⁰

430 The model can exhibit many modes, where the number of modes is 3^n due to all the combinations of the contact modes of the $n-1$ non-ideal stabilizers and the saturation mode of the bit tilt. Even though we have a large number of modes, the LCP framework allows for a compact model description.

Remark 3. Note that delayed values of the borehole inclination are required in (33). In a numerical scheme, the borehole inclination can be calculated via (30) and stored⁴⁹⁰ to obtain the delayed values Θ_i as needed in the right-hand side of (33).

440 **Remark 4.** For a more detailed derivation of the borehole propagation model, we refer to [41].

6. Dynamic Analysis

A dynamic analysis of the response of the drilling system to a step in RSS actuation force is pursued here on⁵⁰⁰

the model derived above. This analysis recognizes that step variations in the RSS actuation force over multiple sections are used in practice to generate complex borehole trajectories. In particular, we elaborate on solutions arising in the long range response in each such section: we characterize these solutions parametrically as a function of the applied constant RSS force; we present methods to analyze their stability; and we highlight in which situations the drilling system is prone to borehole rippling. Furthermore, we show that the parameter group $\eta\Pi$ is key, since it controls both the drilling capabilities as well as the stability of the response [2]. We use numerical simulations to further support our analysis. The goal of the analysis is to gain essential insights in the behavior of directional drilling systems, where we focus on the essential role of the imposed nonlinearities in the response to such an actuation.

Three length scales can be identified in the response of the drilling system to a step in RSS actuation force at borehole length ξ^* , namely a short, medium and long length scale [2]. The short length scale is related to the fast response ($\xi - \xi^* < \lambda_{(1)} = 1$). Due to the imposed step in the RSS force at the start, the borehole kinks and hence the borehole curvature is locally infinite.

The medium length scale starts when the first (non-ideal) stabilizer travels through the initial kink in the borehole ($1 < \xi - \xi^* \lesssim \lambda$). Each stabilizer i , for $i = 1, 2, \dots, n$, passing through this initial kink induces a new discontinuity in the borehole curvature at the bit. Due to the geometric feedback of the borehole sensed through the stabilizers, oscillations can be self-induced at this length scale and possibly lead to borehole rippling.

The long length scale typically appears for $\xi - \xi^* \gtrsim \lambda$. The effect of the initial kink diminishes quickly and the borehole evolves with quasi-uniform curvature when only viewed at the scale of the BHA length. Due to gravity acting on the moving BHA, the borehole curvature evolves slowly over the borehole length ξ . A typical BHA configuration and RSS actuator only allow for the drilling of small curvature boreholes. This implies that the forces acting on the BHA do not change significantly when viewed over a small distance of order $\mathcal{O}(\lambda)$. Hence, the shape of the BHA is said to be quasi-stationary during such a solution and the BHA can be interpreted as a rigid body moving through the rock formation. These solutions are named quasi-stationary solutions (QSSs).

A particular subset of QSSs are the so-called constant inclination solutions (CISs). These steady-state solutions are characterized by a zero curvature and, hence, a constant borehole inclination. They appear asymptotically when the forces acting on the BHA are balanced so that the bit does not experience any moment. CISs come into pairs of two, namely a downward evolving one and an upward evolving one [31, 34]. Due to the imposed nonlinearities, it is possible that multiple pairs of CISs exist for a single setting in terms of model parameters and applied RSS force.

7. Long Range Response

The QSSs and CISs appearing at the long length scale are analyzed in this section. We derive these solutions analytically as a function of the applied constant RSS force. These derivations are given for the general n -stabilizer case and the bit tilt nonlinearity accounted for. The qualitative results presented in this section are applied in Appendix D to a two-stabilizer model introduced in Appendix C. The main conclusions of that analysis are repeated at the end of this section.

7.1. Quasi-Stationary Solutions

The borehole curvature during a QSS evolves slowly over the increasing borehole length due to gravity. Considering that the actual radii of curvature that can be drilled in practice are large compared to the BHA length, we assume uniformity of the borehole curvature when viewed over the BHA length. Furthermore, we take the relative orientation of gravity constant throughout the BHA length. Recall, that we used a similar assumption in the BHA model.

We introduce the borehole curvature $\bar{K} := \Theta'_0$. When calculating QSSs, we assume that the borehole curvature is uniform over the BHA length. This assumption is exploited to derive the geometrical relation

$$\langle \Theta \rangle_{(i)} - \theta_0 = -\psi + \frac{\bar{K}}{2}(\lambda_{(i)} - 2s_i), \quad (34)$$

and results in the QSSs, calculated next, being approximates of the true borehole curvature that are only accurate on a short borehole interval corresponding to the BHA length. Another consequence of this assumption is that the bit tilt ψ remains constant over the BHA length and, therefore, $\bar{K} = \theta'_0$, where θ'_0 is given in (31). The borehole curvature is found by substituting (34) and $\theta'_0 = \bar{K}$ in (31), and solving for the curvature $\bar{K}$, which leaves

$$\bar{K} = \alpha \left(\sum_{i=1}^{n-1} b_i \tilde{F}_i^{\bar{K}} + c_1 \tilde{F}_{\text{RSS}} + c_2 \sin \Theta^{\bar{K}} - \bar{a} \psi^{\bar{K}} \right), \quad (35)$$

where $\alpha := \frac{2\epsilon\eta\Pi}{2\epsilon\eta\Pi+1}$ and $\bar{a} := \sum_{i=1}^n a_i = \frac{1}{\epsilon\eta\Pi\lambda}$. Variables with superscripts $(\cdot)^{\bar{K}}$ are associated with QSSs. The term $\sin \langle \Theta \rangle_{(1)}$ is replaced by $\sin \Theta^{\bar{K}}$ consistently with the assumption of a constant relative inclination of gravity with respect to the BHA. The contact forces $\tilde{F}_i^{\bar{K}}$ and the bit tilt $\psi^{\bar{K}}$ are the solutions of the LCP (B.6) formulated in Appendix B. A QSS for a fixed $\tilde{F}_{\text{RSS}}$ and $\Theta^{\bar{K}}$ is calculated by evaluating (35) after solving the LCP (B.6) to obtain the contact forces and the bit tilt.

The LCP (B.6) can always be solved analytically if we consider a specific mode, i.e., a contact mode for the non-ideal stabilizers and a saturation mode for the bit tilt. To this end, if stabilizer i contacts one of the borehole walls, we know the values of the gap variables γ_{ℓ_i} and γ_{u_i} . On the other hand, if the stabilizer is cleared from both walls, then the contact forces are zero, i.e., $\tilde{F}_{u_i} = \tilde{F}_{\ell_i} = 0$.

similar reasoning can be applied for the bit tilt saturation part of the LCP.

The column vector $q^{\bar{K}}$ in the LCP (B.6) depends on the applied RSS force $\tilde{F}_{\text{RSS}}$ and the assessed inclination $\Theta^{\bar{K}}$. As a result, implicitly, these influence (35) through the contact forces $\tilde{F}_i^{\bar{K}}$ and the bit tilt $\psi^{\bar{K}}$. After solving the LCP analytically for each mode, we can write (35) as

$$\bar{K} = \tilde{c}_1^{(m)} \tilde{F}_{\text{RSS}} + \tilde{c}_2^{(m)} \sin \Theta^{\bar{K}} + \sum_{i=1}^{n-1} \tilde{d}_i^{(m)} \tilde{y}_i^* + \tilde{d}_n^{(m)} \psi^*, \quad (36)$$

where m represents the examined mode. By taking this approach, we can analytically express coefficients $\tilde{c}_1^{(m)}$, $\tilde{c}_2^{(m)}$, $\tilde{d}_1^{(m)}$, $\dots$, $\tilde{d}_n^{(m)}$, and investigate their dependency on the parameters defining the model, in particular the parameter group $\eta\Pi$.

Coefficient $\tilde{c}_1^{(m)}$ can be interpreted as the steering efficiency of the RSS, where a large absolute value corresponds to a large influence of the RSS actuation on the obtained borehole curvature. However, this coefficient can also become negative for some model parameters, implying that the borehole evolves in the opposite direction of the applied RSS force. In case $\tilde{c}_1^{(m)} = 0$, the RSS force has no influence on the borehole curvature, hence, the drilling system is said to be RSS-independent [2]. The parameter $\eta\Pi|_{\text{RSS}}^{(m)}$ is defined as the $\eta\Pi$ value for which $\tilde{c}_1^{(m)} = 0$.

Similarly, an analysis can be pursued for the coefficient $\tilde{c}_2^{(m)}$, which is a measure of the effect of gravity on the borehole curvature. Again, this coefficient takes positive and negative values, depending on the model parameters. In case $\tilde{c}_2^{(m)} = 0$, gravity has no influence on the borehole curvature, hence, the system is said to be gravity-independent [2]. In the stability analysis in Section 8, it is shown that the sign of this coefficient controls the stability of the QSSs and CISs. The parameter $\eta\Pi|_{\text{g}}^{(m)}$ is the $\eta\Pi$ value for which $\tilde{c}_2^{(m)} = 0$.

If non-ideal stabilizer i is cleared from both walls, then $\tilde{d}_i^{(m)} = 0$ and if the bit tilt is not saturated, then $\tilde{d}_n^{(m)} = 0$. We do not perform an analysis on coefficients $\tilde{d}_i^{(m)}$, for $i = 1, \dots, n-1$ and $\tilde{d}_n^{(m)}$. These represent the dependency of the examined solutions on the nominal borehole clearance $\tilde{y}_i^*$ and the bit tilt saturation boundary ψ^* , respectively. However, in our case, we restrict ourselves to constant values of these model parameters.

In general, the expressions for these coefficients are rather complex and are, therefore, not given explicitly for the general case of n stabilizers. However, we give these coefficients in Appendix C for a two-stabilizer model to support our analysis. It is recommended to use symbolic software to find these expressions.

7.2. Constant Inclination Solutions

A particular subset of QSSs are the CISs, which are characterized by a constant borehole inclination over a borehole length of at least the BHA length. These solutions

appear asymptotically in the long length scale response. Since the BHA orientation is invariant, the relative inclination of gravity naturally does not change during these solutions, which makes the obtained solutions exact. Variables with a superscript $(\cdot)^\infty$ refer to CISs, where ∞ is used as a reference to the asymptotic appearance of these solutions.

Substituting a zero curvature $\bar{K} = 0$ in (35) and solving for $\sin \Theta^\infty (= \sin \Theta^{\bar{K}})$ results in

$$\sin \Theta^\infty = \frac{-1}{c_2} \left(\sum_{i=1}^{n-1} b_i \tilde{F}_i^\infty + c_1 \tilde{F}_{\text{RSS}} - \bar{a} \psi^\infty \right). \quad (37)$$

Again, the contact forces $\tilde{F}_i^\infty$ experienced at the non-ideal stabilizers and the bit tilt ψ^∞ are the solutions of the LCP (B.10) presented in Appendix B. A CIS only exists when the forces acting on the BHA are balanced such that the moment acting on the bit is zero, which translates to the absolute value of the right-hand side of (37) not exceeding one. For a given $\tilde{F}_{\text{RSS}}$, first the LCP (B.10) should be solved to obtain ψ^∞ and $\tilde{F}_i^\infty$, which then should be used in (37) to calculate the inclination of the CIS. Naturally, $\text{asin}(\cdot)$ returns two solutions, one corresponding to a downward directed propagating borehole and one corresponding to the upward case. Consequently, if Θ^∞ is a CIS, then so is $\pi - \Theta^\infty$.

Alternatively, the CISs can be derived from (36) with $\bar{K} = 0$:

$$\sin \Theta^\infty = \frac{-1}{\tilde{c}_2^{(m)}} \left(\tilde{c}_1^{(m)} \tilde{F}_{\text{RSS}} + \sum_{i=1}^{n-1} \tilde{d}_i^{(m)} \tilde{y}_i^* + \tilde{d}_n^{(m)} \psi^* \right) \quad (38)$$

for mode m . Here, the influence of coefficients $\tilde{c}_1^{(m)}$ and $\tilde{c}_2^{(m)}$ can clearly be seen. In the RSS-independent case (i.e., $\tilde{c}_1^{(m)} = 0$), the RSS force has no influence on the response. For example, if all the stabilizers are cleared from the borehole walls and the bit tilt is not saturated, i.e., $\tilde{d}_i^{(m)} = 0$ for all $i \in \{1, \dots, n\}$, only vertical (upward and downward directed) boreholes can be drilled. In the gravity-independent case where $\tilde{c}_2^{(m)} = 0$, i.e.,

$$0 \cdot \sin \Theta^\infty = \tilde{c}_1^{(m)} \tilde{F}_{\text{RSS}} + \sum_{i=1}^{n-1} \tilde{d}_i^{(m)} \tilde{y}_i^* + \tilde{d}_n^{(m)} \psi^*$$

gravity has no influence on the response. Consequently, any inclination can be maintained as long as the right-hand side is balanced.

7.3. Quantitative Results

The analysis presented above is applied in Appendix D to a two-stabilizer model introduced in Appendix C. Such a simple model is still relevant because the lateral force and moment at the bit, which are responsible for the directional tendency of the system, are predominantly influenced by the first few contact points between the BHA

and the borehole walls [27]. Furthermore, it limits the complexity of the exposition. The QSSs and CISs are there given in the form of maps of these solutions as a function of the RSS force for several values of key parameter $\eta\Pi$. The modeling of the nonlinearities leads to the model exhibiting multiple modes. Consequently, in each such mode, the dynamics of the model are significantly different, which gives rise to many interesting phenomenon directly related to the imposed nonlinearities. The main conclusions drawn in Appendix D are summarized below.

