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Abstract

A non-Laplacian integral is an integral of which a complete asymptotic expansion cannot be obtained by the so-called "Method of Laplace". A subclass of these non-Laplacian integrals is shown to have an asymptotic expansion in terms of (mostly) nonelementary functions. These functions pose new asymptotic problems which in many cases are easier to handle than the original problem. For example, a complete asymptotic expansion can be derived for \( \int_0^1 (\epsilon + f(x))^\alpha dx \) where \( \alpha \in \mathbb{R} \) and \( f \in C[0, 1], f(x) \approx x^{\beta_1} + ax^{\beta_2} + \ldots \ (x \downarrow 0) \) with increasing \( \beta_1 > 0 \).

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INTRODUCTION

The asymptotics of many integrals can be determined by the so-called "method of Laplace" (See [2], ch.5 and [3], ch.4). An essential feature of these integrals is the property that the asymptotic behaviour is determined by the contribution of a (in most cases relatively small) part of the interval of integration on which the integrand can be approximated by a simple function which allows integration in terms of simple functions. More precisely, let

\[ F(t) = \int_a^b \phi(x, t) \, dx \]

be the function of which the asymptotic behaviour has to be determined for \( t \to \infty \). The bounds \( a \) and \( b \) can be dependent of \( t \), \( a \) may be \( \infty \) and \( b \) may be \( -\infty \). Suppose that \( J \subseteq (a, b) \), such that

\[ F(t) \sim \int_J \phi(x, t) \, dx \quad (t \to \infty) \]
and that furthermore \( \phi(x, t) \) can be approximated on \( J \) by a function \( \psi(x, t) \) in such a way that

\[
\int_J \phi(x, t) \, dx \sim \int_J \psi(x, t) \, dx
\]

and the asymptotic behaviour of \( \int_J \psi \) is rather simple to determine, say \( \int_J \psi(x, t) \, dx \sim A(t) \). Then clearly

\[
F(t) \sim A(t) \quad (t \to \infty).
\]

Remark. Of course the original interval \((a, b)\) of integration may contain more than one subinterval for each of which the above procedure has to be carried out.

In many cases the difference of the original and approximating integral is, with a relatively small error, again an integral which can be handled by the foregoing method. If the repeated execution of this procedure never breaks down, then in principle a complete asymptotic expansion can be obtained. In many cases it is clear from the start that the above procedure never fails; then often it is possible to short-cut the whole process and to find in one step the complete asymptotic expansion. Many integrals can be treated in this way. A few examples for \( t \to \infty \) are

\[
\Gamma(t + 1) = \int_0^\infty e^{-x} x^t \, dx, \quad \int_0^1 \left(\frac{\sin x}{x}\right)^t \, dx, \quad \int_0^\infty \exp\left(-\frac{t}{2}x^2\right) \, dx.
\]

But there are integrals where the above procedure breaks down; i.e., at the very start or after a few steps of the above procedure one keeps up with an integral for which the Method of Laplace does not work any more. The cause of the trouble is in most of such cases the fact that a relatively large part of the whole interval of integration is contributing to the terms of the asymptotic expansion. Such integrals will be called non-Laplacian integrals. The occurrence of such integrals in applications will be the subject of the next section.

A few examples of non-Laplacian integrals are

\[
I(t) := \int_0^\infty e^{-tx^2}(1 + tx)^{-1} \, dx \quad (t \to \infty), \quad J(t) := \int_0^1 (t + \sin x^2)^{-1} \, dx \quad (t \downarrow 0),
\]

where \((t \to \infty)\) or \((t \downarrow 0)\) indicates the asymptotic problem.

The class of NON-LAPLACIAN INTEGRALS of which the asymptotics for \( \varepsilon \downarrow 0 \) will be studied is the union of the following two classes of integrals (1) and (2). The main interest concerns integrals of type (1), which in certain circumstances can be reduced to integrals of type (2).
The first class consists of the integrals

\[
\int_0^1 (\varepsilon + f(x) + g(x) + \varepsilon h(x)) p(x) \, dx,
\]

where \(f, g, h, \Phi\) and \(p\) satisfy the following conditions:

- \(f\) is a non-decreasing continuous function on \([0, 1]\) with \(f(0) = 0\) and \(f\) is positive on \((0, 1]\);
- \(g\) is a continuous function on \([0, 1]\) with \(g(x) = o(f(x))\) \((x \downarrow 0)\);
- \(h\) is a continuous function on \([0, 1]\) such that \(h(x) = o(1)\) \((x \downarrow 0)\);
- \(\varepsilon + f(x) + g(x) + \varepsilon h(x) > 0\) \((0 < x \leq 1, \varepsilon \leq \varepsilon_0)\) for some positive \(\varepsilon_0\);
- \(\Phi\) is \(C^\infty\) on an interval \((0, a]\) where \(a\) is large enough in order for \(\Phi(\varepsilon + f(x) + g(x) + \varepsilon h(x))\) and \(\Phi((\varepsilon + f(x))/ (1 - u(x)))\) with \(u := g/(f + g), u(0) := 0\) to be defined for \(0 \leq x \leq 1, 0 < \varepsilon \leq \varepsilon_0\), with the further property that for every \(k \in \{0, 1, 2, \ldots\}\)
  \(\forall x \in (0, a], \Phi^{(k)}(x) \neq 0\), and \(x \left| \frac{\Phi^{(k+1)}(x)}{\Phi^{(k)}(x)} \right| \text{ is bounded on } (0, a]\);
- \(p\) is a continuous function on \((0, 1)\) such that \(\int_0^1 |p(x)| \, dx \text{ exists}\).

Remark. Since "\(\Phi(k)(x)\) is identically zero on \((0, a]\) for some \(k > 0\)" leads to the trivial case \(\Phi(x) = ax^l\) with \(l\) a positive integer, this possibility is excluded.

Remark. The method was developed originally for integrals with \(\Phi(x) = x^{-\alpha}\) with \(\alpha\) not a negative integer and \(f(x) = x^\beta\). Such functions \(\Phi\) and \(f\) are multiplicative. Theorem (6) holds for the larger class of functions \(\Phi\) and \(f\) as described above in the definition of Class (1). In the case that \(\Phi\) is multiplicative the integral (1) can be written as a sum of integrals all of which are of type (1) with \(g(x) = 0\) and \(h(x) = 0\) according to Theorem (6). If \(f\) is also multiplicative then these latter integrals can be written in the form \(\int_0^1 p(x)f(x)K(\varepsilon/x)dx\); such integrals can be calculated asymptotically as explained in Theorem (8). These integrals constitute the class (2) which will be defined after the following remark.

