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Data-Driven Iterative Inversion-Based Control: Achieving Robustness Through Nonlinear Learning [★]

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Abstract

Learning from past data enables substantial performance improvement for systems that perform repeating tasks. Achieving high accuracy and fast convergence in the presence of unknown disturbances typically imposes requirements on the available system knowledge. The aim of this paper is to develop a data-driven approach that achieves high tracking performance through learning for Linear Time-Invariant (LTI) systems whose dynamics are unknown and that are subject to unknown disturbances. This is achieved by developing an Iterative Inversion-based Control (IIC) framework that employs a nonlinear input updating strategy to ensure fast and robust convergence. The developed method is applied to an experimental desktop printer and is compared to a pre-existing approach, which shows that the performance is significantly improved by imposing smoothness properties on the iteration dynamics.

Key words: Learning control, Frequency response methods, Nonlinear control systems, Convergence analysis, System identification.

1 Introduction

High tracking performance can be achieved for systems that perform repeating tasks by learning from past data. For instance, Iterative Learning Control (ILC) [1, 5] and Repetitive Control (RC) [14] exploit the identical nature of the tasks to anticipate disturbances, resulting in superior performance compared to traditional feedback control. Many successful applications have been reported including robotics [8, 29, 2], additive manufacturing [18], printing systems [4], and distributed parameter systems [15, 16].

Achieving convergence is essential in learning and is typically aided by incorporating system knowledge in the form of a model [23]. By designing learning filters that are based on an inverse dynamics model of the system [27, chap. 6], fast convergence can be achieved if the model is sufficiently accurate. In contrast, when the model is of poor quality, the learning process may not

converge or converges with poor transient learning behavior [5]. This behavior spurred the development of robust learning approaches [33, 32] that aim to ensure certain convergence properties. Ensuring these properties generally involves a trade-off between performance and robustness, and leads to nonzero asymptotic errors [26]. Furthermore, these approaches typically require approximate system models and a detailed model of the uncertainties over a large frequency range [3].

The effort associated with developing approximate models and robustness filters has led to the development of data-driven learning approaches [19]. A data-driven ILC method is presented in [7, 6], where an estimate of the system dynamics is updated based on previous trial data. This approach is specifically developed to achieve a desired terminal output using constant inputs, making it less suitable for tracking applications. In [4], a data-driven steepest-descent ILC method is developed and applied to a printing system. In this approach, dedicated experiments are required and many iterations are needed before convergence is achieved. Similarly, in [21], the impulse response of a Linear Time-Invariant (LTI) system is estimated from a weighted combination of previous trials. In these approaches, the modeling effort is replaced by system identification techniques that are integrated into the iterative process. Averaging techniques

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are used to reduce the effect of unmeasured disturbances, although the effect of disturbances on the convergence behavior of the iterative process has not yet been fully investigated.

Beyond typical ILC and RC approaches, Iterative Inversion-based Control (IIC) is a related class of iterative algorithms that aims to improve the performance of LTI systems through learning without using a parametric model. In [31], a nonparametric Frequency Response Function (FRF) model of the inverse system dynamics is used to achieve convergence. Such models are typically accurate, inexpensive and fast to obtain [28]. Since the FRF model is commonly estimated on a finite frequency grid, severe requirements are imposed on the compatibility of the grid and the task length. Moreover, similar to model-based ILC, suitably designed frequency-dependent learning coefficients are still required in IIC to ensure convergence in the presence of modeling errors [34, 9].

In Model-Free IIC (MF-IIC) [22], the modeling requirement is further reduced by removing the need for a pre-determined FRF model. In particular, the inverse FRF model is estimated from the data that results from the previous trial. This results in increased tracking performance and fast convergence if the reference is large compared to the output disturbances. However, if the reference and output disturbance are of comparable magnitude, the performance can deteriorate significantly, as is shown in [10]. The loss of performance is caused by substantial amplification of the input signal from one trial to the next, which is caused by the inverse model error that is induced by the corrupted output signal. Ad hoc attempts to avoid excessive amplification of the input signal by using a threshold learning function are presented in [24, 20], although a formal convergence analysis is not provided.

Although the estimation of the inverse plant dynamics during control iterations reduces the prior modeling requirement in learning control, presently, unmeasured disturbances can negatively affect the convergence behavior when the estimation is based on data from previous trials. The aim of this paper is to develop a data-driven learning approach that ensures high performance and bounded input amplification in the presence of disturbances. This is achieved by developing a Nonlinear IIC (NL-IIC) approach through the following contributions.

- (C1) A general NL-IIC approach is developed that is shown to achieve robust learning by employing learning-gain functions.
- (C2) The limitation and advantage of traditional MF-IIC, which is recovered as a special case of NL-IIC, are revealed through analysis of the nonlinear iteration dynamics.

- (C3) The obtained insights are applied in the development of a specific learning-gain function that ensures bounded input amplification in the presence of disturbances.
- (C4) The developed method is successfully applied experimentally and is compared to pre-existing MF-IIC, revealing superior performance.

A commercial desktop printer is used to demonstrate the developed method since it is a typical example of a system that performs a repetitive positioning task. The developed data-driven method is specifically suitable for such systems since the available prior knowledge is typically limited. It is shown that due to unmeasured disturbances, traditional MF-IIC results in erratic convergence behavior, whereas NL-IIC converges smoothly.

The present paper extends the initial research in [10] by: (i) explicitly bounding the input amplification for any possible measured output, (ii) generalizing the results to bounded trial-varying disturbances, (iii) extending previous local stability results to global results, (iv) providing proofs, and (v) supporting these claims with extensive experimental tests.