Firstly, multiple branches of solutions exist for both QSSs and CISs for the same $\eta\Pi$ setting. Each branch corresponds to a specific contact mode of the non-ideal stabilizer and saturation mode of the bit tilt. The transition points between branches are those points where the non-ideal stabilizers start/lose contact the borehole walls or the bit tilt starts/ends saturating, i.e., a transition in mode.

Secondly, for the same RSS setting and $\eta\Pi$ value, multiple, coexisting downward and upward evolving CISs exist. The stability of these coexisting solutions and the initial conditions of the drilling system determine towards which of these solutions the response converges.

Thirdly, it is shown that the bit tilt saturation deteriorates the ability to drill large curvature boreholes. When the bit tilt is saturated, an increase in RSS force results in a decrease of the borehole curvature, which is an undesired effect. Enlarging the RSS force can even result in negative curvature boreholes. In case the bit tilt is not saturated, a small $\eta\Pi$ is favored in terms of RSS efficiency, which allows drilling of large curvature borehole by only applying a small RSS force.

Fourthly, when the bit tilt saturates, the CISs are shown to be $\eta\Pi$ -independent. Consequently, for any $\eta\Pi$ value, the same CISs exist.

The observations listed above show the importance of the modeled nonlinearities, since it entirely changes the long range response. A more detailed quantitative analysis can be found in Appendix D and in [41].

8. Stability Analysis

A stability analysis is performed for the long range solutions presented above. Naturally, key model parameter $\eta\Pi$ influences the stability of the aforementioned solutions. For unstable QSSs and CISs, two types of instability are distinguished: drift-type and oscillatory-type. In the drift-type instability, appearing in the long range response, perturbations grow in a drift-type manner, caused by the effect of gravity. This corresponds to at least one real pole being located in the complex right half plane (CRHP). In the oscillatory-type instability, perturbations grow in an oscillatory fashion due to at least one complex pole pair being located in the CRHP. This type of instability manifests in the medium range response and causes borehole rippling in practice. It is induced by geometric (delayed) feedback sensed by the stabilizers.

Two methods to assess stability are presented. The first method involves linearization of the dynamics local to the nominal path of the examined solution. This method, relying on a numerical algorithm to compute poles of the underlying linearized DDE, predicts both types of instabilities. The second method is based on an ordinary differential equation (ODE) formulation of long length scale solutions. This method allows to analytically compute the parameter settings for which the drift-type instability occurs, which gives valuable insight. Both methods are applied to the two-stabilizer model introduced in Appendix C.

8.1. Stability Based on Linearization

For linear delay complementarity systems, sufficient and necessary conditions for local exponential stability as well as sufficient conditions for global exponential stability exist [40]. To our best knowledge, there are no stability conditions for the nonlinear delay complementarity system we are facing. Therefore, we assess local stability by linearizing the dynamics local to the nominal path of the considered solution. We emphasize that this method is only valid as long as the trajectory stays sufficiently close to the nominal path and the mode of the solution does not change, i.e., the contact mode of the non-ideal stabilizers and the bit tilt mode remain unchanged, locally around the considered solution.

The delay differential equations in (31) and (33) can be written as a set of first-order nonlinear delay differential equations

$$x'(\xi) = \sum_{i=0}^n (A_i x(\xi_i) + D_i \psi(\xi_k)) + B \tilde{F}(\xi) + C_1 \tilde{F}_{\text{RSS}}(\xi) + C_2 \sin \langle \Theta \rangle_{(1)}(\xi) \quad (39)$$

with the state $x(\xi_i) \in \mathbb{R}^{n+1}$ defined as

$$x(\xi_i) := [\theta_0(\xi_i) \quad \langle \Theta \rangle_{(1)}(\xi_i) \quad \dots \quad \langle \Theta \rangle_{(n)}(\xi_i)]^\top \quad (40)$$

at the borehole position $\xi_i := \xi - \tilde{s}_i$, for $i = 0, 1, \dots, n$. The contact forces are collected in the column vector $\tilde{F}(\xi)$ as defined in (26). The delay differential equation (39) is subject to LCP (27). The matrices $A_i \in \mathbb{R}^{n+1, n+1}$, $B \in \mathbb{R}^{n+1, n-1}$, $C_1 \in \mathbb{R}^{n+1}$, $C_2 \in \mathbb{R}^{n+1}$ and $D_i \in \mathbb{R}^{n+1}$ are given in Appendix E for any number of stabilizers n . For a constant RSS actuation, the linearized dynamics local to the trajectory $\hat{x}(\xi_i)$ in mode m read as

$$z'(\xi) = \sum_{i=0}^n \mathcal{A}_i^{(m)}(\xi_i) z(\xi_i) \quad (41)$$

with the perturbation state $z(\xi) := x(\xi) - \hat{x}(\xi)$ and the

Jacobian matrices

$$\begin{aligned} \mathcal{A}_0^{(m)}(\xi_i) &:= \left[A_0 + D_0 \frac{\partial \psi(\xi)}{\partial x(\xi)} \right. \\ &\quad \left. + B \frac{\partial \tilde{F}(\xi)}{\partial x(\xi)} + C_2 \frac{\partial \sin \langle \Theta \rangle_{(1)}(\xi)}{\partial x(\xi)} \right]_{\hat{x}(\xi)} \quad (42) \\ \mathcal{A}_i^{(m)}(\xi_i) &:= A_i + D_i \frac{\partial \psi(\xi_k)}{\partial x(\xi_i)} \Big|_{\hat{x}(\xi)}. \end{aligned}$$

The RSS force $\tilde{F}_{\text{RSS}}$ is constant and, therefore, not present in $\mathcal{A}_i^{(m)}$ in (42). A consequence of linearization is the ξ -varying nature of the linearized dynamics, since each $\mathcal{A}_i^{(m)}$, for $i = 0, 1, \dots, n$, varies as a function of the borehole length ξ .

In case of QSSs, the nominal path is given by

$$\hat{x}(\xi) = \xi \bar{K} + \begin{bmatrix} \theta_0^{\bar{K}} \\ \Theta^{\bar{K}} - \frac{1}{2} \bar{K} \\ \vdots \\ \Theta^{\bar{K}} - \bar{K}(\tilde{s}_{i-1} + \frac{1}{2} \lambda_{(i)}) \\ \vdots \\ \Theta^{\bar{K}} - \bar{K}(\tilde{s}_{n-1} + \frac{1}{2} \lambda_{(n)}) \end{bmatrix} \in \mathbb{R}^{n+1}, \quad (43)$$

which evolves over ξ , since each element of the state x increases (or decreases) with curvature $\bar{K}$. Consequently, the relative orientation of gravity with respect to the BHA evolves over the borehole length in the linearized dynamics. However, the BHA orientation does not change significantly on a small interval in borehole length ξ , which is a consequence of the curvature of the borehole $\bar{K}$ being small. Therefore, we adopt the assumption that gravity has a constant influence on the response, as we did in the first place to calculate the QSSs. Substituting $\Theta^{\bar{K}}$ for $\langle \Theta \rangle_{(1)}(\xi_k)$ leaves ξ -independent linearized dynamics in (41). In case of CISs, the nominal path is given by

$$\hat{x}(\xi) := [\theta^\infty \quad \Theta^\infty \quad \dots \quad \Theta^\infty]^\top \in \mathbb{R}^{n+1}, \quad (44)$$

which is constant and thus ξ -independent, since the relative orientation of gravity with respect to the BHA is naturally constant during a CIS. Consequently, the $\mathcal{A}_i^{(m)}$ matrices in (42), for $i = 0, 1, \dots, n$, are all ξ -independent, which leaves the linearized dynamics in (41) ξ -independent.

The linearized ξ -independent dynamics examined near a QSSs or a CIS read as

$$z'(\xi) = \sum_{i=0}^n \mathcal{A}_i^{(m)} z(\xi_i). \quad (45)$$

By substituting solutions of the form $z(\xi) = C_0 e^{\nu \xi}$ in this DDE, where $C_0 \in \mathbb{R}^{n+1}$ is a constant column vector accounting for initial conditions, the characteristic equation is deduced to be:

$$\det \left(\nu I - \sum_{i=0}^n \mathcal{A}_i^{(m)} e^{-\nu \tilde{s}_i} \right) = 0. \quad (46)$$

This is a transcendental equation, which has an infinite number of roots ν_i with $i \in \{1, 2, \dots, \infty\}$ [42]. The roots of this characteristic equation represent the poles of (45). It is known that all these poles are located in the complex half plane characterized by $\mathcal{R}(\nu_i) < v$, where $v \in \mathbb{R}$ can be calculated numerically using the publicly available Matlab toolbox developed by [43]. The linearized dynamics (45) are globally exponentially stable if and only if all the poles ν_i , for $i = 1, 2, \dots, \infty$, are located in the open complex left half plane. It is noted that (46) has roots with multiplicity n at the origin of the complex plane. These poles do not affect the stability of the QSSs and CISs [27] as these are induced by the introduction of the states $\langle \Theta \rangle_{(i)}$ in (40).

In previous literature [11, 2, 28], the effect of gravity is neglected by taking the BHA weight $\tilde{w} = 0$ in the scope of stability analysis, which also omits the ξ -dependency, in the linearized dynamics. This method captures the oscillatory-type of instability, but does not account for the drift-type instability. In contrast, the effect of gravity is taken constant in this analysis, which also captures the drift-type instability. Drift-type instability was previously addressed in [27].

Remark 5. *The ξ -independent linearized dynamics in (45) are equivalent for a QSSs with inclination Θ^K and a CIS with inclination Θ^∞ , where $\Theta^\infty = \Theta^K$. As a result, local stability properties of a QSS with inclination Θ^K are equivalent to local stability properties of a CIS with inclination Θ^∞ .*

8.2. Stability Based on ODE Formulation

During the long range response, i.e., a QSS or CIS, the borehole curvature is (quasi-) uniform. Consequently, the dependency on the delayed borehole geometry vanishes and the BHA can be viewed as a rigid body moving through the rock formation [2]. This observation allows to rewrite the delay complementarity system (39) into an ODE for the active mode m of the examined solution. To this extent, the variable $\bar{\Theta}(\xi)$ is introduced as an average borehole inclination at borehole length ξ and its derivative with respect to the borehole length ξ is naturally $\bar{K}$, i.e., the relation $\frac{d\bar{\Theta}(\xi)}{d\xi} = \bar{K}$ holds.

The quasi-stationary solution $\bar{K}$ given in (36) is rewritten to

$$\frac{d\bar{\Theta}(\xi)}{d\xi} = \tilde{c}_1^{(m)} \tilde{F}_{\text{RSS}} + \tilde{c}_2^{(m)} \sin \bar{\Theta}(\xi) + \sum_{i=1}^{n-1} \tilde{d}_i^{(m)} \tilde{y}_i^* + \tilde{d}_n^{(m)} \psi^*,$$

which is an ODE being active in mode m with state $\bar{\Theta}(\xi)$ and independent variable ξ . A linearization around a fixed inclination $\bar{\Theta}$ reads as

$$z'(\xi) = \tilde{c}_2^{(m)} \cos \bar{\Theta} z(\xi), \quad (47)$$

where $z(\xi) := \bar{\Theta}(\xi) - \bar{\Theta}$. The RSS force $\tilde{F}_{\text{RSS}}$, nominal clearances $\tilde{y}_i^*$ and bit tilt saturation boundary ψ^* do not play a role since these are constants. The dynamics (47)

have a pole located at $\tilde{c}_2^{(m)} \cos \bar{\Theta}$. Therefore, for local stability, it is necessary and sufficient that this term is negative. A sign change in $\tilde{c}_2^{(m)}$ happens for $\eta \Pi|_{\mathcal{g}}^{(m)}$, meaning that the effect of gravity on the angular penetration φ changes in mode m . A sign change in $\cos \bar{\Theta}$ happens for $\bar{\Theta} = \pm\pi/2$, which are the boundaries of the upward and downward QSSs and CISs.

The explanation for this kind of instability is straightforward. If gravity has a dropping tendency on the drilling direction, i.e., an increase in gravity forces results in a smaller curvature, which is true for $\tilde{c}_2^{(m)} < 0$ in mode m , then downward solutions are stable. For example, imagine a CIS with $\Theta^\infty = \pi/4$. For a small perturbation $\Theta^\infty + \iota$ with $\iota > 0$ the gravity forces are increased. This results in a negative curvature due to the dropping tendency of the drilling system, which stabilizes the downward solutions. A similar reasoning applies for upward solutions. For example, imagine a CIS with $\Theta^\infty = 3\pi/4$. For a small perturbation $\Theta^\infty - \iota$ with $\iota > 0$ the gravity forces are increased. This results in a negative curvature due to the dropping tendency of the drill system, which destabilizes the upward solution. If $\tilde{c}_2^{(m)} > 0$, the effect of gravity on the angular penetration reverses and, consequently, downward solutions become unstable while upward solutions become stable. Thus the parameter $\eta \Pi|_{\mathcal{g}}^{(m)}$ determines this type of instability.

8.3. Stability Analysis of the Two-Stabilizer Model

The two methods to assess stability of the long range response, as presented in the previous sections, are applied to the two-stabilizer model introduced in Appendix C. This model accounts for a non-ideal stabilizer above the bit and an ideal stabilizer at the end of the BHA. First, we focus on the case where the bit tilt is not saturated. After that, stability for the full two-stabilizer model is assessed by considering both nonlinearities.