Remark. The above integral (1) is not the most general case for which (eventually slightly modified) Theorem (6) holds. For instance, the function \(g\) may also depend on \(\varepsilon\), i.e. \(g(x)\) in the integral may be replaced by \(g(x, \varepsilon)\). The latter function must then satisfy the condition \(g(x, \varepsilon) = o(f(x))\) \((x \downarrow 0)\) uniformly in all positive \(\varepsilon\) sufficiently small, say \(\varepsilon \in (0, \varepsilon_0)\) with \(\varepsilon_0 > 0\). Also \(h\) may depend on \(\varepsilon\). For instance \(h(x, \varepsilon) = \varepsilon h_1(x) + \varepsilon^\beta h_2(x)\) with \(\beta > 1\), \(h_1(x) = o(1)\) \((x \downarrow 0)\) and \(h_2(x) = O(1)\) \((x \downarrow 0)\) is possible. The function \(p\) also allows for generalizations; for instance \(p(x, \varepsilon) = \sum_0^\infty p_k(x)\varepsilon^k\) is an obvious possibility. To make things not to complicated only integrals of type (1) will be considered; adaptations for more general cases are rather obvious.
The second class of integrals consists of

\begin{equation}
\int_0^1 p(x)f(x)K\left(\frac{\varepsilon}{x}\right) \, dx ,
\end{equation}

where the functions $p, f, K$ satisfy the conditions:

$f \in C[0, 1]$ such that $f(x) \approx \sum_0^\infty a_k x^{\alpha_k} \; (x \downarrow 0)$, where $0 \leq \alpha_0 < \alpha_k < \alpha_{k+1}$ \; ($k > 0$) and $\alpha_k \to \infty$ \; ($k \to \infty$).

$K \in C[0, \infty)$ such that $K(y) \approx \sum_0^\infty b_k y^{\beta_k} \; (y \downarrow 0)$, where $0 \leq \beta_k < \beta_{k+1}$ \; ($k > 0$) and $\beta_k \to \infty$ \; ($k \to \infty$).

$p \in C(0, 1)$ such that $\int_0^1 |p(x)| \, dx$ exists.

Moreover $p, K$ together satisfy the condition (*) : $\int_0^1 |p(x)K(\varepsilon/x)| \, dx$ exists and is uniformly bounded for all $\varepsilon > 0$ sufficiently small.

**Remark.** Integrals of type (2) are treated in [2],[4] and [9] : see the second remark after (13). Several generalizations of (2) are possible. For instance the function $K$ may depend on $\varepsilon$. For example the coefficients $b_k$ may be asymptotic powerseries in $\varepsilon$. This is the case in the second example in the section OCCURRENCES. We do not try to formulate a very general version of Theorem (8) ; in many cases the necessary modifications are obvious.

**Remark.** If $|xp(x)|$ is nondecreasing on some interval $(0, \gamma)$ with $\gamma > 0$ and $\int_0^1 |p(x)K(1/x)| \, dx$ exists then condition (*) in the definition of class (2) is satisfied.

**Examples of class (1) integrals**

\[ \int_0^1 \frac{dx}{\varepsilon + x + x^3} \; \left( \Phi(x) = \frac{1}{x}, f(x) = x, g(x) = x^3, h(x) = 0, p(x) = 1 \right) . \]

\[ \int_0^1 \frac{dx}{x + \varepsilon e^x} \; \left( \Phi(x) = \frac{1}{x}, f(x) = x, g(x) = x(e^x - 1), h(x) = 0, p(x) = e^{-x} \right) . \]

\[ \int_0^1 \frac{dx}{\sqrt{\varepsilon + \sin x^2}} \; \left( \Phi(x) = \frac{1}{\sqrt{x}}, f(x) = x^2, g(x) = \sin x^2 - x^2, h(x) = 0, p(x) = 1 \right) . \]

**Examples of class (2) integrals**

\[ \int_0^1 e^{-\frac{\varepsilon}{x^2}} \, dx \; \left( p(x) = 1, f(x) = e^{-\frac{1}{2}x^2}, K(y) = e^{-y} \right) . \]
\[ \frac{1}{\varepsilon} \int_{0}^{1} e^{-x^2} \frac{dx}{\varepsilon + x} \quad (p(x) = 1, f(x) = e^{-x^2}, K(y) = \frac{y}{1 + y}). \]

This paper is organized as follows:
After this INTRODUCTION follows a section OCCURRENCES where several non-Laplacian integrals are given which are of importance in physical applications. Then follows a short section PREREQUISITES which gives the necessary notions in asymptotics in order for the main theorems in the section "RESULTS" to be understandable. The section "APPLICATIONS" presents a few worked out examples. In the section "PROOFS" one can find the proofs of the main theorems. Finally there follows a section QUESTIONS with some open problems about integrals of type (1).

**OCCURRENCES**

In [8] are studied approximations to the Drift-Diffusion equations in semiconductors. On page 48, formula (3.53), the following integral \( I_1(x) \) occurs of which the asymptotic behaviour for \( x \downarrow 0 \) is needed.

\[ I_1(x) := \int_{0}^{1} \frac{ds}{\sqrt{s - 1 + e^{-s} + x}} ds \quad (x > 0). \]

The author of [4] is able to derive a complete asymptotic expansion of \( I_1(x) \) by means of the Mellin transform. This integral \( I_1(x) \) originates from (1) with \( \Phi(x) = 1/\sqrt{x} \), \( f(x) = \frac{1}{2}x^2 \), \( g(x) = e^{-x} + x - 1 - \frac{1}{2}x^2 \), \( h(x) = 0 \), \( p(x) = 1 \). Hence it is possible to find a complete asymptotic expansion by using Theorem (6) from the section RESULTS. The occurring functions \( F_k(x) \) can be handled by Theorem (8).

In [5] appears the integral \( I_0^+ \) which from an asymptotical point of view essentially resembles an integral of the following type

\[ I := \int_{0}^{1} \frac{e^{i\sqrt{\varepsilon^2 + x^2}}}{\sqrt{x^2 + \varepsilon^2}} F(x) \, dx. \]

The author of [3] is able to expand \( I \) asymptotically up to \( O(\varepsilon^3) \) by using a method invented in [7]. Taking in (1) \( \Phi(x) = e^{i\sqrt{x}/\sqrt{2}}, f(x) = x^2, g(x) = 0, h(x) = 0, p(x) = F(x) \) and replacing \( \varepsilon \) by \( \varepsilon^2 \) one gets the above integral \( I \). Using that (with \( y := \varepsilon/x) \)

\[ \sqrt{\varepsilon^2 + x^2} - x = \frac{\varepsilon y}{\sqrt{1 + y^2} + 1}, \]
in the exponent of the integrand of I we can write

$$I = \frac{1}{\varepsilon} \int_{0}^{1} e^{ix} F(x) K(\varepsilon/x) \, dx,$$

where

$$K(y) := \frac{y}{\sqrt{1 + y^2}} \left( 1 + \exp \left( \frac{i\varepsilon y}{\sqrt{1 + y^2 + 1}} \right) \right).$$

In this last form it can be handled by Theorem (8) ; the extra $\varepsilon$ in $K(y)$ poses no problem.