This paper is structured as follows. The iterative tracking problem is formalized in Section 2. In Section 3, the NL-IIC framework is developed as a solution to this problem. In Section 4, the limitations and advantages of traditional MF-IIC are identified. In Section 5.1, the developed insights are utilized by formulating a specific NL-IIC update-law, and implementation and design aspects are addressed. In Section 6, the developed approach is applied to a commercial printer.

1.1 Preliminaries

The sets of real numbers, nonnegative real numbers, and complex numbers are denoted by $\mathbb{R}$, $\mathbb{R}_{\geq 0} \triangleq [0, \infty)$, and $\mathbb{C}$ respectively. A closed ball with radius $r \in \mathbb{R}_{\geq 0}$ around a point $z \in \mathbb{C}$ is denoted by $\mathcal{B}_r(z) \triangleq \{x \in \mathbb{C} \mid |x - z| \leq r\}$. A sequence $\{x_i\}$ is said to converge to an $\bar{r}$ neighborhood of z , if $|x_i - z| \leq r_i \forall i$, $\lim_{i \rightarrow \infty} r_i = \bar{r}$, which by slight abuse of notation is denoted by $\lim_{i \rightarrow \infty} x_i \in \mathcal{B}_{\bar{r}}(z)$. A real function $f(x)$, $x \in X \subseteq \mathbb{R}$ is bounded if $\exists c \in \mathbb{R}_{\geq 0}$, such that $|f(x)| \leq c \forall x \in X$. The transfer function of a Discrete Time (DT) LTI system G is denoted by $G(z)$, $z \in \mathbb{C}$. The N -point Discrete Fourier Transform (DFT) and its inverse are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$, i.e., $S(k) = \mathcal{F}\{s(t)\}$, $s(t) = \mathcal{F}^{-1}\{S(k)\}$, $k = 0, 1, \dots, N - 1$. Throughout, $\{X_i(k)\}$ denotes a sequence of signals $X_i(k)$ as a function of the iteration number i .

2 The iterative tracking control problem

In this section, three key requirements for iterative tracking control methods are formulated and motivated, and the iterative tracking problem is formulated.

Fig. 1. An illustration of two input sequences that converge to a region around U_∞ (—). $U^{(1)}$ (—•) displays large variations, whereas $U^{(2)}$ (—•) remains close to U_∞ . The convergence smoothness can be quantified by κ_i as given by (1).

2.1 Requirements in iterative tracking control

Consider the output of an open loop system $y_i = Gu_i + d_i$, where d_i is a trial-varying output disturbance, and define the tracking error as $e_i \triangleq r - y_i$, where r is a trial-independent reference signal. The iterative tracking control problem is to iteratively determine the input in view of the following requirements.

- (R1) Increased performance: after several iterations, the error is small in an appropriate sense.
- (R2) Input convergence: after a number of iterations, the input is close to a certain bounded signal.
- (R3) Input amplification: the trial-to-trial amplification of the input is limited, such that severe performance loss is avoided from one trial to the next.

Requirement R1 states the main control objective, and R2 reflects the constraint that the input converges and remains bounded, which is an important aspect of learning control [26]. The importance of R3 is illustrated in Figure 1, which displays an erratically converging input sequence. In practice, such variations in the input result in unpredictable behavior and unreliable performance. The following measure is introduced to quantify this behavior.

Definition 1 For a given sequence $\{X_i(k)\}$, define the propagation factor at frequency k as

$$\kappa_i(k) \triangleq \frac{|X_{i+1}(k)|}{|X_i(k)|}. \quad (1)$$

The bottom plot of Figure 1 reveals that bounding κ_i addresses requirement R3, i.e., $\kappa_i \leq \bar{\kappa} \forall i$. Furthermore, if $\kappa_i < 1 \forall i$, then the sequence converges monotonically to 0. This concept is used throughout and is formalized as follows.

Definition 2 A sequence $\{X_i\}$, $X_i \in \mathbb{C}$ is said to converge monotonically to a point $X_\infty \in \mathbb{C}$ if there exists a $\bar{\kappa} \in [0, 1)$ such that $|X_{i+1} - X_\infty| \leq \bar{\kappa}|X_i - X_\infty| \forall i$.

In the next section, the formulated requirements are formalized in the iterative tracking problem.

2.2 Problem formulation

In this section, the iterative tracking problem is formulated in the frequency domain under the following assumptions.

Assumption 1 Let $r(t)$ be a known trial-invariant and bounded reference signal with DFT coefficients $R(k)$.

Assumption 2 Let $G(z)$ be an unknown stable LTI system with $G(e^{j\omega_k}) \neq 0 \forall k \in \{0, \dots, N-1\}$, $\omega_k = \frac{2\pi k}{N}$.

The assumption that $G(e^{j\omega_k})$ is nonzero on the DFT grid is implied if $G(z)$ has no zeros on the unit disc, as is a standard assumption in system inversion [11].

Assumption 3 The tracking error satisfies

$$E_i(k) \triangleq R(k) - Y_i(k), \quad (2)$$

where the output $Y_i(k)$ is measured and given by

$$Y_i(k) = G(e^{j\omega_k})U_i(k) + D_i(k), \quad (3)$$

with $D_i(k)$ an unknown disturbance.

The disturbance D_i comprises input-output noise, the effect of nonlinearities, and the transient response, which induces leakage [28]. It is assumed that $|D_i|$ is bounded to enable stability guarantees of the iterative control algorithm for any sequence of disturbances $\{D_i\}$.

Assumption 4 The unknown output disturbance is bounded by $\delta(k)$, i.e., $|D_i(k)| \leq \delta(k) \forall i$.

A similar assumption is commonly employed in system identification for control [25] to enable robustness against worst-case estimation errors. In contrast to [10] where D_i is assumed to be trial-invariant, i.e., $D_i = D \forall i$, here, D_i is allowed to vary over the iterations.

The iterative tracking problem is formulated as follows.