First, we consider the case where the bit tilt saturation is not saturated. We assess stability by linearizing dynamics local to QSSs and CISs, as presented in Section 8.1. The following observations allow to limit the complexity of the stability analysis:

- The linearized dynamics local to a QSSs with inclination Θ^K are equivalent to those near a CIS with the same inclination, i.e., $\Theta^K = \Theta^\infty$, see Remark 5. Hence, we can check stability of one type of solutions to conclude for the other.
- For stability, it does not matter which wall the non-ideal stabilizer contacts, since the non-ideal stabilizer experiences geometrical feedback from the borehole wall in both cases [41]. A distinction between contact or no contact for each stabilizer is sufficient.
- Upward and downward solutions only interchange the drift-type instability at $\pm\pi/2$ for a fixed $\eta \Pi$, see Section 8.2 and [2, 41]. Therefore, a distinction between

upward and downward solutions should be made. The oscillatory-type instability is irrespective of the borehole inclination [2, 41].

To conclude, for the case where the bit tilt does not saturate, we only need to distinguish between the upward and downward solutions and between the non-ideal stabilizer contacting and not contacting a borehole wall to conclude stability for all QSSs and CIs.

Figure 12 contains four subplots corresponding to the aforementioned distinction. Each subplot shows stability regions over the dimensionless group $\eta\Pi$ and the BHA length λ . The two left subplots correspond to solutions where the non-ideal stabilizer is cleared from the borehole walls. The right subplots refer to solutions where the non-ideal stabilizer contacts either the upper or lower borehole wall. The two top subplots correspond to upward propagating solutions, while the bottom subplots refer to downward propagating solutions. In all situations, the bit tilt is not saturated.

The numerical results depicted in the two left subplots in Figure 12 agree with the observations made in Section 8.2, where it was shown that the sign of the term $\tilde{c}_2 \cos \Theta^K$ in (47) is decisive for stability. This coefficient changes sign exactly for $\eta\Pi = \eta\Pi_{\mathcal{G}}^{(\text{nc,ns})}$. Moreover, stability is interchanged for upward and downward solutions, as the cosine term in the linearized dynamics in (47) changes sign. This type of instability leads to a borehole drifting away from the examined solution and corresponds to at least one real pole crossing the imaginary axis into the CRHP. Exactly at $\eta\Pi_{\mathcal{G}}^{(m)}$, gravity has no effect and solutions are marginally stable. In case the non-ideal stabilizer contacts one of the borehole walls, i.e., the two subplots on the right in Figure 12, again the crossing of a real pole through the imaginary axis can be observed. In this case, the sign of $\tilde{c}_2^{(\text{c,ns})}$ changes at $\eta\Pi = \eta\Pi_{\mathcal{G}}^{(\text{c,ns})}$, which changes the areas for which the drift-type instability occurs entirely compared to the no-contact case.

Stability lobes, indicated by the red areas in Figure 12, are observed in case the non-ideal stabilizer contacts one of the borehole walls. Such lobes have earlier been observed in [2, 27, 28] and are upper bounded by [2]

$$\eta\Pi = \frac{1}{\pi} \lambda_{(2)}^{-3/2}. \quad (48)$$

At the boundaries of the red areas, a Hopf-bifurcation takes place, resulting in a complex pole pair crossing the imaginary axis, hence, generating oscillatory responses. The instability is induced by geometric feedback of the borehole on the BHA deformation via interaction with the (non-ideal) stabilizers. Perturbations in the borehole trajectory being sensed by the stabilizers are amplified/attenuated progressively at the bit. For small $\eta\Pi$, a small $\tilde{F}_0$ generates a large bit tilt ψ , see (17a). Consequently, a small $\eta\Pi$ realizes a larger amplification, which results in this type of instability. The branches of lobes

intersect the λ axis (i.e., at $\eta\Pi = 0$) at integer values $\lambda = k \in \mathbb{N}$ for $\lambda > 1$.

When the non-ideal stabilizer is cleared from both walls, the BHA behaves as a one-stabilizer BHA. Such a BHA is equivalent to a two-stabilizer BHA with $\lambda_{(2)} \rightarrow \infty$ [2]. As can be seen in the right subplots of Figure 12, indeed, the stability lobes vanish for large $\lambda_{(2)} \rightarrow \infty$ (i.e., $\lambda = \lambda_{(1)} + \lambda_{(2)} \rightarrow \infty$). This observation agrees with the numerical results given in the two left subplots, where such stability lobes are not observed. Furthermore, this observation agrees with previous results in the literature, see [2], where it is stated that one-stabilizer BHAs do not induce borehole rippling.

The parametric stability analysis presented here and the results for calculating the long range response presented earlier show a tradeoff. To be robust against borehole rippling, a sufficiently large $\eta\Pi$ should be selected. However, in terms of RSS efficiency in the long range response, a small $\eta\Pi$ is preferred. These two objectives interfere with each other and need to be taken into account carefully in applications of directional drilling.

Remark 6. *The results presented here can be extended to the general n -stabilizer BHA. As is shown in Section 8.2 and supported by the numerical analysis, the drift-type instability is determined by $\eta\Pi_{\mathcal{G}}^{(m)}$ in mode m and interchanges for downward and upward solutions. In principle, one can find analytical expressions for $\eta\Pi_{\mathcal{G}}^{(m)}$ for any mode m . The oscillatory-type instability can be found using the numerical algorithm presented in Section 8.1, not restricted to the number of stabilizers.*

Next, consider the case where the bit tilt is saturated. Figure 13 plots stability regions over the parameters $\eta\Pi$ and λ . When the bit tilt is saturated, $\psi(\xi)$ is constant over the borehole length implying $\partial\psi(\xi_i)/\partial x(\xi_i) = 0$ in (42), for $i = 0, 1, \dots, n$. When the non-ideal stabilizer is cleared from the walls, the coefficient $\tilde{c}_2^{(\text{nc,s})}$ in (C.3) is positive for any parameter setting, which concludes that upward solutions are stable and downward solutions are unstable, see the two left subplots of Figure 13. In case the non-ideal stabilizer contacts one of the borehole walls, gravity has an additional influence through the contact force at the non-ideal stabilizer. The coefficient $\tilde{c}_2^{(\text{c,s})}$ is given by the expression in (C.4), whose sign is independent of $\eta\Pi$. However the sign does change at $\lambda_s = (3 + \sqrt{5})/2 = 2.618$. For a BHA with length smaller than λ_s , this coefficient is positive, which concludes that upward solutions are stable and downward solutions are unstable. In case $\lambda > \lambda_s$, this coefficient is negative, which concludes that stability is reversed, caused by the dropping tendency of gravity, see the two right subplots of Figure 13. Complex pole pairs have not been observed in the CRHP in case the bit tilt saturates.

The stability analysis clearly emphasizes the hybrid nature of the model. In some modes, the long range response is stable, while in others its not, even for the same param-

Figure 12: Local stability of the QSSs and CISs for a grid of parameters $\eta\Pi$ and λ for the two-stabilizer model without bit tilt saturation. A distinction in the four subplots is made with respect to their mode and the direction of borehole propagation.

eter setting. In the next section, we address the switching between stable and unstable modes using numerical simulations and study how potentially steady-state oscillating solutions (related to borehole rippling) may arise.

9. Numerical Simulation Results

A numerical scheme is implemented in Matlab Simulink [44]. In this implementation, θ'_0 , as in (31), and $\langle\Theta\rangle'_{(i)}$, for $i = 1, 2, \dots, n$, as in (33), are integrated with respect to the independent variable ξ using the variable step solver ODE45 [45]. The borehole inclination at the bit Θ_0 , given in (30), is calculated, stored and delayed to serve in the expressions for the delayed terms in $\langle\Theta\rangle'_{(i)}$ given in (33). Furthermore, using the algorithm given in [46], the LCP in (27) is solved during the simulation to update the contact forces $\tilde{F}$ and the bit tilt ψ . For various $\eta\Pi$ and $\tilde{F}_{\text{RSS}}$ forces, the response of the borehole propagation model is simulated for a step in $\tilde{F}_{\text{RSS}}$ using this implementation.

The two-stabilizer model introduced in Appendix C is once more used for the numerical simulations. The simulation settings are chosen in a way to emphasize the role of $\eta\Pi$, which influences both the directional capabilities as well as the directional stability of the system. Furthermore, we focus on the behavior of the imposed nonlinearities and, in particular, we show how these are responsible for limit cycles.

9.1. Three Length Scales

Four simulations are performed with the settings given in Table 1. The results for a short/medium borehole length of $\xi \in \{0, 10\}$ are depicted in Figure 14 and 15. In all simulations, a vertical borehole is used as an initial condition. At borehole length $\xi^* = 1$, the RSS force $\tilde{F}_{\text{RSS}}$ is increased from 0 to 10^{-2} . In simulation one in Figure 14, $\eta\Pi = 0.2$ is used and a large bit tilt saturation boundary is used, i.e., the bit tilt cannot saturate. A kink is observed in the borehole inclination at the step moment $\xi = \xi^* = 1$. Every time a stabilizer passes through this initial kink, a new discontinuity (of lesser order) in the borehole curvature is induced at the bit. Due to the unitary segment lengths $\lambda_{(1)} = \lambda_{(2)} = 1$, this occurs at borehole lengths $\xi = 2, 3, \dots$. However, these discontinuities smooth out as ξ grows. This is in line with the expectation, since the right subplot in the stability map in Figure 12 reveals that the response should be stable for such a large $\eta\Pi$. In simulation two in Figure 14, we use $\eta\Pi = 0.05$, which is significantly smaller than the setting in simulation one. According to the right subplot in Figure 12, the response should be unstable in an oscillatory fashion, which we also observe in the simulation results, where oscillations grow unbounded. Simulation three in Figure 15 also uses $\eta\Pi = 0.05$, but now the bit tilt saturation is in place with $\psi^* = 2^\circ$. It is observed that the bit tilt indeed saturates and prevents oscillations in the borehole path to grow unbounded. In simulation four in Figure 15, again $\eta\Pi = 0.05$ is used, however, now a smaller $\tilde{F}_{\text{RSS}} = 10^{-4}$ is applied. Such a small actuation magnitude results in the bit tilt

Figure 13: Local stability of the QSSs and CISs for a grid of parameters $\eta\Pi$ and λ for the two-stabilizer model with bit tilt saturation. A distinction in the four subplots is made with respect to their mode and the direction of borehole propagation.

not saturating and the non-ideal stabilizer being cleared from the borehole walls making the BHA behave as if it was equipped with only one stabilizer. Consequently, the bottom left subplot of Figure 12 applies now and reveals that the response should be stable, which is also concluded from the simulation results. The set of simulations clearly shows the effect of the nonlinearities on the short/medium range response and, in particular, its stability properties.

A simulation triggering the drop of curvature observed when transitioning from no saturation to a saturation of the bit tilt in QSSs (the gray circles in Figure Appendix D.2 are the transition points) is presented in Figure 16 for $\eta\Pi = 0.4$ and the saturation boundary $\psi^* = 0.5^\circ$. The simulation starts with a vertical downward borehole. The RSS force $\tilde{F}_{\text{RSS}} = 0.015$ is applied in the first part of the simulation up to borehole length $\xi = 500$. In Figure Appendix D.2, this corresponds to a $\tilde{F}_{\text{RSS}}$ slightly to the left of the gray circle. At borehole length $\xi = 500$, the RSS force is doubled to $\tilde{F}_{\text{RSS}} = 0.03$ for the remainder of the simulation, which corresponds to a $\tilde{F}_{\text{RSS}}$ to the right of this gray circle. The second subplot shows the rate of change of bit inclination θ'_0 , which can be interpreted as an accurate approximate of the borehole curvature Θ'_0 , since ψ' is of order $\mathcal{O}(10^{-6})$. The observations in this simulation agree with previous conclusions: (i) the borehole curvature drops in the second part with respect to the first part of the simulation even though the RSS force is increased (doubled) in the second part with respect to the first part; and (ii) the bit tilt in the first part is not saturated (but stays close to the boundary), while in the second part it is saturated. Note that the results in Figure 16 show the drilling of approximately 2.5 full circular loops, which is unrealistic in practice. These results are only used to illustrate the observed curvature drop.

The curvature in Figure 16 is not quite constant over the borehole length ξ . However, the analytical results do agree

Table 1: Settings for the simulations in Figure 14 and 15. The initial condition is a vertical downward borehole in all simulations.

Simulation	$\eta\Pi$	$\tilde{F}_{\text{RSS}} \cdot 10^{-2}$	ψ^*
1	0.2	1	90°
2	0.05	1	90°
3	0.05	1	2°
4	0.05	0.01	2°

Figure 14: Simulation results with the settings given in Table 1. Top: Borehole inclination Θ_0 . Middle: Bit tilt ψ . Bottom: Contact force F_1 . The initial condition is a vertical downward borehole in both simulations.

Figure 15: Simulation results with the settings given in Table 1. Top: Borehole inclination Θ_0 . Middle: Bit tilt ψ and saturation boundaries $\psi^* = 2^\circ$ in black dashed lines. Bottom: Contact force $\tilde{F}_1$. The initial condition is a vertical downward borehole in both simulations.

Table 2: Settings for the simulations presented in Figure 17. The bit tilt saturation boundary takes the value $\psi^* = 0.1$ in all simulations.

Simulation	$\eta\Pi$	$\tilde{F}_{\text{RSS}} \cdot 10^{-3}$	Initial Condition
1	0.5	1.1	0
2	0.5	1.1	$3\pi/2$
3	0.5	1.4	0
4	0.5	1.4	$3\pi/2$
5	0.5	1.8	0
6	0.5	10	0
7	2	10	0

with the simulation results in the sense that the method described in Section 7.1 to calculate $\bar{K}$ leaves (approximately) a curvature of 0.0175 (for $0 < \xi < 500$) and 0.014 (for $500 < \xi < 1000$) for inclination $\Theta^K = 0$. The error in curvature between this analytical calculation and the simulation result is in the order of $\mathcal{O}(10^{-4})$, which validates the assumption on uniformity of QSSs on a scale of the BHA length used in Section 7.1 to calculate QSSs.