In [1] and [6] the problem of the asymptotics of the function

$$G(t) := \int_{0}^{\infty} e^{-\frac{t^2}{\varepsilon^2} - \frac{1}{2} x^2} \, dx$$

which plays an important role in percolation theory, comes up. The asymptotic behaviour of $G(t)$ for $t \to \infty$ is easily determined with the method of Laplace. The asymptotics for $t \downarrow 0$ can be found by splitting the integral in two parts: $\int_{0}^{1}$ and $\int_{1}^{\infty}$. The second integral is easily handled by expanding $e^{-t/x}$; the integral $\int_{0}^{1}$ can be treated straightforward by application of Theorem (8).

Elliptic integrals play an important role in many applications. For example a complete asymptotic expansion for $\varepsilon \downarrow 0$ of the integral

$$\int_{0}^{1} \frac{dx}{\sqrt{\varepsilon + (\sin x)^2}}$$

can be derived. This integral is of class (1) with $\Phi = 1/\sqrt{x}$, $f(x) = x^2$, $g(x) = (\sin x)^2 - x^2$, $h(x) = 0$, $p(x) = 1$.

**PREREQUISITES**

The necessities from asymptotics are the $\mathcal{O}$, $o$ and $\sim$ symbols and the notions of asymptotic sequence and asymptotic series in the simple case of real functions on $\mathbb{R}$.

Notation: $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$.

Let $f(x)$ and $g(x)$ be defined on $I := (0, a)$ with $a > 0$. Then

$$f(x) = \mathcal{O}(g(x)) \quad (x \in I) \text{ means: } \exists_{B > 0} |f(x)| \leq B|g(x)| \quad (x \in I);$$
\[ f(x) = O(g(x)) \quad (x \downarrow 0) \text{ means: } \exists B > 0 \ \exists \delta > 0 \ |f(x)| \leq B|g(x)| \quad (x \in (0, \delta)); \]

\[ f(x) = o(g(x)) \quad (x \downarrow 0) \text{ means: } \forall \varepsilon > 0 \ \exists \delta > 0 \ |f(x)| \leq \varepsilon |g(x)| \quad (x \in (0, \delta)); \]

\[ f(x) \sim g(x) \quad (x \downarrow 0) \text{ means: } f(x) - g(x) = o(g(x)) \quad (x \downarrow 0). \]

A sequence \((\phi_n)\) of functions all of which are defined on at least an interval \((0, a)\) with \(a > 0\) is called an asymptotic sequence for \(x \downarrow 0\) iff

\[ \forall n \in \mathbb{N} \quad [\phi_n(x) \neq 0 \quad (x \in (0, a)) \land \phi_{n+1}(x) = o(\phi_n(x)) \quad (x \downarrow 0)]. \]

Let \(F\) be a function defined on an interval \((0, a)\) with \(a > 0\). Let \((\phi_n)\) be an asymptotic sequence for \((x \downarrow 0)\). Then

\[ F(x) \approx \sum_{k=1}^{\infty} f_k(x) \quad (x \downarrow 0), \quad (\phi_n), \]

means: There is a sequence \((f_k)\) of functions defined on some interval \((0, b)\) with \(b > 0\) such that

\[ (*) \quad \forall N \in \mathbb{N}_0 \quad F(x) = \sum_{k=1}^{N} f_k(x) + O(\phi_{N+1}(x)) \quad (x \downarrow 0). \]

One says that \(F(x)\) has an (extended) asymptotic expansion with respect to \((\phi_n(x))\) for \(x \downarrow 0\). From this definition it follows that \(\forall k \in \mathbb{N} \quad f_k(x) = O(\phi_k(x)) \quad (x \downarrow 0). \)

If the development \((*)\) holds only for a fixed value of \(N\), then one says that \(F(x)\) has an asymptotic expansion to \(N\) terms.

If the functions \(f_k\) are equal to \(c_k\phi_k\), where the \(c_k\) are constants, then one writes

\[ F(x) \approx \sum_{k=1}^{\infty} c_k \phi_k(x) \quad (x \downarrow 0), \]

and one says that \(F(x)\) has an asymptotic expansion (with respect to \((\phi_k)\) for \(x \downarrow 0)\).

**RESULTS**

The method to tackle the integrals of type \((1)\), especially if \(\Phi\) is multiplicative, is as follows:

Firstly the integral is split up in integrals \(F_k(\varepsilon)\) defined for all \(k \in \mathbb{N}_0\) by

\[ F_k(\varepsilon) := \varepsilon^k \int_0^1 (\bar{u}(x))^{k-1} \Phi(\varepsilon) \left(\frac{\varepsilon + f(x)}{1 - u(x)}\right) p(x) \, dx, \]

where \(u := g/(f + g), \bar{u} := \bar{h} - h, \bar{h} := g/f\).

Secondly in the case that both \(\Phi\) and \(f\) are multiplicative these integrals can be reduced to integrals of type \((2)\) which can be handled iteratively as described in Theorem (8).
Theorem

Let \( f, g, h, \Phi \) and \( p \) satisfy the conditions described in the definition of class (1). Then the sequence of functions \( F_k(\varepsilon) \quad (k \in \mathbb{N}_0) \), defined by

\[
F_k(\varepsilon) := \varepsilon^k \int_0^1 \left( \frac{(u(x))^k}{1-u(x)} \right) \cdot \left( \frac{\varepsilon + f(x)}{1-u(x)} \right) \left| p(x) \right| \, dx ,
\]

forms an asymptotic sequence for \( \varepsilon \downarrow 0 \).

Theorem

Under the same conditions on \( f, g, h, \Phi \) and \( p \) as in Theorem (4)

\[
\int_0^1 \Phi(\varepsilon + f(x) + g(x) + \varepsilon h(x)) \, dx \approx \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cdot F_k(\varepsilon) \quad (\varepsilon \downarrow 0) , \quad (F_k).
\]

For application of formula (7), the functions \( F_k(\varepsilon) \) are to be calculated asymptotically. In the case that both \( \Phi \) and \( f \) are power functions, i.e. \( \Phi(x) = x^\beta \), and \( f(x) = x^\gamma \), the functions \( F_k(\varepsilon) \) can be written in the form (9) in Theorem (8), whence the asymptotic behaviour of the functions \( F_k(\varepsilon) \) can be determined by means of formula (10) or (13).

Theorem

Let \( p, f, K \) satisfy the conditions described in the definition of class (2). Then clearly

\[
I(\varepsilon) := \int_0^1 p(x) f(x) K\left( \frac{\varepsilon}{x} \right) \, dx = \mathcal{O}(1) \quad (\varepsilon \downarrow 0) .
\]

Further the following iterative formula holds:

\[
\int_0^1 p(x) f(x) K\left( \frac{\varepsilon}{x} \right) \, dx =
\]

\[
K(0) \int_0^1 p(x) f(x) \, dx + f(0) \int_0^1 p(x) \left( K\left( \frac{\varepsilon}{x} \right) - K(0) \right) \, dx +
\]

\[
\varepsilon^\gamma \int_0^1 p(x) \frac{\partial f(x)}{dx} K\left( \frac{\varepsilon}{x} \right) \, dx,
\]

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where $\tilde{f}(x) := (f(x) - f(0))x^{-\gamma}$, $\tilde{K}(y) := (K(y) - K(0))y^{-\gamma}$, where $\gamma := \min\{\alpha_1, \beta_1\}$. Of course $\tilde{f}(0)$ and $\tilde{K}(0)$ are defined by taking limits. Clearly the functions $\tilde{f}$ and $\tilde{K}$ satisfy the same conditions as imposed on $f$ and $K$ with of course modified numbers $a_k, b_k, \alpha_k, \beta_k$ in the asymptotic expansions of $\tilde{f}$ and $\tilde{K}$.