Problem 1 Determine an update law $U_{i+1} = f(U_i, Y_i, R)$ that generates sequences $\{U_i(k)\}$ such that for any given frequency $k \in \{0, \dots, N-1\}$ the following criteria are satisfied under Assumption 1 to 4.

- (P1) The error converges to a neighborhood of zero, i.e., $\lim_{i \rightarrow \infty} E_i(k) \in \mathcal{B}_{r_e(k)}(0)$ for a certain $r_e(k) \in \mathbb{R}_{\geq 0}$.

- (P2) *The input converges to a bounded set, i.e., $\lim_{i \rightarrow \infty} U_i(k) \in \mathcal{B}_{r_u(k)}(U_\infty(k))$, for a certain $|U_\infty(k)| < \infty$ and $r_u(k) \in \mathbb{R}_{\geq 0}$.*
- (P3) *The input propagation factor, as given by (1) where $X_i(k) = U_i(k)$, is always bounded, i.e., $\kappa_i(k) \leq \bar{\kappa}(k) \in \mathbb{R}_{\geq 0} \forall i$.*

Note that subproblems P1, P2, and P3 directly reflect requirements R1, R2 and R3 as presented in the previous subsection. This frequency domain formulation enables a frequency-by-frequency evaluation of the signals, which significantly aids the forthcoming analysis and synthesis.

In the next section, a general class of update laws is presented that aims to solve the iterative tracking problem.

3 Nonlinear IIC for unmodeled LTI systems

In this section, a general NL-IIC approach is developed that encompasses MF-IIC [22] as a special case. Bounded input amplification is achieved by incorporating learning-gain functions. By establishing a global analysis framework, a class of learning-gain functions is formulated that solves Problem 1, thereby constituting contribution C1. In the remainder, the results are presented for a specific frequency k , which is occasionally omitted from the notation to facilitate the exposition.

3.1 The disturbance-free case

First, the disturbance-free case is considered, i.e., $\delta = 0$. This occurs if the steady-state output of an LTI system is measured without noise.

Lemma 1 *If $\delta(k) = 0$ under Assumption 1 to 4, then for any $U_0(k) \neq 0$, the sequences generated by*

$$U_{i+1}(k) = U_i(k) + \frac{U_i(k)}{Y_i(k)} E_i(k), \quad (4)$$

result in $E_i(k) = 0$ and $U_i(k) = G^{-1}(e^{j\omega_k})R(k) \forall i \geq 1$.

PROOF. Substituting (3) into (4) with $D = 0$ and computing $E_1(k)$ results in $E_1(k) = 0 \forall U_0(k) \neq 0$. Since $U_{i+1}(k) = U_i(k)$ for $E_i(k) = 0$, it holds that $E_i(k) = 0 \forall i \geq 1$ and $U_i(k)$ follows directly from (3) and (2).

Consequently, (4) presents a solution to Problem 1 for $\delta(k) = 0$. The update law (4) corresponds to the MF-IIC approach [22], which essentially perfectly estimates the inverse dynamics $G^{-1}(e^{j\omega_k})$ from the initial trial $i = 0$ and employs it to compute the input that results in perfect tracking during the next trial.

Addressing the effect of disturbances on the iteration dynamics, described by (3) and (4), is the key challenge. Considering (3) in combination with (4) for the case that $\delta > 0$, shows that if D tends to $-GU_i$, then Y_i tends to zero. This results in an unbounded input update, hence $|U_{i+1}|$ in (4) may become arbitrarily large.

3.2 Nonlinear Iterative Inversion-based Control

The following class of NL-IIC update laws is considered to ensure limited trial-to-trial variations in the input in case $\delta > 0$.

Definition 3 *Let $U_{i+1}(k) = f(U_i(k), Y_i(k), R(k))$ for $i \geq 1$, with*

$$f(U_i, Y_i, R) = \begin{cases} U_i + \rho(|Y_i|) \frac{U_i}{Y_i} E_i, & \text{for } Y_i \neq 0, \\ U_i, & \text{for } Y_i = 0, \end{cases} \quad (5)$$

where, $\rho : \mathbb{R}_{\geq 0} \mapsto [\rho, \bar{\rho}] \subseteq \mathbb{R}_{\geq 0}$, is a learning-gain function, $U_0(k) \in \mathbb{C}$, and $Y_i(k)$ and $E_i(k)$ are given by (3) and (2).

The rationale is to let $\rho(|Y_i|)$ become small as $|Y_i|$ becomes small to avoid large input amplification. In the remainder, the following assumption is made to accommodate a clear exposition.

Assumption 5 *The output is nonzero, i.e., $Y_i(k) \neq 0$.*

Assumption 5 only excludes the trivial case and is nonrestrictive. Indeed, if $Y_i = 0 \forall i$, then $U_i = U_0 \forall i$. Furthermore, if $Y_i = 0$ for some i , then removing these occurrences from the analysis results in identical asymptotic results.

The aim of the next sections is to derive conditions on the function $\rho(|Y|)$ such that Problem 1 is solved by (5).

3.3 Asymptotic performance

In this section, conditions are established under which (5) results in a tracking error that converges to a neighborhood of zero, such that subproblem P1 is solved.

By definition, $|E_i|$ cannot converge to 0 if the disturbances D_i vary across the trials. The rate of change $\Delta\delta$ is defined to investigate the influence of trial-varying disturbances.

Definition 4 *Let $\Delta\delta(k) \triangleq \max_{i \geq 0} |D_i(k) - D_{i+1}(k)|$.*

Evidently, $\Delta\delta \leq 2\delta$ due to Assumption 4. The following theorem presents the conditions under which NL-IIC results in increased tracking performance.