Next, the results of a set of seven simulations involving CISs are presented to (i) indicate the influence of the initial condition on the response of the borehole propagation model; (ii) indicate the behavior near a transition from one mode to another; and (iii) show that CISs are $\eta\Pi$ -independent when the bit tilt saturates. In these simulations, we use the same saturation boundary $\psi^* = 0.1^\circ$ as we used to generate Figure Appendix D.6, which contains maps of CISs as a function of the applied RSS force. The simulations are performed with the settings given in Table 2 and the results are depicted in Figure 17.

Both simulations 1 and 2 are performed with $\eta\Pi = 0.5$ and $\tilde{F}_{\text{RSS}} = 1.1 \cdot 10^{-3}$, these only differ in initial condi-

Figure 16: Drop in borehole curvature after a transition from no saturation to saturation of the bit tilt, triggered by increasing $\tilde{F}_{\text{RSS}}$ from 0.015 to 0.03 at borehole length $\xi = 500$. The dimensionless group $\eta\Pi = 0.4$ and the saturation boundary ψ^* is set to 0.5° . The initial condition is a vertical downward borehole. The borehole inclination Θ_0 , rate of change of the bit inclination θ_0 , the bit tilt ψ (and its boundary ψ^* in dashed black) and the contact force $\tilde{F}_1$ are shown in respective orders from the top to the bottom subplots.

Figure 17: Results of a set of 7 simulations for the long length scale step response with the settings given in Table 2. The simulation number in Table 2 corresponds to the Sim number in the legend. Top: Borehole inclination Θ_0 . Middle: Bit tilt ψ and its boundaries ψ^* in black dashed lines. Bottom: Contact force $\tilde{F}_1$ at the non-ideal stabilizer.

Figure 18: Orientation of CISs for the simulations in Figure 17 with the settings in Table 2. The green arrows represent stable solutions, while the red arrows represent unstable solutions. The simulations start in the orientation of the blue arrow, where the number of the arrow corresponds to the simulation number.

tion. In this setting, two distinct downward CISs exist, see Figure 18a. The initial condition in simulation 1 is a vertical downward borehole. The response converges in a counterclockwise fashion to the stable solution in mode $m = nc, ns$. Simulation 2 starts with a horizontal borehole with inclination $3\pi/2$. The response converges in a clockwise fashion to the stable upward CIS in the mode $m = c, s$. Since multiple CISs coexist, the initial condition determines to which CISs the response converges.

Next, simulations 3 and 4 are considered. These are performed with $\eta\Pi = 0.5$, $\tilde{F}_{RSS} = 1.4 \cdot 10^{-3}$ and again distinct initial conditions. Two distinct downward CISs exist, see Figure 18b. The initial condition in simulation 3 is a vertical downward borehole. The response converges in a counterclockwise fashion to the stable solution in mode $m = c, ns$. Compared to simulation 1, where a slightly smaller $\tilde{F}_{RSS}$ was applied, the response now converges to a slightly larger inclination and the non-ideal stabilizer contacts a borehole wall. Simulation 4 starts with a horizontal borehole with inclination $3\pi/2$. The response converges in a clockwise fashion to the stable upward CIS in mode $m = c, s$. Again, we observe that the initial condition is important in the response.

Simulation 5 is performed with $\eta\Pi = 0.5$ and $\tilde{F}_{RSS} = 1.8 \cdot 10^{-3}$. It starts with a vertical downward borehole. In this case, a unique unstable downward CIS exists, see Figure 18c. The response converges in a counterclockwise fashion to the stable upward pair of this solution. Compared to simulations 1 and 3, where the same initial condition was taken, but slightly smaller RSS forces were applied, the response converges to a significantly larger constant inclination. This is related to the bit tilt nonlinearity, which, in this case, ‘creates’ additional solutions, since for a model without bit tilt saturation, no CIS would exist in this setting, see Figure Appendix D.4. Thus, transitioning from no bit tilt saturation to saturation can cause a significantly different response.

Finally, simulations 6 and 7 are considered, where $\tilde{F}_{RSS} = 10^{-2}$ is applied and a vertical downward borehole is taken as initial condition. In Simulation 6, $\eta\Pi = 0.5$

while $\eta\Pi = 2$ in Simulation 7. It can be observed that both responses converge to the same CIS, as CISs are $\eta\Pi$ -independent in case the bit tilt saturates. The two responses, however, differ significantly up to a borehole length of $\xi \approx 3000$, which is expected since earlier results in Appendix D revealed that, even though the bit tilt is saturated, $\eta\Pi$ does influence the QSSs, which characterize this first part of the response. These simulations conclude that indeed the same CIS exists for the same $\tilde{F}_{RSS}$ actuation when the bit tilt is saturated, irrespective of $\eta\Pi$, which was already concluded from Figure Appendix D.6.

9.2. Limit Cycles

This section treats limit cycles observed in simulations of the two-stabilizer model, which can be related to borehole rippling. The imposed nonlinearities make the model exhibit multiple modes, where possibly stable and unstable dynamics are active. To this extent, a threshold $\eta\Pi|_s = 0.147$ is defined which represents the critical $\eta\Pi$ value for which the Hopf-bifurcation takes place, i.e., the boundary of the red area for the fixed $\lambda_{(2)} = 1$ in Figure 12 corresponding to a complex pole pair crossing the imaginary axis.

The simulations presented in Figure 19 and Figure 20 are performed with $\eta\Pi = 0.01 < \eta\Pi|_s$ and for a zero RSS actuation, i.e., $\tilde{F}_{RSS} = 0$. An almost vertical downward borehole is taken as initial condition. First, in Figure 19, we present simulations with different bit tilt saturation boundaries $\psi^* \in \{2^\circ, 1^\circ, 0.5^\circ, 0.1^\circ\}$ to show the effect of this nonlinearity on the obtained limit cycles. After that, the effect of the nominal clearance of the non-ideal stabilizer is addressed in Figure 20 by taking $\tilde{y}_1^* \in \{1, 1.5, 2\} \cdot 10^{-3}$.

In Figure 19, it can be observed that the responses, except the one with $\psi^* = 0.1^\circ$, oscillate around the trivial CIS $\Theta^\infty = 0$. During one period, the bit tilt saturates on both sides and the non-ideal stabilizer contacts on both borehole walls. Our stability analysis revealed that in case the bit tilt is not saturated, oscillatory behavior is induced when the non-ideal stabilizer contacts a wall. However, in

all other modes, no oscillatory behavior is induced. Particularly, when the non-ideal stabilizer is cleared from both walls and the bit tilt is not saturated, the response is stable. Taking ψ^* sufficiently small, such as $\psi^* = 0.1^\circ$, generates a stable response. In general, a smaller ψ^* results in smaller oscillations in the bit and borehole inclinations and smaller contact forces $\tilde{F}_1$. This is expected, since the BHA has to deform less inside the borehole in order to saturate the bit tilt. The borehole inclination Θ_0 exceeds the bit tilt saturation boundary ψ^* , which is due to phase mismatch between the bit and borehole inclinations. The oscillation wavelength is approximately 1.3 in dimensionless length, which seems to decrease with smaller ψ^* .

Figure 20 depicts the results of simulations with different nominal clearances $\tilde{y}_1^*$ between the non-ideal stabilizer and the borehole walls. This clearance being small requires a small BHA deflection for the non-ideal stabilizer to contact the borehole walls and, therefore, makes the response more prone to oscillatory behavior for such small $\eta\Pi$ setting. Increasing the clearance gives the non-ideal stabilizer more room to move and, hence, being cleared from both walls. In that case, the BHA acts as if equipped with only one stabilizer. Such BHAs do not generate oscillatory responses. For a sufficiently large $\tilde{y}_1^*$, the response is thus stable and converges to the trivial CIS with inclination 0° . Allowing for a larger clearance makes the response less prone to oscillatory behavior.

The stability map of Figure 12 shows to be accurate and useful in predicting oscillatory responses. However, due to the existence of multiple modes induced by the hybrid nature of the nonlinearities, related to the bit tilt saturation and non-ideal stabilizers, one should rely on numerical simulation to conclude whether the response converges to steady-state oscillations.

In field data representing borehole spiraling [8, 9, 28], it is observed that the oscillation wavelength is related to the distance between the bit and the first contact point of the drillstring with the borehole walls and that the oscillation amplitude is of order $\mathcal{O}(10^{-1} \sim 1)$ cm. In the simulations shown above in Figure 19 and Figure 20, the wavelength is approximately 1.3 in dimensionless length, which we can relate to the dimensionless distance between the bit and the non-ideal stabilizer, which is defined as 1. Furthermore, the oscillation amplitude in the numerical results are roughly between 0.3 and 2 cm. Based on this comparison, we relate the steady-state oscillations observed in simulation to borehole rippling, which is the two-dimensional equivalent of the unwanted borehole spiraling observed in practice.

10. Conclusions

In this paper, we developed a dynamic non-smooth borehole propagation model in the form of a delay complementarity system. The delay nature stems from the delayed geometrical feedback of the borehole on the deformation

Figure 19: Limit cycles for $\eta\Pi = 0.01$ and different bit tilt saturation boundaries $\psi^* = \{2^\circ, 1^\circ, 0.5^\circ, 0.1^\circ\}$. An almost vertically inclined downward borehole is taken as initial condition. No RSS force is applied. Top: Borehole inclination Θ_0 in solid lines and the bit inclination θ_0 in dashed lines. Middle: Bit tilt ψ in solid lines and its corresponding boundaries ψ^* in dashed lines. Bottom: Contact force at the non-ideal stabilizer.

Figure 20: Limit cycles for $\eta\Pi = 0.01$ and different nominal clearances $\tilde{y}_1^* = \{1, 1.5, 2\} \cdot 10^{-3}$. An almost vertically inclined downward borehole is taken as initial condition. No RSS force is applied. Top: Borehole inclination Θ_0 in solid lines and the bit inclination θ_0 in dashed lines. Middle: Bit tilt ψ in solid lines and its corresponding boundaries ψ^* in dashed lines. Bottom: Contact force at the non-ideal stabilizer.

of the drillstring through the stabilizers. A linear complementarity problem (LCP) framework is used to model two essential nonlinearities: the saturation of the bit tilt and any number of non-ideal stabilizers inducing unilateral contact. The LCP framework enables a compact formulation of the model, despite the large number of modes corresponding to the contact mode of the non-ideal stabilizers and the saturation mode of the bit tilt.

Using the model, we analyzed the long range response to a constant RSS actuation. Due to the imposed nonlinearities, the model exhibits multiple modes, where in each mode significantly different dynamics are active. Consequently, in each mode significantly different long range responses are observed.

Methods to assess the stability of the long range response are developed. To this extent, a numerical method, assessing both a drift-type and oscillatory-type of instability, and a analytical method, assessing the drift-type instability, are presented. The drift-type of instability is caused by the effect of gravity, while the oscillatory-type of instability is an effect of geometrical feedback sensed through the stabilizers and amplified by the drill bit.

Our study revealed that in some modes, the system exhibits the oscillatory-type of instability, while in other modes, the system is stable. The combination of such stable and unstable dynamics results in steady-state oscillations, which represent borehole rippling. These are the two-dimensional equivalent of the so-called borehole spiraling observed in practice.

The work presented in this paper provides essential insights in the behavior of directional drilling systems. Here-with, the presented results can be used to support improved directional drilling system design and form the basis for further work on automation techniques for the downhole robotic actuator to track predefined borehole paths while mitigating borehole rippling.

Appendix A. BHA Model

The three model components are introduced in Section 3. This appendix presents the derivation of the BHA model, which aims to fit the BHA inside the borehole and, thereby, obtaining expressions for the forces and moment experienced by the bit.

The beam equation reads as the differential equation (9), which is subject to static equilibrium equations and a set of constraints. We present a dimensionless equivalent system of equations and solve it to obtain the deflection and relative inclination of the BHA. Finally, expressions for the forces and moment experienced by the bit are derived. In these derivations, the contact forces experienced by the non-ideal stabilizers due to the unilateral contact with the borehole walls will be treated as unknowns. A linear complementarity problem is formulated in Section 4 to deal with non-ideal stabilizers and the bit tilt saturation.

The dimensionless equivalent of the beam equation (9) reads as

$$\frac{d\phi(\tilde{s}, \xi)}{d\tilde{s}} = 3\tilde{M}(\tilde{s}, \xi). \quad (\text{A.1})$$

See Section 3.4 for the definition of the scaled variables. For notational convenience, the coordinate ξ is dropped in further elaborations. The static equilibrium equations are written in dimensionless form and, for an n -stabilizer BHA, they read as

$$0 = -\tilde{N}_0 + \tilde{N}_n + \tilde{w}\lambda \cos \langle \Theta \rangle_{(1)}, \quad (\text{A.2a})$$

$$0 = \tilde{F}_0 + \tilde{F}_{\text{RSS}} - \tilde{w}\lambda \sin \langle \Theta \rangle_{(1)} + \sum_{i=1}^{n-1} \tilde{F}_i + \tilde{F}_n, \quad (\text{A.2b})$$

$$0 = -\Delta \tilde{F}_{\text{RSS}} + \frac{1}{2} \tilde{w} \lambda^2 \sin \langle \Theta \rangle_{(1)} - \sum_{i=1}^{n-1} \tilde{F}_i \tilde{s}_i - \tilde{F}_n \lambda + \tilde{M}_0, \quad (\text{A.2c})$$

where the sum of the moments $\tilde{M}$ can be taken with respect to any arbitrary location on the BHA, here with respect to the bit location. The dimensionless beam equation (A.1) together with the dimensionless static equilibrium equations (A.2) constitute the BHA model.