Formula (10) has to be accompanied by an asymptotic formula for the integral in the third term of (10). The asymptotics for $\varepsilon \downarrow 0$ of

\begin{equation}
\int_0^1 p(x)K\left(\frac{\varepsilon}{x}\right) \, dx
\end{equation}

has to be determined. In the case that $p(x) = x^{-\alpha}$ with $0 < \alpha < 1$ condition (*) becomes

\begin{equation}
\int_1^\infty |K(y)|y^{-2+\alpha} \, dy < \infty
\end{equation}

and the asymptotics of (11) can be determined as follows: Write

\begin{equation}
\int_0^1 x^{-\alpha} K\left(\frac{\varepsilon}{x}\right) \, dx = \varepsilon^{1-\alpha} \left(\int_1^{\infty} + \int_1^{\varepsilon}\right) y^{\alpha-2} K(y) \, dy,
\end{equation}

substitute the asymptotic expansion of $K(y)$ into $\int_1^{\varepsilon}$ and integrate termwise.

There are more functions $p$ for which explicit asymptotics of (11) can be given. For instance this is also possible for functions $p(x) = x^\alpha (\log x)^n$ and linear combinations of such functions.

Defining sequences of functions $(f_n)$, $(K_n)$ and numbers $\gamma_n$ by

$$f_{n+1}(x) = \frac{f_n(x) - f_n(0)}{x^{\gamma_{n+1}}}, \quad f_0 = f; \quad K_{n+1}(y) = \frac{K_n(y) - K_n(0)}{y^{\gamma_{n+1}}}, \quad K_0 = K, \quad \gamma_0 = 0$$

where $\gamma_n$ is the maximal value such that $f_n$ and $K_n$ are continuous on $[0, 1]$. $f_n(0)$ and $K_n(0)$ are defined by taking limits for $x \downarrow 0$. Using repeatedly formula (10) one finds for every $N \in \mathbb{N}_0$ with $\delta_n := \sum_0^n \gamma_k$ :

\begin{equation}
\int_0^1 p(x)f(x)K(\varepsilon/x) \, dx =
\end{equation}

\begin{equation}
= \sum_{k=0}^N \left(K_k(0) \int_0^1 p(x)f_k(x) \, dx + f_k(0) \int_0^1 p(x)(K_k(\varepsilon/x) - K_k(0)) \, dx\right) \varepsilon^{\delta_n}
\end{equation}

\begin{equation}
+ \varepsilon^{\delta_{N+1}} \int_0^1 p(x)f_{N+1}(x)K_{N+1}(\varepsilon/x) \, dx.
\end{equation}
Remark. If one of the functions $f$, $K$ has only a finite asymptotic expansion of the kind
\[ \sum_{0}^{M} d_k \varepsilon^{\alpha_k} + \mathcal{O}(\varepsilon^{\alpha M + 1}) , \]
then formula (13) is of course only valid for $N \in \{0, 1, 2, \ldots, M\}$.

Remark. Integrals of the kind $\int_{0}^{\infty} f(x) K(x) \, dx$ are thoroughly treated via the Mellin
transform in [2], ch.4, section 6, in [4], the appendix, and in [9], ch.III. In [2] the functions $f$
and $K$ are allowed to have terms $e^{-\beta x - \alpha x \log x} n$ with $\beta \geq 0, \alpha \geq 0$ in their asymptotic expansions. Since it is possible to calculate the asymptotic behaviour of (11) if $p(x)$ is equal
to such a term we can derive results for $\int_{0}^{1} f(x) K(x) \, dx$ similar to the results in [2] via (10)
although it requires painstaking bookkeeping. The iterative formula (10) and also (13) seem new.

APPLICATIONS

(14) Application

As a first example the integral
\[ F(\varepsilon) := \int_{0}^{1} \frac{dx}{\varepsilon + x + x^3} \]
will be treated. The functions $\Phi$, $f$, $g$, $h$, $p$ are $\Phi(x) = 1/x$, $f(x) = x$, $g(x) = x^3$, $h(x) = 0$, $p(x) = 1$. Then $u(x) = g(x)/(f(x) + g(x)) = x^2/(1 + x^2)$ and $\tilde{u}(x) = x^2$. The functions $F_k$ are by definition (2):
\[ F_k(\varepsilon) = (-1)^k k! \varepsilon^k \int_{0}^{1} \frac{x^{2k}}{(1 + x^2)^{k+1}} \frac{1}{(\varepsilon + x)^{k+1}} \, dx. \]
The ratio $R$ of the integrands of $F_{k+1}$ and $F_k$ is $R = -(k + 1) \frac{\varepsilon x^2}{(1 + x^2)^2 (\varepsilon + x)}$ and it is clear
that $|R| < (k + 1) \varepsilon$. Hence $F_{k+1}(\varepsilon) = \mathcal{O}(\varepsilon F_k(\varepsilon))$ ($\varepsilon \downarrow 0$). If one decides to calculate the asymptotic behaviour up to terms which are $\mathcal{O}(\varepsilon^m)$ times the main term then it is not necessary to treat $F_k$ with $k > m$. For the choice $m = 2$ the calculations go as follows. Application of formula (13) in Theorem (8) to $\varepsilon F_0$ goes as follows:
\[ f_0 = -f_2 = \frac{1}{1 + x^2}, f_1 = -f_3 = -\frac{x}{1 + x^2}, K_0 = \frac{y}{1 + y}, K_1 = -K_2 = K_3 = \frac{1}{1 + y}, \]
\[ \int_{0}^{1} K_0(\varepsilon/x) \, dx = -\varepsilon \log \varepsilon + \varepsilon^2 - \frac{1}{2} \varepsilon^3 + \mathcal{O}(\varepsilon^4), \]
\[ \int_{0}^{1} f_1(x) \, dx = \int_{0}^{1} f_3(x) \, dx = -\frac{1}{2} \log 2, \]
\[ \int_{0}^{1} f_2(x) \, dx = \frac{\pi}{4}, -\int_{0}^{1} K_2(\varepsilon/x) \, dx = 1 + \varepsilon \log \varepsilon + \mathcal{O}(\varepsilon^2), \]
whence

\[ F_0 = -\log \varepsilon - \frac{1}{2} \log 2 + (2 + \frac{\pi}{4}) \varepsilon + \varepsilon^2 \log \varepsilon - \left( \frac{1}{2} \log 2 \right) \varepsilon^2 + O(\varepsilon^3 \log \varepsilon) \]