Theorem 1 Consider the update law given by (5) under Assumption 1 to 5. If for a certain $\alpha > 1$, it holds that;

(i) $\rho(|Y|)$ is bounded as

$$\inf \rho(|Y|) \triangleq \underline{\rho} > \frac{\alpha - 1}{\alpha}, \quad (6a)$$

$$\sup \rho(|Y|) \triangleq \bar{\rho} \leq 1, \quad (6b)$$

(ii) and the reference is such that

$$|R| \geq \alpha\delta + \Delta\delta \frac{\alpha}{\alpha - 1}, \quad (7)$$

(iii) and the initial error is such that

$$|E_0| \leq |R| - \alpha\delta, \quad (8)$$

then, the sequences $\{U_i\}$ generated by (5) result in

$$\lim_{i \rightarrow \infty} E_i \in \mathcal{B}_{r_e}(0), \quad r_e = \Delta\delta \frac{\alpha}{\alpha - 1}. \quad (9)$$

Moreover, if $\Delta\delta = 0$, $\{E_i\}$ converges monotonically to $E_\infty = 0$.

This theorem states that asymptotic tracking is guaranteed if: (i) the learning gain $\rho(|Y|)$ does not exceed 1 and satisfies a nonzero strict lower bound, (ii) $|R|$ is large enough with respect δ and $\Delta\delta$, as is made explicit by (7), and, (iii) the initial error is small enough with respect to $|R|$ and δ , as is reflected by (8). Inequality (7) can be interpreted as a condition on the signal-to-noise ratio that is required to increase the performance through data-driven learning. Furthermore, $|E_i|$ converges to a neighborhood of 0 that scales with $\Delta\delta$. Consequently, if the disturbance is trial-invariant, i.e., $D_i = D \forall i$ such that $\Delta\delta = 0$, then perfect tracking is obtained.

A proof of Theorem 1 requires the following lemmas.

Lemma 2 Consider (5) under Assumption 1 to 5. Each sequence $\{E_i\}$ satisfies the following bounds

$$|E_{i+1}| \leq \kappa_i |E_i| + \Delta\delta \quad i \geq 0, \quad (10)$$

$$|E_i| \leq \bar{\kappa}^i |E_0| + \Delta\delta \frac{\bar{\kappa}^i - 1}{\bar{\kappa} - 1} \quad i \geq 0, \quad (11)$$

where κ_i is given as follows where ρ_i denotes $\rho(|Y_i|)$

$$\kappa_i \triangleq \left| \frac{(1 - \rho_i)(R - E_i) + \rho_i D_i}{R - E_i} \right|, \quad (12)$$

and $\bar{\kappa} \in \mathbb{R}_{\geq 0}$ is any number that satisfies $\kappa_i \leq \bar{\kappa} \forall i$.

A proof of this lemma is provided in Appendix A.

Lemma 3 If for a certain $\alpha > 1$, it holds that $\bar{\rho} \leq 1$, $\underline{\rho} > \frac{\alpha-1}{\alpha}$, $|R| \geq \alpha\delta$, and

$$|E_i| \leq |R| - \alpha\delta, \quad (13)$$

then $\kappa_i \leq \frac{1}{\alpha}$, where κ_i is given by (12).

A proof of this lemma is given in Appendix B. Consider the following proof of Theorem 1.

PROOF. [Proof of Theorem 1] It is given that (6a), (6b), (8) and (7) hold for a certain $\alpha < 1$. It is shown by induction that this implies that (13) holds $\forall i$. The base case $i = 0$ follows directly from Lemma 3 in combination with (8). Next, it is shown that if (13) holds for i , then (13) holds for $i + 1$. To see this, assume that (13) holds for i , such that by Lemma 3, $\kappa_i \leq \frac{1}{\alpha}$. In combination with (7), this implies that $|R| \geq \alpha\delta + \Delta\delta \frac{1}{1-\kappa_i}$, which is identical to $\kappa_i(|R| - \alpha\delta) + \Delta\delta \leq |R| - \alpha\delta$. Since (13) holds for i , this implies that $\kappa_i |E_i| + \Delta\delta \leq |R| - \alpha\delta$. Using (10), this is identical to $|E_{i+1}| \leq |R| - \alpha\delta$. Consequently, if (13) holds for i , then it also holds for $i + 1$. Induction implies that (13) holds for $\forall i$ since (13) holds for $i = 0$. Combining this result with Lemma 3 implies that $\kappa_i \leq \frac{1}{\alpha} < 1 \forall i$. Hence, for $i \rightarrow \infty$, (11) evaluates to

$$\lim_{i \rightarrow \infty} |E_i(k)| \leq \Delta\delta \frac{\alpha}{\alpha - 1},$$

which proves (9). In addition, if $\Delta\delta = 0$, then (10) in Lemma 2 directly yields that $\{E_i\}$ converges monotonically to 0, which is concluded by recalling Definition 2 for $\bar{\kappa} < 1$. This concludes the proof.

The main result of this section is established by Theorem 1, which considers the asymptotic tracking performance. Next, the aspect of input convergence is considered.

3.4 Input convergence

In this section, conditions on (5) are established under which the input converges to a neighborhood of a limit point, thereby providing a solution to subproblem P2.

Under the conditions presented in Theorem 1, the input converges to a bounded signal.

Theorem 2 Consider (5) under Assumption 1 to 5. If for a certain $\alpha > 1$, (6), (7), and (8) hold, then

$$\lim_{i \rightarrow \infty} U_i \in \mathcal{B}_{r_u}\{|U_\infty|\}, \quad U_\infty = G^{-1}R, \quad (14)$$

$$r_u = |G^{-1}| \left(\Delta\delta \frac{\alpha}{\alpha - 1} + \delta \right).$$

Moreover, in case $\Delta\delta = 0$, $\{U_i\}$ converges to $U_\infty = G^{-1}(R - D)$.

A proof to this theorem is provided in Appendix C.