The inclination of the BHA is measured relative to $\langle \Theta \rangle_{(1)}(\mathbf{L})$, which is the inclination of the undeformed reference configuration of the BHA evolving over the borehole length $\mathbf{L}$. Using the method of sections [47], the internal moment $\tilde{M}_{(i)}(\tilde{s})$ is formulated for each BHA segment:

$$\begin{aligned} \tilde{M}_{(0)}(\tilde{s}) &= \tilde{M}_{(1)}(\tilde{s}) + \tilde{F}_{\text{RSS}}(\Delta - \tilde{s}), \\ \tilde{M}_{(i)}(\tilde{s}) &= \tilde{M}_{(i+1)}(\tilde{s}) + \tilde{F}_i(\tilde{s}_i - \tilde{s}), \\ \tilde{M}_{(n)}(\tilde{s}) &= \tilde{F}_n(\lambda - \tilde{s}) - \frac{1}{2} \tilde{w} (\lambda^2 - 2\lambda\tilde{s} + \tilde{s}^2) \sin \langle \Theta \rangle_{(1)}, \end{aligned} \quad (\text{A.3})$$

for $i = 1, 2, \dots, n-1$. It should be noted that the recursive functions $\tilde{M}_{(i)}(\tilde{s})$ can be evaluated for $\tilde{s} \in \mathbb{R}$; however, these functions can only be interpreted as internal moments when evaluated in their corresponding segment i , i.e., evaluated for $\tilde{s} \in \tilde{S}_{(i)}$ with $\tilde{S}_{(i)}$ defined in (10). The moment for the last segment, i.e., $\tilde{M}_{(n)}$ which is furthest away from the bit, is calculated first, which is then consecutively used in the calculation of the internal moment of segments closer to the bit.

Integration of the beam equation (A.1) with $\tilde{M}_{(i)}(\tilde{s})$ in (A.3) results in the relative inclination $\phi_{(i)}(\tilde{s})$ and deflection $\tilde{y}_{(i)}(\tilde{s})$ of the BHA for each BHA segment i . Other than the contact forces experienced by the non-ideal stabilizers, expressions $\phi_{(i)}(\tilde{s})$ and $\tilde{y}_{(i)}(\tilde{s})$ contain $2n+3$ unknowns, which consist of $2(n+1)$ integration constants and the contact force F_n experienced by the last stabilizer.

The $2(n+1)$ integration constants are eliminated by imposing continuity of the BHA inclination and deflection over the ends of the BHA segments ($2n$ constraints) and by centering the bit and last stabilizer inside the borehole (2 constraints). After that, by imposing the BHA inclination

at the bit to equal the inclination of the bit θ_0 (a constraint following from the interface laws), we obtain an expression for the contact forces experienced by the last stabilizer

$$\begin{aligned} \tilde{F}_n = & \frac{3}{8} \tilde{w} \lambda \sin \langle \Theta \rangle_{(1)} - \frac{1}{\lambda^3} \sum_{i=1}^n \lambda_{(i)} \left(\langle \Theta \rangle_{(i)} - \theta_0 \right) \\ & + \frac{1}{2\lambda^3} \left(\Delta^2 (\Delta - 3\lambda) \tilde{F}_{\text{RSS}} + \sum_{i=1}^{n-1} \tilde{s}_i^2 (\tilde{s}_i - 3\lambda) \tilde{F}_i \right). \end{aligned} \quad (\text{A.4})$$

The BHA deflection is explicitly given by

$$\begin{aligned} \tilde{y}_{(i)}(\tilde{s}) = & \left(\langle \Theta \rangle_{(1)} - \theta_0 \right) \tilde{s} + \sum_{k=1}^n \left(\zeta_k(\tilde{s}) \left(\langle \Theta \rangle_{(k)} - \theta_0 \right) \right) \\ & + \zeta_{2n}(\tilde{s}) \tilde{F}_{\text{RSS}} + \zeta_{2n+1}(\tilde{s}) \sin \langle \Theta \rangle_{(1)} \\ & + \sum_{k=1}^i \left(\zeta_{k+n}(\tilde{s}) \tilde{F}_k \right) + \sum_{k=i+1}^{n-1} \left(\zeta_{i+n}(\tilde{s}_k) \tilde{F}_k \right) \end{aligned} \quad (\text{A.5})$$

with functions $\zeta_k(\tilde{s})$, for $k = 1, 2, \dots, 2n+1$, defined as

$$\begin{aligned} \zeta_k(\tilde{s}) &:= \frac{\tilde{s}^2(\tilde{s} - 3\lambda)}{2\lambda^3} \lambda_{(k)} \quad \text{for } k = 1, 2, \dots, n, \\ \zeta_{k+n}(\tilde{s}) &:= \frac{\tilde{s}_k^2}{4\lambda^3} \left(\tilde{s}^2(3\lambda - \tilde{s})(\tilde{s}_k - 3\lambda) + 2\lambda^3(3\tilde{s} - \tilde{s}_k) \right) \\ &\quad \text{for } k = 1, 2, \dots, n-1, \\ \zeta_{2n}(\tilde{s}) &:= \frac{\Delta^2}{4\lambda^3} \left(\tilde{s}^2(3\lambda - \tilde{s})(\Delta - 3\lambda) + 2\lambda^3(3\tilde{s} - \Delta) \right), \\ \zeta_{2n+1}(\tilde{s}) &:= \frac{1}{16} \tilde{w} \tilde{s}^2 (-3\lambda + 2\tilde{s})(\lambda - \tilde{s}). \end{aligned} \quad (\text{A.6})$$

In (A.5), a summation where the lower bound is higher than the upper bound is an empty sum, which has a zero outcome. To arrive at (A.5), the relation

$$\zeta_{k+n}(\tilde{s}_i) + \frac{1}{2} (\tilde{s}_k - \tilde{s}_i)^3 = \zeta_{i+n}(\tilde{s}_k), \quad \text{for } k > i, \quad (\text{A.7})$$

is used, which provides valuable insights as shown in Section 4.

The lateral force at the bit can be deduced from the static equilibrium (A.2b), where we substitute (A.4) for $\tilde{F}_n$ to leave

$$\begin{aligned} \tilde{F}_0 = & \frac{1}{\lambda^3} \left(\sum_{i=1}^n \lambda_{(i)} \left(\langle \Theta \rangle_{(i)} - \theta_0 \right) \right) \\ & - \frac{1}{2\lambda^3} \left(\sum_{i=1}^{n-1} (2\lambda^3 - 3\lambda \tilde{s}_i^2 + \tilde{s}_i^3) \tilde{F}_i \right) \\ & - \frac{\Delta^3 - 3\lambda \Delta^2 + 2\lambda^3}{2\lambda^3} \tilde{F}_{\text{RSS}} + \frac{5}{8} \tilde{w} \lambda \sin \langle \Theta \rangle_{(1)}. \end{aligned} \quad (\text{A.8})$$

In the same way, (A.4) is used in (A.2c) to obtain an ex-

pression for the moment at the bit given by

$$\begin{aligned} \tilde{M}_0 = & -\frac{1}{\lambda^2} \left(\sum_{i=1}^n \lambda_{(i)} \left(\langle \Theta \rangle_{(i)} - \theta_0 \right) \right) \\ & + \frac{1}{2\lambda^2} \left(\sum_{i=1}^{n-1} \tilde{s}_i (\lambda - \tilde{s}_i) (2\lambda - \tilde{s}_i) \tilde{F}_i \right) \\ & + \frac{\Delta}{2\lambda^2} (\lambda - \Delta) (2\lambda - \Delta) \tilde{F}_{\text{RSS}} - \frac{1}{8} \tilde{w} \lambda^2 \sin \langle \Theta \rangle_{(1)}. \end{aligned} \quad (\text{A.9})$$

We comment on the different terms influencing the lateral force (A.8) and the moment (A.9) at the bit. Firstly, we identify the average borehole inclinations $\langle \Theta \rangle_{(i)}$ and the bit inclination θ_0 . These terms arise because the BHA has to conform to the borehole geometry, i.e., the bit is surrounded by rock formation and the last stabilizer is centered inside the borehole. Next, we observe in (A.8) and (A.9) the effect of contact forces $\tilde{F}_i$ experienced by the non-ideal stabilizers. These forces obviously influence the lateral force and moment that is experienced by the bit. Moreover, we identify a term related to the applied RSS actuator force $\tilde{F}_{\text{RSS}}$. Finally, we observe a sine term, which is related to gravity acting on the BHA, which is inclined over $\langle \Theta \rangle_{(1)}$.

Appendix B. LCP for QSSs and CISs

The presented qualitative results for the long range response in Section 7 require the contact forces experienced by the non-ideal stabilizers and the bit tilt during such a response. This appendix presents the derivations of the LCPs that are associated to these solutions. First, we present the derivations regarding the QSSs, after which, we present derivations concerning the CISs. Similar to the derivation in Section 4, we can clearly distinguish between a part concerning the bit tilt saturation nonlinearity, a part treating the unilateral contact of the non-ideal stabilizers and a part for combining both aforementioned parts.

Appendix B.1. LCP for QSSs

The QSSs are calculated via (35) given an applied RSS force $\tilde{F}_{\text{RSS}}$ and an inclination $\Theta^{\bar{K}}$. However, (35) also requires the contact forces $\tilde{F}_i^{\bar{K}}$, for $i = 1, \dots, n-1$, and the bit tilt $\psi^{\bar{K}}$ during such a solutions. Therefore, we want to formulate an LCP, similar to (27), which returns these required variables. In essence, we eliminate the terms $\langle \Theta \rangle_{(i)} - \theta_0$, for $i = 1, \dots, n-1$, in the column vector q in the LCP in (27) using their geometrical relation with the curvature $\bar{K}$ given in (34).

First, the part on the bit tilt is treated. Using the LCP in (27) and relation (28), we retrieve the bit tilt as

$$\psi^{\bar{K}} = \frac{\bar{c}}{\eta \Pi} \tilde{F}^{\bar{K}} + \frac{1}{2\eta \Pi} \hat{F}^{\bar{K}} + \hat{q},$$

where

$$\hat{q} = \sum_{i=1}^n \frac{a_i}{\eta \Pi} \left(\langle \Theta \rangle_{(i)} - \theta_0 \right) + \frac{c_1}{\eta \Pi} \tilde{F}_{\text{RSS}} + \frac{c_2}{\eta \Pi} \sin \Theta^{\bar{K}}$$

and the coefficients given in (25). The terms $\langle \Theta \rangle_{(i)} - \theta_0$, for $i = 1, 2, \dots, n$, are eliminated using the geometrical relation in (34) and the expression for the curvature in (35), which leaves

$$\psi^{\bar{K}} = \tau \tilde{F}^{\bar{K}} + \frac{1}{2\tau_{n+2}} \hat{F}^{\bar{K}} + \underbrace{\tau_n \tilde{F}_{\text{RSS}} + \tau_{n+1} \sin \Theta^{\bar{K}}}_{\hat{q}^{\bar{K}}} \quad (\text{B.1})$$

with coefficients

$$\begin{aligned} \tau &:= [\tau_1, \dots, \tau_{n-1}], \\ \tau_i &:= \frac{1}{\tau_{n+2}} (\ell_i + \alpha b_i \mu), \quad i = 1, \dots, n-1, \\ \tau_n &:= \frac{1}{\tau_{n+2}} (c_1 + \alpha c_1 \mu), \\ \tau_{n+1} &:= \frac{1}{\tau_{n+2}} (c_2 + \alpha c_2 \mu), \\ \tau_{n+2} &:= \eta \Pi + \bar{a} + \alpha \bar{a} \mu, \\ \mu &:= \frac{-1}{2\lambda}, \end{aligned}$$

and

$$\bar{a} := \sum_{i=1}^n a_i = \frac{1}{\epsilon \eta \Pi \lambda}, \quad (\text{B.2a})$$

$$\bar{a} := \sum_{i=1}^n a_i = \frac{1}{\lambda^2}. \quad (\text{B.2b})$$

Next, an expression for the BHA deflection at the stabilizer positions during a QSS is derived. In any situation, the BHA deflection δ_i at stabilizer i , for $i = 1, 2, \dots, n-1$, satisfies (13). Again (34) and (35) are used to eliminate the terms $\langle \Theta \rangle_{(i)} - \theta_0$, for $i = 1, 2, \dots, n$, leaving

$$\delta_i^{\bar{K}} = \sum_{k=1}^{n-1} \sigma_{i,k} \tilde{F}_k^{\bar{K}} + \sigma_{i,n} \tilde{F}_{\text{RSS}} + \sigma_{i,n+1} \sin \Theta^{\bar{K}} + \sigma_{i,n+2} \psi^{\bar{K}}$$

with coefficients

$$\begin{aligned} \sigma_{i,k} &:= \begin{cases} \zeta_{k+n}(\tilde{s}_i) + \beta_i \alpha b_k, & k = 1, \dots, i, \\ \zeta_{i+n}(\tilde{s}_k) + \beta_i \alpha b_k, & k = i+1, \dots, n-1, \end{cases} \\ \sigma_{i,n} &:= \zeta_{2n}(\tilde{s}_i) + \beta_i \alpha c_1, \\ \sigma_{i,n+1} &:= \zeta_{2n+1}(\tilde{s}_i) + \beta_i \alpha c_2, \\ \sigma_{i,n+2} &:= -\tilde{s}_i - \bar{\zeta}(\tilde{s}_i) - \beta_i \alpha \bar{a}, \end{aligned}$$