Similarly

\[ F_1 = -\frac{1}{4} \varepsilon - (\log 2 + \frac{1}{2}) \varepsilon^2 + O(\varepsilon^3 \log \varepsilon), \]

\[ F_2 = \frac{3}{8} \varepsilon^2 + O(\varepsilon^3 \log \varepsilon). \]

Substituting these results in \( F_0 + F_1 + F_2 + O(\varepsilon^3 \log \varepsilon) \) one gets

\[ F(\varepsilon) = -\log \varepsilon - \frac{1}{2} \log 2 + \left( \frac{\pi}{4} + \frac{7}{8} \right) \varepsilon + \varepsilon^2 \log \varepsilon - \left( \frac{5}{8} \log 2 \right) \varepsilon^2 + O(\varepsilon^3 \log \varepsilon) \quad (\varepsilon \downarrow 0). \]

**Remark.** The asymptotics for both \( t \downarrow 0 \) and \( t \to \infty \) of

\[ \int_{0}^{\infty} \frac{dx}{t + x + x^3} \]

are of non-Laplacian type and can be treated as follows. In both cases the integral is splitted
in two integrals \( I_1 = \int_{0}^{1} \) and \( I_2 = \int_{1}^{\infty} \).

The case \( t \downarrow 0 \): Then integral \( I_1 \) is treated above; \( I_2 \) is handled as follows

\[ I_2 = \int_{1}^{\infty} (u - tu^2 + t^2 u^3 - \ldots + O(t^N u^{N+1})) \, dx, \]

where \( u := (x + x^3)^{-1} \).
The case \( t \to \infty \): Then

\[ I_1 = \int_{0}^{1} (t^{-1} - t^{-2}v + t^{-3}v^2 - \ldots O(t^{-N-1}v^N)) \, dx, \]

where \( v := x + x^3 \). The integral \( I_2 \) is treated as follows: substitution of \( x = y^{-1} \), \( t = \varepsilon^{-3} \)
leads to the integral

\[ \varepsilon^3 \int_{0}^{1} \frac{p(y)}{\varepsilon^3 + y^3 + g(y)} \, dy, \]

where \( p(y) := y(1 + y^2)^{-1} \), \( g(y) := -y^5(1 + y^2)^{-1} \), \( h(y) = 0 \) which is covered by Theorems (6) and (8).
(15) **Application**

A second example is the integral

\[ F(\varepsilon) := \int_0^1 \frac{dx}{\varepsilon + x + x\sqrt{2}}. \]

The functions \( \Phi, f, g, h, p \) are \( \Phi(x) = 1/x, f(x) = x, g(x) = x\sqrt{2}, h(x) = 0, p(x) = 1 \). Then \( u(x) = x^\alpha/(1 + x^\alpha) \), \( u(x) = x^\alpha \), where \( \alpha = \sqrt{2} - 1 \). The functions \( F_k \) are

\[ F_k(\varepsilon) := (-1)^k k! \varepsilon^k \int_0^1 \frac{x^k}{(1 + x^\alpha)^k + (\varepsilon + x)^{k+1}} dx. \]

The ratio \( R \) of the integrands of \( F_{k+1} \) and \( F_k \) is \( -(k + 1)\varepsilon x^\alpha (1 + x^\alpha)^{-1}(\varepsilon + x)^{-1} \) from which it follows that \( |R| = O(\varepsilon^\alpha) \quad (\varepsilon \downarrow 0) \). Hence \( F_{k+1}(\varepsilon) = O(\varepsilon^\alpha F_k(\varepsilon)) \quad (\varepsilon \downarrow 0) \). If one decides to calculate the asymptotic behaviour up to terms which are \( O(\varepsilon^m F_0(\varepsilon)) \) then it is not necessary to treat \( F_k \) with \( k > m \). For the choice \( m = 2 \) the calculations go as follows. \( \varepsilon F_0 \) originates from (2) by \( p(x) = 1, f(x) = (1 + x^\alpha)^{-1}, K(y) = y/(1 + y) \). For use in (13) one calculates:

\( f_0 = -f_1 = f_2 = \frac{1}{1 + x^\alpha}, \quad f_3 = \frac{x^3 - 1}{1 + x^\alpha}, \quad f_4 = f_0, \)

\( K_0 = \frac{y}{1 + y}, \quad K_1 = \frac{y^{1-\alpha}}{1 + y}, \quad K_2 = \frac{y^{1-2\alpha}}{1 + y}, \quad K_3 = \frac{1}{1 + y}, \quad K_4 = \frac{s^{3-6\alpha}}{1 + s}, \)

\( \gamma_1 = \alpha, \gamma_2 = \alpha, \gamma_3 = 1 - 2\alpha, \gamma_4 = 3\alpha - 1, \delta_1 = \alpha, \delta_2 = 2\alpha, \delta_3 = 1, \delta_4 = 3\alpha. \)

\( \int_0^1 K_0(\varepsilon/x)dx = -\varepsilon \log \varepsilon + \varepsilon^2 + O(\varepsilon^3) \quad (\varepsilon \downarrow 0), \)

\( \int_0^1 K_1(\varepsilon/x)dx = \varepsilon \int_\varepsilon^\infty K_1(s)s^{-2}ds = \varepsilon \int_\varepsilon^\infty K_1(s)s^{-2}ds = \varepsilon \left( \int_\varepsilon^1 + \int_1^\infty \right) s^{-1-\alpha}(1 + s)^{-1}ds \)

\[ = A\varepsilon + \varepsilon \int_\varepsilon^1 s^{-1-\alpha}ds - \varepsilon(\int_0^1 + \int_\varepsilon^\infty) s^{-\alpha}/(1 + s)ds \]

\[ = A\varepsilon + \frac{1}{\alpha} \varepsilon^{1-\alpha} + \varepsilon \int_0^\infty s^{-\alpha}(1 - s + O(\varepsilon^2))ds \]

\[ = \frac{1}{\alpha} \varepsilon^{1-\alpha} + A\varepsilon + \frac{1}{1-\alpha} \varepsilon^{2-\alpha} + O(\varepsilon^{3-\alpha}) \quad (\varepsilon \downarrow 0), \]
Applying (13) one gets
\[ h \simeq -1 + 4 + \frac{2 - 2\alpha}{1 - 2\alpha} \varepsilon^{2\alpha} - \frac{1}{2 - 2\alpha} \varepsilon^{3 - 2\alpha} + \mathcal{O}(\varepsilon^{4 - 2\alpha}) \quad (\varepsilon \downarrow 0) , \]
where \( B = \int_0^\infty s^{-1 - 2\alpha} (1 + s)^{-1} ds - \int_0^1 s^{-2\alpha} (1 + s)^{-1} ds - 1/(2\alpha) \). And
\[ \int_0^1 K_4(\varepsilon/x) dx = C\varepsilon + \mathcal{O}(\varepsilon^{5 - 6\alpha}) \quad (\varepsilon \downarrow 0) . \]