This theorem confirms that if the error converges to a neighborhood of 0, then the input converges to a neighborhood of U_∞ . The following theorem reveals a key result for the case where the reference is relatively small with respect to the disturbances.

Theorem 3 Consider (5) under Assumption 1 to 5. If $\bar{\rho} \leq 1$, and $\rho(|Y|) > 0$ for $|Y| \neq 0$, and if for a certain $\beta > 1$ the reference is such that

$$\beta|R| \leq |D_i| \quad \forall i,$$

and the initial input is such that

$$|U_0| < (\beta - 1)|G^{-1}R|, \quad (15)$$

then, the sequences $\{U_i\}$ generated by (5) converge monotonically to $U_\infty = 0$.

This theorem provides the following insight. If $|R|$ is small with respect to $|D_i|$, then U_i converges to zero. In other words, if the disturbances dominate the output, then the control action is automatically removed. This is in stark contrast to iterative learning methods that employ an inverse-model-based update law. In such approaches, unstable iteration dynamics may lead to unbounded inputs if the convergence criterion is not met, as is concluded in [5, 31]. In NL-IIC, iterative updating does not result in unbounded inputs when rejection is not achieved, instead, the input converges to zero.

A proof of Theorem 3 employs the following lemmas.

Lemma 4 Consider (5) under Assumption 1 to 5. Each sequence $\{U_i\}$ satisfies

$$U_{i+1} = \left(\frac{(1 - \rho_i)(GU_i + D_i) + \rho_i R}{GU_i + D_i} \right) U_i, \quad (16)$$

where ρ_i denotes $\rho(|Y_i|)$. Moreover, each sequence $\{U_i\}$ satisfies the following trial-to-trial bound

$$|U_{i+1}| = \kappa_i |U_i| \quad i \geq 0, \quad (17)$$

where κ_i is given by

$$\kappa_i \triangleq \left| \frac{(1 - \rho_i)(GU_i + D_i) + \rho_i R}{GU_i + D_i} \right|, \quad (18)$$

and $\bar{\kappa} \in \mathbb{R}_{\geq 0}$ is any number that satisfies $\kappa_i \leq \bar{\kappa} \quad \forall i$.

A proof of this lemma is provided in Appendix D.

Lemma 5 If $\bar{\rho} \leq 1$, $\rho(|Y|) > 0$ for $Y \neq 0$, and if for a certain $\beta > 1$ it holds that $\beta|R| \leq |D_i| \quad \forall i$, and

$$|U_i| < (\beta - 1)|G^{-1}R| \quad \forall i,$$

then $\kappa_i < 1$, where κ_i is given by (18).

A proof of this lemma is provided in Appendix E. These results enable the following proof of Theorem 3.

PROOF. [Proof of Theorem 3] It is given that $\bar{\rho} \leq 1$, $\rho(|Y|) > 0$ for $Y \neq 0$, and that for a certain $\beta > 1$ it holds that $\beta|R| \leq |D_i| \quad \forall i$, and (15). Hence, Lemma 5 implies that $\kappa_0 < 1$. For $i = 0$, (17) equals $|U_1| = \kappa_0|U_0|$. Using that $\kappa_0 < 1$, in combination with (15) yields, $|U_1| \leq (\beta - 1)|G^{-1}R|$. Consequently, Lemma 5 implies that also $\kappa_1 < 1$. Repeating this process shows that $\kappa_i < 1 \quad \forall i$, i.e., $\bar{\kappa} < 1$. Consequently, combining (17) with Definition 2 shows that $\{U_i\}$ converges monotonically to $U_\infty = 0$.

The results in this section are readily connected to the stability analysis of nonlinear systems. To see this, consider the iteration dynamics given by (16), and regard the disturbance D_i as a parameter and U_i as the dependent variable. In this way, (16) can be classified as an autonomous nonlinear time-varying discrete time recursion with complex coefficients. Theorem 3 equivalently states that if $|R| < |D_i| \quad \forall i$, then $U_\infty = 0$ is an asymptotically stable fixed point [17, Prop. 3.3.2], and relation (15) explicitly provides a region of attraction [17, Prop. 3.1.25]. Furthermore, if $\Delta\delta = 0$, then (16) is time-invariant, and Theorem 2 states that if $|R| > \delta$, then $U_\infty = G^{-1}(R - D)$ is an asymptotically stable fixed point. Note that these stability results are local since stability is not guaranteed globally, i.e., for all initial inputs $U_0 \in \mathbb{C}$. Note that these results extend [10] by providing regions of attraction and allowing time-varying dynamics, which are induced by the trial-varying disturbances D_i .

The main result of this section considers input convergence and is established by Theorem 2 and 3. Next, the aspect of bounded input amplification is considered.

3.5 Bounded input amplification

In this section, conditions on $\rho(|Y|)$ are derived such that the input propagation factor is bounded. In this way, NL-IIC, as given by (5), presents a solution to subproblem P3. The following theorem establishes this main result.

Theorem 4 Consider (5) under Assumption 1 to 5. If: (i) $\rho(|Y|)$ is continuous on a neighborhood of 0, (ii) $\lim_{|Y| \rightarrow 0} \rho(|Y|) = 0$, and (iii) $\frac{d\rho}{d|Y|} \big|_{|Y|=0} \in \mathbb{R}$ exists, then $|U_{i+1}| \leq \bar{\kappa}|U_i| \quad \forall i$, with

$$\bar{\kappa} = 1 + \bar{\rho} + \tau|R|, \quad \tau \triangleq \sup_{|Y| \neq 0} \frac{\rho(|Y|)}{|Y|}. \quad (19)$$

This theorem shows that if $\rho(|Y|)$ is continuous around 0, $\rho(|Y|) \rightarrow 0$ as $|Y| \rightarrow 0$, and if $\rho(|Y|)$ is differentiable

around 0, then the input propagation factor satisfies an upper bound. Consequently, in the developed NL-IIC method, the trial-to-trial amplification of the input is guaranteed to be bounded, as is discussed in Section 2.1. In contrast to [10], which focuses on $|E_i|$ for $|Y_i| \rightarrow 0$, Theorem 4 provides global guarantees on the convergence behavior, i.e., $\forall Y_i$.