$\bar{a}$ as defined in (B.2a), $\zeta_k(\cdot)$ as defined in (A.6), and

$$\begin{aligned} \bar{\zeta}(\tilde{s}_i) &:= \sum_{k=1}^n \zeta(\tilde{s}_i) = \frac{\tilde{s}_i^2(\tilde{s}_i - 3\lambda)}{2\lambda^2}, \\ \beta_i &:= \frac{\tilde{s}_i^2}{4\lambda} (\lambda - \tilde{s}_i). \end{aligned}$$

Using these derivations, we can write the column vector of deflections as

$$\delta^{\bar{K}} = \sigma \tilde{F}^{\bar{K}} + \sigma_n \tilde{F}_{\text{RSS}} + \sigma_{n+1} \sin \Theta^{\bar{K}} + \sigma_{n+2} \psi^{\bar{K}}, \quad (\text{B.3})$$

where $\sigma_j := [\sigma_{j,1} \ \dots \ \sigma_{j,n-1}]^\top$ for all $j \in \{n, n+1, n+2\}$ and σ is the matrix with elements $\sigma_{i,j}$ with index $i, j \in \{1, \dots, n-1\}$. The right-hand side of (B.3) contains the bit tilt $\psi^{\bar{K}}$, which is eliminated using (B.1), yielding

$$\delta^{\bar{K}} = P \tilde{F}^{\bar{K}} + \rho_{n+2} \hat{F}^{\bar{K}} + \underbrace{\rho_n \tilde{F}_{\text{RSS}} + \rho_{n+1} \sin \Theta^{\bar{K}}}_{\hat{q}^{\bar{K}}}, \quad (\text{B.4})$$

where

$$\begin{aligned} P &:= \sigma + \sigma_{n+2} \tau, \\ \rho_n &:= \sigma_n + \sigma_{n+2} \tau_n, \\ \rho_{n+1} &:= \sigma_{n+1} + \sigma_{n+2} \tau_{n+1}, \\ \rho_{n+2} &:= \frac{\sigma_{n+2}}{2\tau_{n+2}}. \end{aligned}$$

Now we are ready to formulate the LCP. To this extent, we introduce a similar contact law as (14) and a similar saturation law as (20):

$$\begin{aligned} \tilde{F}_{\ell_i}^{\bar{K}} &\geq 0, \quad \gamma_{\ell_i}^{\bar{K}} := \tilde{y}_i^* + \delta_i^{\bar{K}} \geq 0, \quad \tilde{F}_{\ell_i}^{\bar{K}} \gamma_{\ell_i}^{\bar{K}} = 0, \\ \tilde{F}_{u_i}^{\bar{K}} &\geq 0, \quad \gamma_{u_i}^{\bar{K}} := \tilde{y}_i^* - \delta_i^{\bar{K}} \geq 0, \quad \tilde{F}_{u_i}^{\bar{K}} \gamma_{u_i}^{\bar{K}} = 0, \\ \hat{F}_{\ell}^{\bar{K}} &\geq 0, \quad \psi_{\ell}^{\bar{K}} := \psi^* + \psi^{\bar{K}} \geq 0, \quad \hat{F}_{\ell}^{\bar{K}} \psi_{\ell}^{\bar{K}} = 0, \\ \hat{F}_u^{\bar{K}} &\geq 0, \quad \psi_u^{\bar{K}} := \psi^* - \psi^{\bar{K}} \geq 0, \quad \hat{F}_u^{\bar{K}} \psi_u^{\bar{K}} = 0. \end{aligned} \quad (\text{B.5})$$

By using these laws together with the linear equations for $\delta^{\bar{K}}$ (B.4) and for the bit tilt $\psi^{\bar{K}}$ (B.1), we formulate the LCP as

$$w^{\bar{K}} = M^{\bar{K}} z^{\bar{K}} + q^{\bar{K}}, \quad 0 \leq w^{\bar{K}} \perp z^{\bar{K}} \geq 0, \quad (\text{B.6})$$

where the column vector $w^{\bar{K}}$ consists of the column vectors of gap variables $\gamma_{\ell}^{\bar{K}}, \gamma_u^{\bar{K}}, \psi_u^{\bar{K}}$ and $\psi_{\ell}^{\bar{K}}$ and the column vector $z^{\bar{K}}$ consists of the column vectors of force variables $\tilde{F}_{\ell}^{\bar{K}}, \tilde{F}_u^{\bar{K}}, \hat{F}_{\ell}^{\bar{K}}$ and $\hat{F}_u^{\bar{K}}$. We refer to Section 4 for an interpretation of these variables. The matrices in (B.6) are given by

$$\begin{aligned} M^{\bar{K}} &= \begin{bmatrix} P & -P & \rho_{n+2} & -\rho_{n+2} \\ -P & P & -\rho_{n+2} & \rho_{n+2} \\ \tau & -\tau & \frac{1}{2\tau_{n+2}} & -\frac{1}{2\tau_{n+2}} \\ -\tau & \tau & -\frac{1}{2\tau_{n+2}} & \frac{1}{2\tau_{n+2}} \end{bmatrix}, \\ q^{\bar{K}} &= \begin{bmatrix} \tilde{q}^{\bar{K}} + \tilde{y}^* \\ -\tilde{q}^{\bar{K}} + \tilde{y}^* \\ \hat{q}^{\bar{K}} + \psi^* \\ -\hat{q}^{\bar{K}} + \psi^* \end{bmatrix}. \end{aligned}$$

By solving this LCP for a given $\tilde{F}_{\text{RSS}}$ and inclination $\Theta^{\bar{K}}$, the contact forces $\tilde{F}^{\bar{K}}$ at the non-ideal stabilizers and the bit tilt $\psi^{\bar{K}}$ during a QSS are retrieved. The constant curvature $\bar{K}$ can then be calculated via (35).

Appendix B.2. LCP for CISOs

The CISOs are calculated via (37) given an applied RSS force $\tilde{F}_{\text{RSS}}$. In these calculations, the contact forces $\tilde{F}_i^{\infty}$,

for $i = 1, \dots, n-1$ and the bit tilt ψ^∞ during such a solutions are required. Therefore, we formulate an LCP, similar to (27) and (B.6), which returns these required variables.

First, we deal with the part on the bit tilt. Consider (B.1) and substitute (37) for $\sin \Theta^{\bar{K}}$ to arrive at

$$\psi^\infty = \tau^\infty \tilde{F}^\infty + \tau_{n+1}^\infty \hat{F}^\infty + \underbrace{\tau_n^\infty \tilde{F}_{\text{RSS}}}_{\hat{q}^\infty} \quad (\text{B.7})$$

with coefficients

$$\begin{aligned} \tau^\infty &:= [\tau_1^\infty, \dots, \tau_{n-1}^\infty], \\ \tau_i^\infty &:= \frac{1}{\tau_{n+2}^\infty} \left(\tau_i - \tau_{n+1} \frac{b_i}{c_2} \right), \quad i = 1, \dots, n-1, \\ \tau_n^\infty &:= \frac{1}{\tau_{n+2}^\infty} \left(\tau_n - \tau_{n+1} \frac{c_1}{c_2} \right), \\ \tau_{n+1}^\infty &:= \frac{1}{2\tau_{n+2}\tau_{n+2}^\infty}, \\ \tau_{n+2}^\infty &:= 1 - \tau_{n+1} \frac{\bar{a}}{c_2}, \end{aligned}$$

and $\bar{a}$ as defined in (B.2a).

In the same manner, an expression for the BHA deflection at stabilizer positions is derived. To this extent, consider (B.4) and eliminate $\sin \Theta^{\bar{K}}$ using (37) to leave ¹²⁸⁰

$$\delta^\infty = \sigma^\infty \tilde{F}^\infty + \sigma_{n+1}^\infty \hat{F}^\infty + \sigma_n^\infty \tilde{F}_{\text{RSS}} + \sigma_{n+2}^\infty \psi^\infty, \quad (\text{B.8})$$

where the coefficients are defined as

$$\begin{aligned} \sigma^\infty &:= P - \rho_{n+1} \frac{b}{c_2}, \\ \sigma_n^\infty &:= \rho_n - \rho_{n+1} \frac{c_1}{c_2}, \\ \sigma_{n+1}^\infty &:= \rho_{n+2}, \\ \sigma_{n+2}^\infty &:= \rho_{n+1} \frac{\bar{a}}{c_2}, \end{aligned} \quad 1290$$

and where $\bar{a}$ is given in (B.2a). Now, substitute in (B.8) for the bit tilt ψ^∞ the expression (B.7) to yield

$$\delta^\infty = P^\infty \tilde{F}^\infty + \rho_{n+2}^\infty \hat{F}^\infty + \underbrace{\rho_n^\infty \tilde{F}_{\text{RSS}}}_{\hat{q}^\infty}, \quad (\text{B.9})$$

where

$$\begin{aligned} P^\infty &:= \sigma^\infty + \sigma_{n+2}^\infty \tau^\infty, \\ \rho_n^\infty &:= \sigma_n^\infty + \sigma_{n+2}^\infty \tau_n^\infty, \\ \rho_{n+1}^\infty &:= \sigma_{n+1}^\infty + \sigma_{n+2}^\infty \tau_{n+1}^\infty. \end{aligned} \quad 1300$$

Finally, we combine both parts. Consider a similar contact and saturation law as (B.5), where we replace $\bar{K}$ with ∞ . These laws together with the linear equations for δ^∞

(B.9) and for the bit tilt ψ^∞ (B.7), allow to formulate the LCP as

$$w^\infty = M^\infty z^\infty + q^\infty, \quad 0 \leq w^\infty \perp z^\infty \geq 0 \quad (\text{B.10})$$

with the corresponding matrices given by

$$\begin{aligned} M^\infty &= \begin{bmatrix} P^\infty & -P^\infty & \rho_{n+1}^\infty & -\rho_{n+1}^\infty \\ -P^\infty & P^\infty & -\rho_{n+1}^\infty & \rho_{n+1}^\infty \\ \tau^\infty & -\tau^\infty & \tau_{n+1}^\infty & -\tau_{n+1}^\infty \\ -\tau^\infty & \tau^\infty & -\tau_{n+1}^\infty & \tau_{n+1}^\infty \end{bmatrix}, \\ q^\infty &= \begin{bmatrix} \tilde{q}^\infty + \tilde{y}^\star \\ -\tilde{q}^\infty + \tilde{y}^\star \\ \hat{q}^\infty + \psi^\star \\ -\hat{q}^\infty + \psi^\star \end{bmatrix}. \end{aligned}$$

¹²⁷⁰ The contact forces $\tilde{F}^\infty$ at the non-ideal stabilizers and the bit tilt ψ^∞ during a solution CIS are retrieved by solving the LCP (B.10) for a given $\tilde{F}_{\text{RSS}}$. The constant inclination Θ^∞ can then be calculated via (37).

Remark 7. In contrast to the LCP in (27), which has a block lower triangular M -matrix, the LCPs derived in this appendix both contain a full M -matrix. Consequently, the LCP cannot be solved in a two-step fashion as was possible in (27), but rather has to be solved in one single shot. This implies a bidirectional dependency between the two nonlinearities in both LCPs (B.6) and (B.10), which arises due to the imposed geometrical relation (34) (for CIS $\bar{K} = 0$).

Appendix C. Two-Stabilizer Model

The analytical results throughout this paper are applied to a two-stabilizer model. This appendix introduces this model and also given expressions for coefficients $\tilde{c}_1^{(m)}$, $\tilde{c}_2^{(m)}$, $\tilde{d}_1^{(m)}$ and $\tilde{d}_2^{(m)}$ as needed in (36) and (38).

A simple two-stabilizer model is still relevant because the lateral force and moment at the bit, which are responsible for the directional tendency of the system, are predominantly influenced by the first few contact points between the BHA and the borehole walls [27]. Furthermore, it limits the complexity of the exposition. The last stabilizer is ideal, while the stabilizer above the bit is non-ideal. Furthermore, the bit tilt saturation is included. This model exhibits 9 modes, corresponding to all the combinations of the 3 non-ideal stabilizer modes (lower wall contact, upper wall contact and no contact) and the 3 bit tilt modes (saturated on either of boundaries and not saturated).

The parameters of this model are introduced next. The length of the BHA segments are $\ell_{(i)} = 3.66$ m, for $i = 1, 2$. The actuation pads of the RSS are located at $\Delta\ell_{(1)} = 0.6$ m away from the bit. The parameters characterizing the BHA are the distributed weight $w = 1.08 \cdot 10^3$ N/m and the flexural stiffness $EI = 7.2 \cdot 10^6$ Nm². As a result, the characteristic force $F^\star$ takes the value $1.61 \cdot 10^6$ N. The active weight on the bit takes a value of $W_a \in [20, 45]$

kN. The RSS actuation limits are ± 16 kN. The non-ideal stabilizer has a nominal clearance $y_1^* \in [0.1, 1] \cdot 10^{-2}$ m, which, in general, could evolve over the borehole length ξ . The bit tilt saturation boundary is $\psi^* = 1^\circ$.

These parameters lead to the following dimensionless quantities: BHA segment lengths $\lambda_{(i)} = 1$, $i = 1, 2$; the position of the RSS actuation pads behind the bit $\Delta = 1/6$; the distributed BHA weight $\tilde{w} = 2.45 \cdot 10^{-3}$; the dimensionless group $\eta\Pi \in [3.7, 8.37] \cdot 10^{-1}$; the RSS actuation limits $\pm 10^{-2}$ and the nominal clearance $\tilde{y}_1^* \in [0.27, 2.7] \cdot 10^{-3}$ of the non-ideal stabilizer. However, the nominal clearance $\tilde{y}_1^*$ is fixed to 10^{-3} . Finally, the ratio ϵ of the lateral and the angular steering resistances is taken to be 0.1.