Applying (13) one gets
\[ F_0 = -\log \varepsilon + C - A\varepsilon^\alpha + B\varepsilon^{2\alpha} + \frac{2 - 2\alpha}{1 - 2\alpha} \varepsilon + \mathcal{O}(\varepsilon^{3\alpha}) , \]
where \( C = \int_0^1 \frac{x^{3\alpha - 1} (1 + x^\alpha)^{-1} dx}{1 - \frac{1}{2\alpha}} \). Similarly
\[ F_1 = D\varepsilon^\alpha + E\varepsilon^{2\alpha} + \left( -\frac{1}{1 - \alpha} - \frac{4}{1 - 4\alpha} \right) \varepsilon + \mathcal{O}(\varepsilon^{3\alpha}) , \]
where \( D = -\int_0^\infty s^{-\alpha} (1 + s)^{-2} ds, E = 2\int_0^\infty s^{-2\alpha} (1 + s)^{-2} ds \). And
\[ F_2 = G\varepsilon^{2\alpha} + \mathcal{O}(\varepsilon^{2 - 3\alpha}) , \]
where \( G = -6\int_0^\infty s^{3 - 3\alpha} (1 + s)^{-3} ds \). Hence
\[ F = -\log \varepsilon + C + (D - A)\varepsilon^\alpha + (B + E + G)\varepsilon^{2\alpha} + L\varepsilon + \mathcal{O}(\varepsilon^{3\alpha}) \quad (\varepsilon \downarrow 0) , \]
where \( L := -\frac{1}{1 - \alpha} - \frac{4}{1 - 4\alpha} + \frac{2 - 2\alpha}{1 - 2\alpha} \).

(16) Application
The next example is an integral originating from (1) with \( \Phi(x) = 1/x \), \( f(x) = x \), \( g(x) = x(e^{-x} - 1) \), \( h(x) = 0 \), \( p(x) = e^{-x} \), and is already mentioned in the introduction.

\[ E(\varepsilon) := \int_0^1 \frac{dx}{x + \varepsilon e^x} . \]

Without comment the results are given: \( u(x) = 1 - e^x \), \( \bar{u}(x) = e^{-x} - 1 \) and
\[ F_k(\varepsilon) = k!\varepsilon^k \int_0^1 \frac{(e^x - 1)^k}{(\varepsilon + x)^{k+1}} dx \quad (k \in \mathbb{N}) . \]
Again $F_{k+1}(\varepsilon) = O(\varepsilon F_k(\varepsilon)) \quad (0 < \varepsilon \leq 1)$, since the ratio of the integrands of $F_{k+1}$ and $K_k$ is smaller than $2\varepsilon$.

$$F_0 = \int_0^1 \frac{dx}{x + x} = -\log \varepsilon + \log(1 + \varepsilon)$$

$$F_1 = \int_0^1 f(x) K(\varepsilon/x) \, dx \quad \text{with} \quad f(x) = (e^x - 1)/x, \ K(y) = y(1 + y)^{-2},$$

By using formula (13) with $N = 2$ one finds

$$F_1 = \frac{1}{2} \varepsilon \log(1 + \varepsilon^{-2}) + C_1 \varepsilon + O(\varepsilon \log \varepsilon),$$

where $C_1 = \int_0^1 (e^x - 1) x^{-2} \, dx$. With $C := C_1 + 1$ one finds

$$E(\varepsilon) = \log \frac{1}{\varepsilon} + C \varepsilon + \varepsilon \log \frac{1}{\varepsilon} + O \left( \varepsilon^2 \log \frac{1}{\varepsilon} \right) \quad (\varepsilon \downarrow 0).$$

(17) **Application**  
The integral to be treated now is also an integral already mentioned in the introduction.

$$I(t) := \int_0^\infty e^{-tx^2} \, dx \frac{1}{1 + tx},$$

where it is asked to determine the asymptotic behaviour for $t \to \infty$. The substitution $\varepsilon = t^{-1/2}, \ x = \varepsilon y$ transforms the integral into

$$I(t) = \varepsilon \int_0^\infty e^{-s^2} K_1(\varepsilon/y) \, dy,$$

where $K_1(s) := s(1 + s)^{-1}$. One splits the integral in two parts $I_1 := \int_0^1$ and $I_2 := \int_1^\infty$. The integral $I_2$ can be treated quite elementary. For $n \geq 0$ and with $q := \varepsilon/y$

$$\varepsilon I_2 = \varepsilon \int_1^\infty e^{-s^2} q(1 - q + q^2 - \ldots + (-1)^{n-1} q^{n-1} + (-1)^n q^n (1 + q)^{-1}) \, dy$$

$$= \sum_{k=2}^{n+1} C_k \varepsilon^k + O(\varepsilon^{n+2}) \quad (\varepsilon \downarrow 0),$$
where \( C_k := (-1)^k \int_1^\infty e^{-y^2} y^{-k+1} \, dy \quad (k = 2, 3, \ldots) \).

The integral \( I_1 \) can be treated by applying formula (13). Starting with \( f_0(x) = e^{-x^2} \), \( K_0(y) = y(1+y)^{-1} \), it is easily seen that

\[
 f_n(x) = x^{-n} \left( e^{-x^2} - \sum_{0 \leq k < n/2} \frac{(-1)^k}{k!} x^{2k} \right) \quad (n \geq 1) ,
\]

\[
 K_n(y) = (-1)^{n+1} \frac{1}{1+y} \quad (n \geq 1) .
\]

The occurring integrals \( A_k := \int_0^1 f_k(x) \, dx \) can be calculated by means of a recurrence relation to be obtained as follows: substituting \( x = \sqrt{y} \) and integrating by parts one obtains

\[
 A_n = \frac{1}{n-1} (-f_n(1) - A_{n-2}) \quad (n \geq 2) .
\]

Initial values \( A_0 \) and \( A_1 \) are to be calculated, either explicitly or numerically. The integrals \( \int_0^1 (K_n(x)-K_n(0)) \, dx \) are equal to \((-1)^{n+1} (-\varepsilon \log(1 + \varepsilon) + \varepsilon \log \varepsilon) \). In principle one can find a complete asymptotic expansion; a few terms are

\[
 I = \varepsilon^2 \log \frac{1}{\varepsilon} + a \varepsilon^2 + b \varepsilon^3 + O(\varepsilon^4 \log \varepsilon) \quad (\varepsilon \downarrow 0) .
\]

where

\[
 a := -\int_0^1 \frac{1 - e^{-x^2}}{x} \, dx , \quad b := 1 + \int_0^1 \frac{1 - e^{-x^2}}{x^2} \, dx .
\]

(18) \ Application

Also the integral \( J(\varepsilon) \) appears in the introduction.