A proof of Theorem 4 utilizes the following result.

Lemma 6 *The real function $f(x) \triangleq \frac{\rho(x)}{x}$, with $\rho(x)$ as given in Definition 3, is bounded $\forall x \in (0, \infty)$ if: (i) $\rho(x)$ is continuous on a neighborhood of 0, (ii) $\lim_{x \rightarrow 0} \rho(x) = 0$, and (iii) $\frac{d\rho}{d|Y|}|_{|Y|=0} \in \mathbb{R}$ exists.*

A proof to this lemma is provided in Appendix F.

PROOF. [Proof of Theorem 4] Consider the input propagation factor κ_i as given by (18). It follows that $\kappa_i \leq \frac{|Y_i| + \rho(|Y_i|)|Y_i| + \rho(|Y_i|)|R|}{|Y_i|} = 1 + \rho(|Y_i|) + |R| \frac{|\rho(|Y_i|)|}{|Y_i|} \leq 1 + \bar{\rho} + |R| \sup_{|Y| \neq 0} \frac{\rho(|Y|)}{|Y|}$. Since $|R|$ is bounded due to Assumption 1, the third term is bounded if and only if $f(x) \triangleq \frac{\rho(x)}{x}$, is bounded $\forall x \in (0, \infty)$. Applying Lemma 6 directly results in Theorem 4.

In this section, NL-IIC is developed, which is a generalization of MF-IIC that regulates the input amplification by means of learning-gain functions $\rho(|Y|)$. It is shown that if $\rho(|Y|)$ and $|R|$ satisfy specific properties, NL-IIC presents a solution to Problem 1. In the next section, the aspects of traditional MF-IIC are investigated to aid the design of a specific gain function $\rho(|Y|)$ in Section 5.1.

4 The limitation and advantage of MF-IIC

In this section, the limitation and advantage of pre-existing MF-IIC [22], as presented by (4), are revealed by explicitly solving the nonlinear iteration dynamics. This enables the development of additional design guidelines regarding $\rho(|Y|)$ and constitutes contribution C2.

4.1 The limitation of MF-IIC

Consider (18) for $\rho(|Y|) = 1$, such that NL-IIC reduces to traditional MF-IIC. Clearly, the input propagation cannot be bounded in case $Y_i \rightarrow 0$.

Corollary 1 *For MF-IIC, i.e., for $\rho(|Y|) = 1$, it holds that $\lim_{|Y_i| \rightarrow 0} \kappa_i = \infty$, where κ_i is given by (18).*

Consequently, bounded input amplification, as discussed in Section 2 and formalized by subproblem P3, cannot be guaranteed by MF-IIC, which is a major limitation.

4.2 The advantage of MF-IIC

The advantage of MF-IIC is that the stability properties can be shown to hold globally under trial-invariant disturbances. To see this, note that (16) reduces to a Riccati difference equation if (18) is considered for $\rho(|Y|) = 1$ and $\Delta\delta = 0$. The solutions of the nonlinear Riccati equation reveal its behavior [13].

Lemma 7 *Consider (5) under Assumption 1 to 5 for $\rho(|Y|) = 1$ and $\Delta\delta = 0$. For $i \geq 0$, U_i satisfies*

$$U_i = \frac{U_0 G^{-1}(R - D)}{(G^{-1}(R - D) - U_0) \left(\frac{D}{R}\right)^i + U_0} \quad \text{for } R \neq D, \quad (20a)$$

$$U_i = \frac{U_0 G^{-1}R}{U_0 \cdot i + G^{-1}R} \quad \text{for } R = D. \quad (20b)$$

PROOF. Under the posed assumptions, (16) reduces to $U_{i+1} = \frac{RU_i}{GU_i + D}$. By substituting (20a) for U_{i+1} and U_i shows that equality holds, which validates that (20a) is a solution. Similarly, (20b) is shown to be a solution to for $R = D$.

Lemma 7 enables the following results.

Theorem 5 *Under the conditions presented in Lemma 7, the sequences $\{U_i\}$ generated by (5) solve the subproblems of Problem 1 under the following conditions.*

- (i) *If $|R| > |D|$, then $\lim_{i \rightarrow \infty} E_i = 0 \forall U_0 \neq 0$.*
- (ii) *If $|R| > 2|D|$ and U_0 is such that*

$$|U_0| > \frac{2|D|}{|R| - 2|D|} |G^{-1}(R - D)|,$$

then $\{E_i\}$ converges to 0 monotonically.

- (iii) *If $|R| \neq |D|$ or if $R = D$, then $\lim_{i \rightarrow \infty} U_i = U_\infty \forall U_0$, where $U_\infty = 0$ or $U_\infty = G^{-1}(R - D)$.*

A proof of this theorem is presented in Appendix G.

Statements (i) and (iii) show that subproblems P1 and P2, as presented in Section 2, can be guaranteed to be solved globally, i.e., for any initial input $U_0 \in \mathbb{C}$. This is considered an advantage of MF-IIC since the asymptotic results are independent of the initial input. Note that (iii) practically always holds, unless the specific case occurs where R and D have identical magnitudes and different phases, in which case a limit cycle is obtained [13].

In this section, it is shown that although bounded input amplification cannot be guaranteed for MF-IIC, increased tracking performance and input convergence can

be achieved for any initial input U_0 . In the next section, a specific learning gain $\rho(|Y|)$ is developed that combines the favorable aspects of NL-IIC and MF-IIC.