Next, we give analytical expressions for the coefficients $\tilde{c}_1^{(m)}$, $\tilde{c}_2^{(m)}$, $\tilde{d}_1^{(m)}$ and $\tilde{d}_2^{(m)}$ for this two-stabilizer model. The variable m represents the mode of the solutions, where we distinguish between four modes, which are treated below.

The analytical expressions are found by considering (B.1) and (B.4). In these expressions, regarding the non-ideal stabilizer, we either substitute $\delta^{\tilde{K}} = \pm \tilde{y}^*$ (upper or lower wall contact) or $\tilde{F} = 0$ (no contact), depending on the considered mode. Furthermore, considering the bit tilt nonlinearity, we substitute $\psi = \pm \psi^*$ (saturated) or $\tilde{F} = 0$ (not saturated), again depending on the considered mode. The sign of coefficients $\tilde{d}_1^{(m)}$ and $\tilde{d}_2^{(m)}$ depends on the considered mode. Here, these coefficients are given for $y = \tilde{y}_1^*$ and $\psi = \psi^*$. If one considers $y = -\tilde{y}_1^*$ or $\psi = -\psi^*$, then the sign of the respective coefficient reverses.

In case the bit tilt is not saturated, the non-ideal stabilizer can either be cleared from both walls or contact one of the walls. For the no contact case, the coefficients are:

$$\tilde{c}_1^{(\text{nc,ns})} = \frac{(\lambda - \Delta)(\eta\Pi\Delta(\Delta - 2\lambda) + 2)}{\eta\Pi(2\epsilon(1 + \eta\Pi\lambda^2) + \lambda^2)}, \quad (\text{C.1a})$$

$$\tilde{c}_2^{(\text{nc,ns})} = \frac{\tilde{w}\lambda^2(\eta\Pi\lambda^2 - 4)}{4\eta\Pi(2\epsilon(1 + \eta\Pi\lambda^2) + \lambda^2)}, \quad (\text{C.1b})$$

$$\tilde{d}_1^{(\text{nc,ns})} = 0, \quad (\text{C.1c})$$

$$\tilde{d}_2^{(\text{nc,ns})} = 0, \quad (\text{C.1d})$$

where nc means no contact at the non-ideal stabilizer and ns means no bit tilt saturation. The roots of $\tilde{c}_1^{(\text{nc,ns})}$ and $\tilde{c}_2^{(\text{nc,ns})}$ as a function of $\eta\Pi$ are respectively given by

$$\eta\Pi|_{\text{RSS}}^{(\text{nc,ns})} = \frac{2}{\Delta(\Delta - 2\lambda)},$$

$$\eta\Pi|_{\mathcal{F}}^{(\text{nc,ns})} = \frac{4}{\lambda^2}.$$

For the contact case, the coefficients are:

$$\tilde{c}_1^{(\text{c,ns})} = \frac{\Delta(1 - \Delta^2)(\eta\Pi - 2)}{N - \frac{2(\Delta - 1)\lambda((\Delta - 2)\Delta\eta\Pi + 2)}{N}} \quad (\text{C.2a})$$

$$\tilde{c}_2^{(\text{c,ns})} = \frac{\lambda\tilde{w}((\eta\Pi - 2)(3\lambda - \lambda^2) - 2 - \eta\Pi)}{4N}, \quad (\text{C.2b})$$

$$\tilde{d}_1^{(\text{c,ns})} = \frac{2\lambda(\eta\Pi(1 - 2\lambda) + 2)}{(\lambda - 1)N}, \quad (\text{C.2c})$$

$$\tilde{d}_2^{(\text{c,ns})} = 0, \quad (\text{C.2d})$$

with $N = (\eta\Pi + 2)\lambda + \epsilon\eta\Pi(\eta\Pi(4\lambda - 1) + 4\lambda + 2)$ and where c stands for contact at the non-ideal stabilizer. The roots of $\tilde{c}_1^{(\text{c,ns})}$ and $\tilde{c}_2^{(\text{c,ns})}$, respectively, as a function of $\eta\Pi$ are expressed by

$$\eta\Pi|_{\text{RSS}}^{(\text{c,ns})} = -\frac{2(2\lambda - \Delta - \Delta^2)}{\Delta(1 - 4\lambda + \Delta + 2\lambda\Delta)},$$

$$\eta\Pi|_{\mathcal{F}}^{(\text{c,ns})} = 2 - \frac{4}{\lambda^2 - 3\lambda + 1}.$$

The dimensionless group $\eta\Pi$ takes the following critical values for this setting:

$$\eta\Pi|_{\text{RSS}}^{(\text{nc,ns})} = \frac{72}{23} = 3.130, \quad \eta\Pi|_{\mathcal{F}}^{(\text{nc,ns})} = 1,$$

$$\eta\Pi|_{\text{RSS}}^{(\text{c,ns})} = \frac{274}{37} = 7.41, \quad \eta\Pi|_{\mathcal{F}}^{(\text{c,ns})} = 6.$$

Here, abbreviation c and nc refer to contact (any one of the walls) and no contact of the non-ideal stabilizer, respectively, while s and ns refer to saturation (at any boundary) and no saturation of the bit tilt, respectively. Quantities $\eta\Pi|_{\text{RSS}}^{(m)}$ and $\eta\Pi|_{\mathcal{F}}^{(m)}$ represent the critical RSS-independent and gravity-independent $\eta\Pi$ values, respectively, in the corresponding mode m .

In case the bit tilt is saturated, these coefficients can also be derived. In case the non-ideal stabilizer is cleared from both walls, the coefficients read as

$$\tilde{c}_1^{(\text{nc,s})} = -\frac{\Delta(\Delta^2 - 3\Delta\lambda + 2\lambda^2)}{\lambda^2(2\epsilon\eta\Pi + 1)}, \quad (\text{C.3a})$$

$$\tilde{c}_2^{(\text{nc,s})} = \frac{\tilde{w}\lambda^2}{4(2\epsilon\eta\Pi + 1)}, \quad (\text{C.3b})$$

$$\tilde{d}_1^{(\text{nc,s})} = 0, \quad (\text{C.3c})$$

$$\tilde{d}_2^{(\text{nc,s})} = -\frac{2}{\lambda(2\epsilon\eta\Pi + 1)}, \quad (\text{C.3d})$$

where s stands for saturation of the bit tilt. When the non-ideal stabilizer does contact the borehole walls, the

coefficients are given by

$$\tilde{c}_1^{(c,s)} = -\frac{(\Delta - 1)\Delta(\Delta + 2(\Delta - 2)\lambda + 1)}{\lambda + \epsilon\eta\Pi(4\lambda - 1)}, \quad (\text{C.4a})$$

$$\tilde{c}_2^{(c,s)} = -\frac{\tilde{w}\lambda(\lambda^2 - 3\lambda + 1)}{4(\lambda + \epsilon\eta\Pi(4\lambda - 1))}, \quad (\text{C.4b})$$

$$\tilde{d}_1^{(c,s)} = -\frac{2\lambda(2\lambda - 1)}{(\lambda - 1)(\lambda + \epsilon\eta\Pi(4\lambda - 1))}, \quad (\text{C.4c})$$

$$\tilde{d}_2^{(c,s)} = -\frac{4\lambda}{\lambda + \epsilon\eta\Pi(4\lambda - 1)}. \quad (\text{C.4d})$$

All coefficients in (C.3) and (C.4) do not have roots for any value of the parameter group $\eta\Pi$, hence $\eta\Pi|_{\text{RSS}}^{(\cdot,s)}$ and $\eta\Pi|_{\text{g}}^{(\cdot,s)}$ are irrelevant.

Appendix D. Quantitative Analysis for a Two-Stabilizer Model

The presented analytical results in Section 7 are applied to the two-stabilizer model presented in Appendix C, where also expressions for the coefficients $\tilde{c}_1^{(m)}$, $\tilde{c}_2^{(m)}$, $\tilde{d}_1^{(m)}$ and $\tilde{d}_2^{(m)}$ are given for all the 9 possible modes m . First, the quasi-stationary solutions (QSSs) are presented, after which we present the constant inclination solutions (CISs).

Appendix D.1. Quasi-Stationary Solutions

QSSs for the two-stabilizer model are given in a two-step fashion. First, the results for the two-stabilizer model without bit tilt saturation, i.e., for a large value of the parameter ψ^* , are given. After that, results for the two-stabilizer model with the saturation boundary $\psi^* = 0.5^\circ$ are presented.

Figure Appendix D.1 depicts the QSSs without bit tilt saturation for inclinations $\Theta^{\bar{K}} = 0$ (left) and $\Theta^{\bar{K}} = \pi/4$ (right) for various $\eta\Pi$. The horizontal axis represents the applied $\tilde{F}_{\text{RSS}}$ and the vertical axis gives the corresponding borehole curvature $\bar{K}$. The line style refers to the solution's mode and the black circles indicate a mode transition.

The left subplot of Figure Appendix D.1, corresponding to QSSs for $\Theta^{\bar{K}} = 0$, is analyzed first. For this inclination, there is no influence of gravity. The solutions between the black circles (i.e., the solid lines) correspond to the non-ideal stabilizer being cleared from both walls. Here, a linear relationship between $\bar{K}$ and $\tilde{F}_{\text{RSS}}$ is observed. A kink in each $\eta\Pi$ curve occurs at a mode transition, i.e., a black circle, where a new branch of solutions starts in the $\tilde{F}_{\text{RSS}}$ direction. This is due to the unilateral contact of the non-ideal stabilizer with the borehole walls, which brings in an additional influence on the borehole curvature. The direction of the kink is determined by the influence of the contact force on the curvature, which is almost zero for $\eta\Pi = 0.6$. Hence, the kink in this curve is relatively small and hardly distinguishable in these plots.

Next, the right subplot of Figure Appendix D.1 is analyzed, where the inclination $\Theta^{\bar{K}} = \pi/4$ is considered. In

Figure Appendix D.1: QSSs for the two-stabilizer model for inclinations $\Theta^{\bar{K}} = 0$ (left) and $\Theta^{\bar{K}} = \pi/4$ (right) without bit tilt saturation. The line style corresponds to the contact mode of the non-ideal stabilizer: Solid - cleared from both walls; Dashed - contact on one of the walls. The black circles are the transition points between branches of solutions.

contrast to the case $\Theta^{\bar{K}} = 0$, now a nonzero curvature is obtained for $\tilde{F}_{\text{RSS}} = 0$ (except for $\eta\Pi = \eta\Pi|_{\text{g}}^{(\text{nc,ns})} = 1$, which is discussed below). Again, in the interval between the black circles, the non-ideal stabilizer is cleared from both walls. For $\eta\Pi < \eta\Pi|_{\text{g}}^{(\text{nc,ns})} = 1$, gravity has a dropping influence on the curvature, i.e., coefficient $\tilde{c}_2^{(\text{nc,ns})} < 0$ in (36). Hence, not applying a RSS force results in a negative curvature, i.e., a borehole returning to a straight vertical one. For $\eta\Pi > \eta\Pi|_{\text{g}}^{(\text{nc,ns})} = 1$, gravity has a reversed influence resulting in a positive curvature since $\Theta^{\bar{K}} = \pi/4$. The asymmetric shifts of the black circles in the right subplot with respect to the left subplot is explained as follows. Gravity makes the BHA bend towards the lower borehole wall. Therefore, in comparison with $\Theta^{\bar{K}} = 0$, a larger RSS force is needed to make the non-ideal stabilizer contact the upper wall. On the other hand, since gravity already deflects the non-ideal stabilizer towards the lower wall, a less negative RSS force is needed to contact the lower wall.

In the critical gravity-independent case, i.e., $\eta\Pi = \eta\Pi|_{\text{g}}^{(\text{nc,ns})} = 1$, coefficient $\tilde{c}_2^{(m)}$ vanishes in (36). Only in this case, the obtained curvature is not an approximation but exact, since the influence of gravity does not enter in $\bar{K}$. When the non-ideal stabilizer is cleared from both walls, i.e., between the black circles, a linear relationship between $\bar{K}$ and $\tilde{F}_{\text{RSS}}$ exists. Therefore, for $\tilde{F}_{\text{RSS}} = 0$, the curvature is also zero in both the left and right subplot. Another critical case occurs for $\eta\Pi = \eta\Pi|_{\text{RSS}}^{(m)}$, where the influence of the RSS force vanishes in mode m . In this case, the actuator is ineffective, therefore, the obtained curvature only depends on the gravity forces. In both these cases, the respective forces, i.e., gravity and RSS, do affect the BHA deformation and, hence, the shape of the BHA, which determines the mode of contact and saturation.

From these results, it can be observed that the RSS actu-

Figure Appendix D.2: QSSs for the two-stabilizer model with bit tilt saturation boundary $\psi^* = \pm 0.5^\circ$. The line style indicates the mode: Solid - no contact, no saturation (in between the black marks, not visible); Dashed - contact, no saturation (between black and gray marks); Dashed dotted - no contact, saturation (between the green and blue marks); Dotted - contact, saturation (between the gray and green marks and between the blue marks and the limits of applied $\tilde{F}_{RSS}$). The colored circles represent the transition points between these branches.

ator is efficient for a small $\eta\Pi$, i.e., the coefficient $\tilde{c}_1^{(m)}$ is large for a small $\eta\Pi$. The parameter $\eta\Pi$ can, among others, be interpreted as the lateral steering resistance. Enlarging the steering resistance directly deteriorates the ability to drill large curvature boreholes.