\[
 J(\varepsilon) = \int_0^{\frac{1}{\varepsilon}} \frac{dx}{\varepsilon + \sin x^2} .
\]

The functions \( F_k(\varepsilon) \) are

\[
 F_k(\varepsilon) = (-1)^k k! e^k \int_0^{\frac{1}{\sin x^2}} \frac{x^2}{\sin x^2} \left( \frac{\sin x^2 - x^2}{\sin x^2} \right)^k \frac{dx}{(\varepsilon + x^2)^{k+1}} \quad (k = 0, 1, 2, \ldots) .
\]

Also \( F_{k+1}(\varepsilon) = O(\varepsilon F_k(\varepsilon)) \) \( (x \downarrow 0) \). Applying (13) for a few terms one finds

\[
 F_0(\varepsilon) = \frac{\pi}{2\sqrt{\varepsilon}} + A + B \varepsilon + \frac{\pi}{12} \varepsilon^2 + O(\varepsilon^2) \quad (\varepsilon \downarrow 0) ,
\]

\[
 F_1(\varepsilon) = \frac{\pi^2}{4\sqrt{\varepsilon}} + A + B \varepsilon + \frac{\pi}{12} \varepsilon^2 + O(\varepsilon^2) \quad (\varepsilon \downarrow 0) .
\]

\[
 F_2(\varepsilon) = \frac{\pi^3}{8\sqrt{\varepsilon}} + A + B \varepsilon + \frac{\pi}{12} \varepsilon^2 + O(\varepsilon^2) \quad (\varepsilon \downarrow 0) .
\]

\[
 F_3(\varepsilon) = \frac{\pi^4}{16\sqrt{\varepsilon}} + A + B \varepsilon + \frac{\pi}{12} \varepsilon^2 + O(\varepsilon^2) \quad (\varepsilon \downarrow 0) .
\]
where
\[
A = -1 + \int_{0}^{1} \frac{x^2 - \sin x^2}{x^2 \sin x^2} \, dx, \quad B = \frac{1}{3} - \int_{0}^{1} \frac{x^2 - \sin x^2}{x^4 \sin x^2} \, dx;
\]
\[
F_1(\varepsilon) = C \varepsilon - \frac{\pi}{8} \varepsilon^2 + O(\varepsilon^2) \quad (\varepsilon \downarrow 0),
\]
where
\[
C = \int_{0}^{1} \frac{x^2 - \sin x^2}{x^2 (\sin x^2)^2} \, dx.
\]
Hence, by Theorem (6), a few terms in the asymptotics of $J$ are
\[
J(\varepsilon) = F_0(\varepsilon) - F_1(\varepsilon) + O(\varepsilon^2) = \frac{\pi}{2 \sqrt{\varepsilon}} + A + (B - C)\varepsilon + \frac{5\pi}{24} \varepsilon^\frac{3}{2} + O(\varepsilon^2) \quad (\varepsilon \downarrow 0).
\]

(19) **Application**

The asymptotics for $t \to \infty$ of the integral
\[
G_1(t) = \int_{0}^{1} e^{-\frac{t}{4} - \frac{1}{2} x^2} \, dx
\]
was part of an asymptotic problem mentioned in the section OCCURRENCES. Applying (13) starting with $f_0(x) = e^{-\frac{1}{2} x^2}$, $K_0(y) = e^{-y}$ one gets
\[
G_1(t) = C_0 + \varepsilon \log \varepsilon + C_1 \varepsilon + C_2 \varepsilon^2 - \frac{1}{12} \varepsilon^3 \log \varepsilon + O(\varepsilon^3) \quad (\varepsilon \downarrow 0).
\]

The coefficients $C_0, C_1, C_2$ are expressions with definite integrals:
\[
C_0 = \int_{0}^{1} e^{-\frac{1}{2} x^2} \, dx, \quad C_1 = \int_{0}^{1} \frac{e^{y-\frac{1}{2} y^2} - 1}{y^2} \, dy + \int_{1}^{\infty} e^{y-\frac{1}{2} y^2} \, dy - \int_{0}^{1} \frac{e^{-\frac{1}{2} x^2} - 1}{x^2} \, dx - \frac{1}{3}, \quad C_2 = \int_{0}^{1} \frac{e^{-\frac{1}{2} x^2} - 1}{2x^2} \, dx - \frac{1}{6}.
\]

PROOFS

**PROOF OF THEOREM (4)**

Let $f, g, h, \Phi, p$ satisfy the conditions mentioned directly after (1), and define $u = g/(f + g), \bar{h} = g/f, \bar{u} = \bar{h} - h$. The following statement has to be proven:
\[
\forall \delta > 0 \exists \varepsilon_1 > 0 \forall \varepsilon \in (0, \varepsilon_1) \forall k \in \mathbb{N} \quad |\bar{F}_{k+1}(\varepsilon)| < \delta |\bar{F}_k(\varepsilon)|.
\]
Let $0 < \delta < 1$. Let $k \in \mathbb{N}$. Let $Q$ be the quotient of the integrands of $\tilde{F}_{k+1}$ and $\tilde{F}_k$. The proof consists in showing that $Q < \delta$ for all $\varepsilon > 0$ sufficiently small. We have

$$Q := \left| \tilde{u}(x) \left( \frac{\varepsilon + f(x)}{1 - u(x)} \right) \right| \left( 1 - u(x) \right) \frac{\varepsilon}{\varepsilon + f(x)},$$

where $q(x) := x \left| \frac{\phi^{(k+1)}(x)}{\phi^{(k)}(x)} \right|$. Let $M := 1 + \sup\{q(x) : 0 < x \leq \mu\}$, where $\mu := \sup\{\frac{\varepsilon + f(x)}{1 - u(x)} : 0 < x \leq 1\}$. Then clearly $Q \leq M \tilde{Q}(\varepsilon, x)$, where $\tilde{Q}(\varepsilon, x) := \frac{\varepsilon}{\varepsilon + f(x)}$, and $v := (1 - u)\tilde{u}$. Clearly $v(x) = o(1)$ $(x \downarrow 0)$. Let $b = b(\delta) > 0$ be defined by $|v(x)| \leq \delta/(MV)$ $(0 < x \leq b)$, where $V := 1 + \sup\{|v(x)| : 0 \leq x \leq 1\}$. Let $\varepsilon_1 := \delta f(b(\delta))/(MV - \delta)$. Clearly $\varepsilon_1 > 0$. Let $0 < \varepsilon < \varepsilon_1$. Two cases are to be considered.