5 Application of NL-IIC

In this section, the developed theory is employed in the design of an explicit learning-gain function, which constitutes contribution C3. In addition, guidelines are presented for the implementation of NL-IIC.

5.1 An explicit learning-gain function

Based on the results presented in the previous sections, the following learning-gain function is postulated.

Proposition 1 Consider (5) under Assumption 1 to 5 and let $\rho(|Y|)$ be given by

$$\rho(|Y|) = \begin{cases} 1 & |Y| > \gamma, \\ \frac{1}{2} \left(1 - \cos \left(\frac{\pi}{\gamma} |Y| \right) \right) & |Y| \leq \gamma. \end{cases} \quad (21a) \quad (21b)$$

If $\gamma > 0$, then $\rho(|Y|)$ satisfies the criteria in Theorem 3 and 4, where $\lim_{|Y_i| \rightarrow 0} \kappa_i = 1$ and $\bar{\kappa} \leq 2 + \tau|R|$. Here, $\tau = \frac{(1 - \cos(\mu))\pi}{2\gamma\mu}$, with μ the solution to $\cos(\mu) + \mu \sin(\mu) - 1 = 0$ in the interval $(0, \pi]$, which is approximately $\mu \approx 2.3311$.

A proof of this proposition provided in Appendix H.

The motivation for (21) is threefold. First, (21a) is motivated by Theorem 1 and 2, which state that increased tracking performance is achieved when $\rho(|Y|)$ is close to 1. Moreover, drawing on the results presented in the previous section, if $\rho(|Y|) = 1$, Theorem 5 states that subproblem P1 and P2 are solved for any U_0 . Second, (21) satisfies Theorem 3, such that $U_\infty = 0$ is a stable attractor if $|R|$ is small with respect to $|D_i|$. Third, (21b) ensures that Theorem 4 holds, such that the trial-to-trial input amplification is guaranteed to be bounded. This is illustrated in Figure 2, where $\rho(|Y|)$ and $\frac{\rho(|Y|)}{|Y|}$ are compared for $\rho(|Y|) = 1$ and for $\rho(|Y|)$ as given by (21). Since $\frac{\rho(|Y|)}{|Y|}$ contributes to κ_i , as given by (18), Figure 2 shows that (21) results in bounded input propagation, since a maximum occurs at $\frac{|Y|}{\gamma} = \frac{\pi}{2}$ with value τ . Whereas for $\rho(|Y|) = 1$, the input amplification is not bounded.

It is suggested to set the threshold parameter as $\gamma = |R|$. The motivation for this is provided in twofold by (21a), i.e., for $|Y_i| > \gamma = |R|$, $\rho(|Y_i|) = 1$. First, if $|R| > \delta$, then the error propagation factor as given by (12) reduces to $\kappa_i = \frac{|D_i|}{|Y_i|} \leq \frac{\delta}{|Y_i|} \leq \frac{|R|}{|Y_i|} < 1$, such that the tracking performance increases monotonically. Second, regardless of D_i , the input propagation factor as given

Fig. 2. A plot of the NL-IIC learning gain $\rho(|Y|)$ given by (21) (—), and τ (•) which is the maximum of $\frac{\rho(|Y|)}{|Y|}$ (—). Similarly for MF-IIC, $\rho(|Y|) = 1$ (---) and $\frac{1}{|Y|}$ (—) are shown, where the latter is not bounded.

by (18) reduces to $\kappa_i = \frac{|R|}{|Y_i|} < 1$, such that the input will be strictly smaller during the next iteration. In the next section, a number of implementation guidelines are presented.

5.2 NL-IIC algorithm and implementation guidelines

In this section, the NL-IIC algorithm is summarized, and implementation guidelines are presented. The NL-IIC algorithm is formulated as follows.

Algorithm 1 NL-IIC

Select a suitable initial input $u_0(t)$.

- (1) Apply the input $u_i(t)$ to the system and record the tracking error $e_i(t) = r(t) - y_i(t)$.
- (2) Obtain $U_i(k)$ and $E_i(k)$ using the DFT. Determine $U_{i+1}(k)$ for each frequency k using (5) with $\rho(|Y|)$ given by (21). Obtain $u_{i+1}(t)$ using the IDFT.
- (3) Set $i \leftarrow i + 1$ and repeat from (1) until e_i converges.

5.2.1 The initial input

Statement (ii) of Theorem 5 motivates large initial inputs. As in system identification for LTI systems, higher input magnitudes improve the estimation accuracy by increasing the signal-to-noise ratio. Furthermore, if an FRF model of the plant, $G_m(e^{j\omega_k})$, is available, a suitable initial input is given by $U_0 = \alpha G_m^{-1} R$. The frequency dependent coefficient $\alpha(k) \in [0, 1]$ can be used to suppress the effect of model uncertainty as is described in [31]. In this way, prior information can be exploited to increase the convergence speed of the data-driven NL-IIC method.

5.2.2 The reference signal

Theorem 1 reveals that performance is increased if $|R|$ is large with respect to $|D_i|$. An upper bound δ on the spectrum of D_i can be estimated from additional experiments, as described in [28, §12]. If there exists freedom to shape $|R|$, then performance can be increased by increasing the ratio $\frac{|R|}{\delta}$ through reference shaping [30].

Fig. 3. The closed-loop configuration where H and C_{fb} are the printer and the feedback controller, respectively.

Fig. 4. (Top) Application of IIC in a batch-to-batch fashion. (Bottom) Application of IIC in a continuous setting. Transient induced disturbances are reduced by awaiting steady state operation.