Next, the case with bit tilt saturation is considered, where the saturation boundary is taken to be $\psi^* = 0.5^\circ$. Figure Appendix D.2 plots the QSSs calculated for $\Theta^{\tilde{K}} = 0$. Different line styles are used to indicate the contact mode of the non-ideal stabilizer and the saturation mode of the bit tilt.

The QSSs in between the gray circles (for each $\eta\Pi$) correspond to the case where the bit is not saturated. These solutions are identical to QSSs presented earlier without considering the bit tilt saturation. In this sense, the left subfigure of Figure Appendix D.1 is a zoomed plot of the origin of Figure Appendix D.2. Hence, these are not analyzed again here.

Applying a $\tilde{F}_{RSS}$ beyond the gray circles results in the bit tilt being saturated. Consider a positive $\tilde{F}_{RSS}$, result-

ing in a positive curvature, i.e., a borehole evolving counterclockwise, as shown in Figure Appendix D.3a. In this case, the non-ideal stabilizer contacts the ‘upper’ borehole wall and the bit tilt does not saturate. By applying an increased $\tilde{F}_{RSS}$, the BHA tends to deform more, causing the bit to rotate clockwise relative to the previous situation in Figure Appendix D.3a. This implies a smaller borehole curvature due to the saturated relation $\Theta_0 = \theta_0 + \psi^*$, see Figure Appendix D.3b, which explains the drop in borehole curvature despite the increase of $\tilde{F}_{RSS}$. Even negative borehole curvatures can be drilled with a (large) positive RSS force as depicted in Figure Appendix D.3c. When the bit tilt is saturated, an increase in $\tilde{F}_{RSS}$ results in an undesired effect, namely a decrease in borehole curvature. Therefore, it can be stated that a bit tilt saturation decreases the capability of drilling large curvature boreholes.

Results for other inclinations $\Theta^{\tilde{K}}$ are not presented as these are qualitatively similar to the solutions presented for $\Theta^{\tilde{K}} = 0$. For $\Theta^{\tilde{K}} \neq 0$, the QSSs shift with respect to $\Theta^{\tilde{K}} = 0$, both when the bit tilt is not saturated and when

Figure Appendix D.3: Three QSSs for $\eta\Pi = 0.4$ with inclination $\Theta^K = 0$ and bit tilt saturation boundary $\psi^* = 0.5^\circ$. The applied RSS force is increased from (a) to (b) to (c). The non-ideal stabilizer contacts the ‘upper’ borehole wall for all these solutions.

Figure Appendix D.4: Downward CISs as a function of the RSS force $\tilde{F}_{\text{RSS}}$ for $\eta\Pi = 0.1i$, for $i = 1, 2, \dots, 20$, for the two-stabilizer model without bit tilt saturation. The line style represents the contact mode of the solutions: Solid - cleared from both walls; Dashed - contact on either of the walls. The black circles represent the transition points between modes.

the bit tilt is saturated, which is an effect of gravity.

Remark 8. The interested reader is referred to [41] for a more extensive analysis. Among others, it is shown there that QSSs are unique in case the bit tilt is not saturated. Furthermore, a method to calculate the critical RSS force causing a switch in mode, i.e., the $\tilde{F}_{\text{RSS}}$ corresponding to the (colored) circles in Figure Appendix D.1 and Figure Appendix D.2, are given analytically.

Appendix D.2. Constant Inclination Solutions

This section presents CISs for the two-stabilizer model. Again, the analysis is given in a two-step fashion in order to make the role of the nonlinearities clear. First, the case without bit tilt saturation is considered. After that, the case treating both nonlinearities is presented.

Figure Appendix D.4 depicts the CISs for the two-stabilizer model without bit tilt saturation. The horizontal axis represents the applied $\tilde{F}_{\text{RSS}}$, while the vertical axis represents the inclination Θ^∞ of the obtained CISs. Again, the line style indicates the contact mode of the non-ideal stabilizer and the black circles indicate the transition points between modes.

For solutions in the RSS force interval between the black circles, the non-ideal stabilizer is cleared from both walls. For these branches of solutions, increasing the RSS force results in an increased Θ^∞ for $\eta\Pi < 1$. Exactly at $\eta\Pi = \eta\Pi_{\text{gr}}^{(\text{nc,ns})} = 1$, i.e., the gravity independent case, the coefficient $\tilde{c}_2^{(\text{nc,ns})} = 0$ in (38). Hence, the effect of gravity vanishes and the borehole inclination can be maintained without applying a RSS force. For $\eta\Pi > 1$, an increase in RSS force results in a decrease of borehole inclination. This is due to the ratio $-c_2^{(m)}/c_1^{(m)}$ being negative in this mode $m = \text{nc,ns}$. Increasing the amplitude of the RSS force beyond the black circles results in the non-ideal stabilizer contacting one of the borehole walls. Consequently,

each curve kinks. The sign of $-c_2^{(m)}/c_1^{(m)}$ in this new mode $m = \text{c,ns}$, determines the direction of the kink.

Transitioning through $\eta\Pi = 1$ triggers the existence of multiple downward (and upward) CISs for the same actuation force and $\eta\Pi$ setting. For example, consider the curve corresponding to $\eta\Pi = 1.1$ and observe that slope of this curve is negative in case the non-ideal stabilizer is cleared from the walls, but is positive when the non-ideal stabilizer contacts one of the borehole walls. This is a direct consequence of modeling the non-ideal stabilizer, which generates multiple branches of solutions. The following example gives insight in the existence of multiple CISs. Consider a zero RSS force and $\eta\Pi > 1$. All CISs should satisfy (38) for $\tilde{F}_{\text{RSS}} = 0$, i.e.,

$$\sin(\Theta^\infty) = -\frac{\tilde{d}_1^{(m)}}{\tilde{c}_2^{(m)}} \tilde{y}_1^*. \quad (\text{D.1})$$

A trivial solution is a straight vertical borehole with inclination $\Theta_2^\infty = 0$ as depicted in Figure Appendix D.5b, where $d_1^{(\text{nc,ns})} = 0$. Now, for the same $\tilde{F}_{\text{RSS}}$ and $\eta\Pi$, consider an inclination $\Theta_3^\infty > \Theta_2^\infty = 0$ as depicted in Figure Appendix D.5c. Since $\Theta_3^\infty > 0$, gravity deflects the BHA towards the lower borehole wall, where for a sufficiently large Θ_3^∞ , the non-ideal stabilizer contacts the lower borehole wall. In this case, (D.1) has a solution if the sign of the ratio $-\tilde{d}_1^{(m)}/\tilde{c}_2^{(m)}$ is positive. It can be said that the increasing effect of gravity is counteracted by the non-ideal stabilizer contacting the borehole wall. A similar reasoning can be applied for $\Theta_1^\infty < \Theta_2^\infty = 0$, see Figure Appendix D.5a. It should be noted multiple CISs are not restricted to $\tilde{F}_{\text{RSS}} = 0$, but also exist for $\tilde{F}_{\text{RSS}} \neq 0$ as is clearly visible in Figure Appendix D.4.

Next, we consider the two-stabilizer model with the bit tilt saturation in place. To illustrate the role of the bit

Figure Appendix D.5: Coexisting downward CISs the same $\tilde{F}_{RSS} = 0$ and $\eta\Pi$. A gray filled stabilizer contacts (one of) the borehole walls, while a transparent one is cleared from the walls.

tilt saturation in CISs, a conservative saturation value $\psi^* = 0.1^\circ$ is chosen. The results are depicted in Figure Appendix D.6 for $\eta\Pi = \{0.1, 0.5, 1.5, 2\}$. Each line style indicates a branch of solutions and the black circles indicate the transition between these branches of solutions. The solid and dashed lines represent solutions for which the bit tilt does not saturate. These solutions are equivalent to those in Figure Appendix D.4, i.e., for a model without bit tilt saturation, and, hence, not analyzed again.

It can be observed in Figure Appendix D.6 that when the bit tilt saturates, for a range of $\tilde{F}_{RSS}$, the same CISs exist independently of $\eta\Pi$. These solutions are represented by the dotted and dashed dotted lines. It can be shown that when the bit tilt saturates, the expression in (38) indeed becomes $\eta\Pi$ -independent. For the two-stabilizer model, observe that parameter $\eta\Pi$ drops out of the ratios $\tilde{c}_1^{(m)}/\tilde{c}_2^{(m)}$ and $\tilde{d}_i^{(m)}/\tilde{c}_2^{(m)}$, for the modes where the bit tilt saturates, see Appendix C. Equation (38) being $\eta\Pi$ -independent when the bit tilt saturates is true for any n -stabilizer model [41].

Figure Appendix D.6 shows that there are no solutions for which the non-ideal stabilizer contacts the borehole walls for $\eta\Pi = 2$. A relatively large $\tilde{F}_{RSS}$ is needed to make the non-ideal stabilizer contact one of the borehole walls. However, applying such a large $\tilde{F}_{RSS}$ results in a saturation of the bit tilt. For $\eta\Pi = 1.5$, solutions exist for which the non-ideal stabilizer contacts the borehole walls, both with and without the bit tilt saturating. The dotted and dashed line lie almost on top of each other and are, therefore, hardly distinguishable in Figure Appendix D.6 for this specific $\eta\Pi$.

For the case without bit tilt saturation, we observed that multiple solutions exist for $\eta\Pi > \eta\Pi_{\mathcal{G}}^{(nc,ns)} = 1$ for the same $\eta\Pi$ and RSS force. By incorporating the bit tilt nonlinearity, multiple solutions exist even for $\eta\Pi \leq 1$. For example, consider Figure Appendix D.7, where the BHA deflection inside the borehole is depicted for multiple coexisting CISs for the same RSS actuation $\tilde{F}_{RSS} = 0.00125$ and for $\eta\Pi = 1.5$. Each of these solution works in a different mode.

It can be observed that, irrespectively of $\eta\Pi$, horizon-

tal boreholes (with inclination $\pm\pi/2$) can be drilled when the bit tilt saturates. Furthermore, for a sufficiently small $\eta\Pi$, for example $\eta\Pi = 0.1$ and $\eta\Pi = 0.5$, there are two more horizontal CISs. In the first solution, the bit tilt is not saturated, but the non-ideal stabilizer contacts one of the borehole walls. In the second solution, the bit tilt is saturated and the non-ideal stabilizer contacts one of the borehole walls. For $\eta\Pi = 0.5$, the deflection of the BHA inside the borehole is depicted in Figure Appendix D.8 for all horizontal solutions with inclination $\pi/2$.

Remark 9. An extended analysis can be found in [41]. There, analytical results are given for the RSS forces which (i) make the bit tilt change mode; (ii) make the non-ideal stabilizers change mode; and (iii) drill a horizontal borehole. The calculation of these RSS forces relies on solving an LCP which is derived analytically. This LCP returns the contact forces at the non-ideal stabilizers and the bit tilt in these specific cases.

The presented analytical result in Section 7 and qualitative results in this appendix show the important role of both nonlinearities in the large scale response. The existence of multiple branches of QSSs and CISs, and the existence of multiple CISs for the same RSS actuation are directly related to these imposed nonlinearities.

Appendix E. State-Space Matrices

The state-space formulation in (39) contains matrices A_i, D_i , for $i = 0, 1, \dots, n$, and matrices B, C_1 and C_2 . These matrices are given by

$$A_i = \begin{cases} \begin{bmatrix} -\bar{a}, & a_1, & \dots, & a_n \\ 1, & 0, & \dots, & 0 \\ & 0_{n-1, n+1} \end{bmatrix} & \text{for } i = 0, \\ \begin{bmatrix} & 0_{i, n+1} \\ -\frac{1}{\lambda_{(i)}}, & 0, & \dots, & 0 \\ \frac{1}{\lambda_{(i+1)}}, & 0, & \dots, & 0 \\ & 0_{n-i-1, n+1} \end{bmatrix} & \text{for } 0 < i < n, \\ \begin{bmatrix} & 0_{n, n+1} \\ -\frac{1}{\lambda_{(n)}}, & 0, & \dots, & 0 \end{bmatrix} & \text{for } i = n, \end{cases}$$

$$B = \begin{bmatrix} b_1, & \dots, & b_{n-1} \\ & 0_{n, n-1} \end{bmatrix},$$

$$C_i = \begin{bmatrix} c_i, \\ 0_{n, 1} \end{bmatrix} \text{ for } i = 1, 2,$$

Figure Appendix D.6: Downward CISs for the two-stabilizer model for $\eta\Pi = \{0.1, 0.5, 1.5, 2\}$ and a bit tilt saturation boundary $\psi^* = 0.1^\circ$. The line styles indicate branches of solutions: Solid - no contact, no saturation; Dashed - contact, no saturation; Dashed dotted - no contact, saturation; Dotted - contact, saturation. The black circles are the transition points between the branches of solutions.

Figure Appendix D.7: Coexisting CISs for $\tilde{F}_{RSS} = 0.00125$ and $\eta\Pi = 1.5$. A gray filled stabilizer contacts (one of) the borehole walls, while a transparent one is cleared from the walls.

Figure Appendix D.8: Three horizontal CISs for $\eta\Pi = 0.5$ with inclination $\pi/2$. A gray filled stabilizer contact the lower borehole wall; a transparent one is cleared from both walls. A different RSS force is applied in each solutions.

$$D_i = \begin{cases} \begin{bmatrix} 0 \\ -1 \\ 0_{n-1} \end{bmatrix} & i = 0, \\ \begin{bmatrix} 0_k \\ \frac{1}{\lambda_{(i)}} \\ -\frac{1}{\lambda_{(i+1)}} \\ 0_{n-i-1} \end{bmatrix} & 0 < i < n, \\ \begin{bmatrix} 0_n \\ \frac{1}{\lambda_{(n)}} \end{bmatrix} & i = n. \end{cases}$$

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