CASE 1: $0 < x \leq b(\delta)$.
In this case $|v(x)| < \delta/(MV)$, whence

$$Q := \left| v(x)q \left( \frac{\varepsilon + f(x)}{1 - u(x)} \right) \left( \frac{\varepsilon}{\varepsilon + f(x)} \right) \leq \frac{\delta}{MV} \cdot 1 < \delta.$$

CASE 2: $b(\delta) \leq x < 1$.
Then $\varepsilon f(x) \leq \varepsilon f(x) = \varepsilon f(x)(MV - \delta - 1) < \frac{\delta}{MV}$, whence

$$Q = q \left( \frac{\varepsilon + f(x)}{1 - u(x)} \right) \left| v(x) \right| \frac{\varepsilon}{\varepsilon + f(x)} \leq M \cdot v \cdot \frac{\delta}{MV} < \delta.$$

Hence the integrand of $\tilde{F}_{k+1}$ is smaller than $\delta$ times the integrand of $\tilde{F}_k$ on the whole interval of integration. Moreover $\tilde{F}_{k+1} = O(m(\varepsilon))\tilde{F}_k$, with $m(\varepsilon) := \max\{\tilde{Q}(\varepsilon, x) : 0 \leq x \leq 1\}$.

PROOF OF THEOREM (6)
For all real numbers $x, y, z$ with $y > 0$ the following identity holds:

$$x + y + z + w = \frac{(x + y)(y + z)}{y} = \frac{xz}{y} + w.$$

Let $f, g, h, \Phi, p$ satisfy the conditions mentioned after the integral (1). Define $u$ and $\tilde{u}$ as in the proof of Theorem (4). Substitution of $x := \varepsilon$, $y := f$, $z := g$, $w = \varepsilon h$ gives

$$\varepsilon + f + g = \frac{\varepsilon + f}{1 - u} - \varepsilon \tilde{u}.$$

Clearly $1 - u = f/(f + g) \neq 0$ on $(0, 1]$ and $\lim_{x \downarrow 0} (1 - u(x)) = 1$. Now one writes down the Taylor development of $\Phi$ around $\frac{\varepsilon + f}{1 - u}$ up to order $N$.

$$\Phi(\varepsilon + f + g + \varepsilon h) = \sum_{k=0}^{N} \frac{(-1)^k}{k!} (\varepsilon \tilde{u})^k \Phi^{(k)} \left( \frac{\varepsilon + f}{1 - u} \right) + R_{N+1},$$

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where

\[ R_{N+1} = \frac{(-1)^{N+1}}{(N+1)!} (\varepsilon \hat{u})^{N+1} \Phi^{(N+1)} \left( \frac{\varepsilon + f}{1 - u} - \theta \varepsilon \hat{u} \right) \]

for some \( \theta \in (0, 1) \). Multiplication with \( p \) and integration of both sides of (19) gives the asymptotic expansion (7) if it can be shown that \( \int_0^1 R_{N+1} \, dx = O(\hat{F}_{N+1}(\varepsilon)) \quad (\varepsilon \downarrow 0) \).

**LEMMA.** Let \( 0 < x_1 < x_2 \leq a \). Let \( M_k := \sup \{ x \mid \Phi^{(k+1)}(x)/\Phi^{(k)}(x) \mid : 0 < x \leq a \} \quad (k \in \mathbb{N}_0) \). Then

\[
\left( \frac{x_1}{x_2} \right)^{M_k} \leq \left| \frac{\Phi^{(k)}(x_2)}{\Phi^{(k)}(x_1)} \right| \leq \left( \frac{x_2}{x_1} \right)^{M_k} .
\]

**Proof.** Integration over \([x_1, x_2]\) of the terms in the inequalities

\[
-\frac{M_k}{x} \leq \frac{\Phi^{(k+1)}}{\Phi^{(k)}} \leq \frac{M_k}{x}
\]

proves the lemma.

Application of the lemma gives

\[
\left| \Phi^{(N+1)} \left( \frac{\varepsilon + f}{1 - u} - \theta \varepsilon \hat{u} \right) \right| \leq A \left| \Phi^{(N+1)} \left( \frac{\varepsilon + f}{1 - u} \right) \right|
\]

where \( A = \left( 1 - \theta \varepsilon \hat{u}(1-u) \right)^{-\text{sgn}(\hat{u}) M_{N+1}} \leq \left( 1 - \frac{\varepsilon \hat{u}(1-u)}{\varepsilon + f} \right)^{-M_{N+1}} \). Since \( M(\varepsilon) := \max \{ \varepsilon | v(x)|/(\varepsilon + f(x)) : 0 \leq x \leq 1 \} = o(1) \quad (\varepsilon \downarrow 0) \) there exists a \( \varepsilon_2 > 0 \) such that \( M(\varepsilon) \leq 1/2 \quad (0 < \varepsilon \leq \varepsilon_2) \). Hence for \( \varepsilon \) sufficiently small we have that \( A \leq 2^{M_{N+1}} \) independent of \( x \) and \( \varepsilon \). It follows that \( \int_0^1 R_{N+1} \, dx = O(\hat{F}_{N+1}(\varepsilon)) \quad (\varepsilon \downarrow 0) \). Hence Theorem (6) is proved.

**PROOF OF THEOREM (8).**

With \( M := \sup \{|f(x)| : 0 \leq x \leq 1\} \) one has

\[
\left| \int_0^1 x^{-\alpha} f(x) K \left( \frac{\varepsilon}{x} \right) \, dx \right| \leq M \varepsilon^{1-\alpha} \left( \int_1^\infty \! + \int_0^1 \! \right) K(y)y^{-2+\alpha} \, dy = \]

\[
O(\varepsilon^{1-\alpha}) + M \varepsilon^{1-\alpha} \int_0^1 |K(y)|y^{-2+\alpha} \, dy \quad (0 < \varepsilon \leq 1) .
\]

From the conditions on \( K \) it follows that \( K(y) = O(1) \quad (0 \leq y \leq 1) \), whence

\[
\int_0^1 |K(y)|y^{-2+\alpha} \, dy = O(\varepsilon^{1-\alpha}) \quad (0 < \varepsilon \leq 1) .
\]
It follows that

\[ \int_0^1 x^{-\alpha} f(x) K \left( \frac{c}{x} \right) \, dx = O(1) \quad (0 < \varepsilon \leq 1) . \]

Formula (10) follows easily by substitution of the definitions of \( \tilde{f} \) and \( \tilde{K} \).

**QUESTIONS**

The foregoing method is satisfying only for multiplicative functions \( \Phi \) and \( f \). Even an integral like

\[ L = \int_1^\infty \frac{dx}{x + \varepsilon e^x} \]

can not be handled in a satisfying way. After the substitution \( e^{1-x} = y \) one gets

\[ L = \int_0^1 \frac{dy}{e^y + f(y)} , \]

with \( f(y) = y \log(e/y) \). Ad hoc one can find a few terms:

\[ L = \log \log(1/\varepsilon) + O \left( \frac{\log \log(1/\varepsilon)}{\log(1/\varepsilon)} \right) \quad (\varepsilon \downarrow 0) , \]

but it is difficult to see how to get more terms. Thus a first open problem is: Find a systematic way to get an asymptotic expansion of integrals like \( L \), which is an integral of type (1) with multiplicative \( \Phi \) and non-multiplicative \( f \). Of course other and perhaps more difficult problems arise if \( \Phi \) is non-multiplicative.

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