Topological stability of kinetic k-centers

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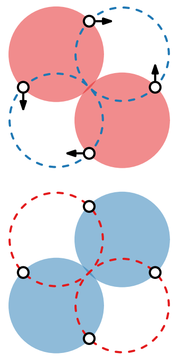
1 Introduction

The *k-center problem* or *facility location problem* asks for a set of k disks that cover a given set of n points, such that the maximum radius of all the disks is as small as possible. Since the introduction of the k-center problem by Sylvester [3] in 1857, the problem has been widely studied and has found many applications in practice.

In recent decades there has been an increased interest, especially in the computational geometry community, to study problems for which the input points are moving, including the k-center problem. These problems are typically studied in the framework of *kinetic data structures* [1], where the goal is to efficiently maintain the (optimal) solution to the problem as the points are moving.



In many practical applications, for example if the disks are represented physically, or if the disks are used for visualization, the disks should move smoothly as the points are moving smoothly. As the optimal k-center (for $k \geq 2$) may exhibit discontinuous changes as points move (see figure), we need to resort to approximations to guarantee stability.

Recently, Meulemans et al. [2] introduced a new framework for algorithm stability, which includes the definition of *topological stability.* An algorithm is topologically stable if its output behaves continuously (albeit with arbitrary speed) as the input is changing. The topological stability ratio ρ_{TS} of a problem is then defined as the ratio between the quality of a topologically stable solution and an optimal but unstable solution. In [2] bounds on ρ_{TS} are given for kinetic Euclidean minimum spanning trees, using various ways of enforcing continuity.

In this abstract we prove the following theorem on the topological stability of k-center.

▶ **Theorem 1.** *For the k-center problem it holds that* $2\sin(\frac{\pi(k-1)}{2k}) \leq \rho_{\mathrm{TS}} \leq 2$ *for* $k \geq 2$.

2 Bounds on topological stability

As illustrated above, some point sets have more than one optimal solution. If we can *transform* an optimal solution into another, by growing the covering disks at least/at most a factor r, we immediately obtain a lower/upper bound of r on the topological stability. The transformations or *morphs* allow (the centers of the) disks to move and radii to change continuously, as long as the points are covered at all times. We first introduce some tools to help us model and reason about these transformations.

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2-colored intersection graphs. Consider a point set P and two sets of k disks, such that each set covers all points in P: we use R to denote the one set (red) and B to denote the other set (blue). We now define the *2-colored intersection* graph $G_{R,B} = (V, E)$: each vertex represents a disk ($V = R \cup B$) and is either red or blue; E contains an edge for each pair of differently colored, intersecting disks. A 2-colored intersection graph always contains equally many red nodes and blue nodes by definition and both colors must cover all points: there may be points only in the area of intersection between a blue and red disk. In the remainder, we use *intersection graph* to refer to 2-colored intersection graphs.

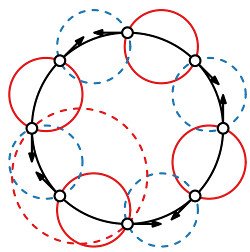
▶ **Lemma 2.** *Let sets R and B of k disks each cover a point set P. If $G_{R,B}$ is a forest, then R can morph onto B without increasing the disk radius, while covering all points in P.*

Proof. From a counting argument, using $|R| = |B|$, it follows that we can always find a red leaf in $G_{R,B}$. This red leaf can then morph onto its blue neighbor. This effectively removes these two nodes from the intersection graph, since the blue disk is fully covered by the red disk; repeating this argument gives a morph from R onto B. ◀

Proving Theorem 1. We are now ready to deal with the k-center problem. The upcoming lemmata each prove one part.

▶ **Lemma 3.** *For the k-center problem it holds that $\rho_{\mathrm{TS}} \geq 2\sin(\frac{\pi(k-1)}{2k})$ for $k \geq 2$.*

Proof. Consider a set of $2k$ points, which are the corners of a regular $2k$-gon with unit radius, i.e., equidistantly spread along the boundary of a unit circle. There are exactly two optimal solutions R and B on these points, for which $G_{R,B}$ forms a cycle (see figure). To morph from R to B, one of the red disks r_1 has to grow to cover the intersection of an adjacent blue disk b with the other (red) neighbor r_2 of b (see dashed red disk). The diameter of the disks in our optimal solution equals the length of a side of this regular $2k$-gon, hence r_1 has to grow with a factor $2\sin(\frac{\pi(k-1)}{2k})$. Once r_1 has grown to overlap the intersection between a blue disk and r_2, r_2 no longer has to cover the points in the intersection and can be treated as a degree-1 vertex in $G_{R,B}$. Since that makes $G_{R,B}$ a tree, we can apply Lemma 2.



If we can show that a set of *moving* points actually forces this swap to happen, the desired bound on the topological stability follows from the above argument. We can place points moving on tangents of the circle defining the $2k$ points, to arrive at the described situation at a time t, while ensuring that a swap before or after t would be only more costly. ◀

▶ **Lemma 4.** *For the k-center problem it holds that $\rho_{\mathrm{TS}} \leq 2$ for $k \geq 2$.*

Proof. Consider a moment in time t where there are two optimal solutions; let R denote the optimal solution at $t - \varepsilon$ and B the optimal solution at $t + \varepsilon$ for arbitrarily small $\varepsilon > 0$. Let C be the maximum radius of the disks in R and in B and let $G_{R,B}$ describe their intersections. First we make a maximal matching between red and blue vertices that are adjacent in $G_{R,B}$. The intersection graph of the remaining red and blue disks has no edges, so we match these red and blue disks in any way. All the red disks that are matched to blue disks they already intersect grow to overlap their initial disk and the matched blue disk. Now the remaining red disks can safely move to the blue disks they are matched to, and adjust their radii to fully cover the blue disks. Finally, to finish the morph all red disks shrink. When all red disks are overlapping blue disks, the maximum of their radii is at most $2C$. ◀

References

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