5.2.3 Leakage

Recall that D_i includes the contribution of leakage. Consequently, $|D_i|$ can be reduced by ensuring that $y(t)$ is approximately in steady state [28]. If $r(t)$ is a time-limited signal, this can be achieved by including trailing zeros, as shown in Figure 4. In this way, NL-IIC can be applied in a batch-to-batch fashion similar to ILC [5]. If $r(t)$ is a periodic signal, then the periodic input $u_0(t)$ can be repeated until $y(t)$ has reached steady state before measuring, as is shown in Figure 4. In this way, NL-IIC can be applied in a continuous fashion similar to RC [14].

5.2.4 Feedback

Feedback control may be required to stabilize the plant and can be used in conjunction with NL-IIC. If a plant $H(z)$ is controlled in closed-loop, as shown in Figure 3, then NL-IIC can be equivalently applied. To see this, note that $Y_i = H(z)S(z)(F_i + C_{fb}(z)R) + S(z)\tilde{D}$, where $S(z) \triangleq (1 + C_{fb}(z)H(z))^{-1}$. Consequently, by defining $G(z) \triangleq H(z)S(z)$, $U_i \triangleq F_i + C_{fb}(z)R$ and $D \triangleq S(z)\tilde{D}$, (5) can be directly applied.

In this section, an explicit learning-gain function is developed, and implementation guidelines are presented. In the next section, NL-IIC is applied in experiments.

6 Application to a desktop printer

In this section, the developed NL-IIC method is applied to a desktop printer, and the results are compared to MF-IIC [22], revealing superior performance, which constitutes contribution C4.

6.1 The desktop printer

The reconfigured A3 printer is shown in Figure 5. The print head is positioned using a current-controlled motor

Fig. 5. The reconfigured desktop printer with: ① direct current motor ② drive belt ③ end stop ④ print head without cartridges ⑤ slide guide ⑥ encoder belt ⑦ motor input ⑧ end stop output ⑨ encoder output.

Fig. 6. (Top) The reference $r(t)$ in the time domain. (Bottom) Power spectral density of the reference up to 100 Hz.

via a cogged belt. The input current is generated using a linear amplifier. The voltage signal is generated by an EtherCAT[®]-based data acquisition unit that facilitates the communication between the printer and a laptop. This unit also processes the pulsed linear encoder output signal, which is used to measure the print head position with a resolution of 600 counts per inch. Simulink Real-Time[™] is used in conjunction with a real-time Ubuntu[®] kernel to facilitate the control implementation.

6.2 The iterative tracking control problem

The objective is to position the print head in a batch-to-batch setting as described in Section 5.2.3. The head is desired to perform a typical printing task as shown in Figure 6. The bottom plot shows that the reference contains mostly low-frequency content up to 100 Hz. The slide guide that constrains the print head motion introduces a significant amount of friction, which severely limits the performance achievable with feedback control. The printer is controlled in feedback as in Figure 3, at a sampling frequency of 2 kHz, using the following controller that achieves a bandwidth of 9 Hz, $C_{fb}(z) = \frac{215.3z - 206.8}{z^2 - 1.65z + 0.7034}$.

Fig. 7. The energy of the error signal as a function of the iterations shows that NL-IIC, as developed in this paper (—●—), results in large performance improvements. Comparatively, a sudden loss of performance can occur in pre-existing MF-IIC (—●—).

6.3 Applying NL-IIC and MF-IIC

The NL-IIC method is applied as described by the algorithm in Section 5.2, where $\rho(|Y|)$ is given by (21), with $\gamma = |R|$. This is compared to the case where $\rho(|Y|) = 1$, in which case NL-IIC reduces to MF-IIC [22]. In an initial experiment that is not shown here, MF-IIC resulted in extreme trial-to-trial variations of the input, and for various trials, the input signals exceeded the actuator limits. This was remedied by restricting MF-IIC learning to 100 Hz by setting $\rho(k) = 0$ for $k > \frac{100}{2000}(N-1)$. For both iterative procedures, the initial input is set as $F_0(k) = 4R(k)$.

6.4 Experimental results

Applying NL-IIC and MF-IIC results in a significant improvement of the tracking performance, as is shown in Figure 7. This figure displays the evolution of the tracking error energy across the trials. This shows that the developed NL-IIC converges in a relatively smooth manner, whereas MF-IIC results in erratic and unpredictable trial-to-trial behavior. Especially during iterations 6 and 19, the performance deteriorates significantly. The time domain performance during the 19th trial is shown in Figure 8, which shows that superior performance is achieved by the NL-IIC method. The power spectral density of the initial error and the error during the 19th trial are shown in Figure 9. This similarly shows a significant difference in the achieved performance and shows that increased tracking performance is achieved across a large frequency range.

The performance degradation of MF-IIC after trial 6 and 19 is attributed to the sudden amplification of the input signal, as is visualized in Figure 10. In the two upper plots, the input DFT coefficients $|U_i(k)|$ are shown for both NL-IIC and MF-IIC for frequencies up to 10 Hz. This frequency-by-frequency visualization shows that for MF-IIC the input components propagate in a non-smooth manner, whereas for NL-IIC these evolve more smoothly. The smoothness is quantified by the

Fig. 8. (Top) The tracking error during the 19th iteration indicates a significantly larger error for MF-IIC (—) than for NL-IIC (—). (Bottom) The corresponding control input signals.

Fig. 9. The Power Spectral Density (PSD) of the tracking error during the 19th iteration indicates that a large reduction is achieved with respect to the initial error (—●—) and that NL-IIC (—●—) outperforms MF-IIC (—●—) for nearly all frequencies.

bottom plots, which display the input propagation factors as given by (1). These plots confirm the relation between the loss of performance and large values of κ_i . Consequently, superior performance and robustness can be obtained by restricting κ_i .

In this section, NL-IIC and MF-IIC are applied to a desktop printer, which confirms that by restricting the input propagation factor, superior performance and increased robustness is achieved.

7 Conclusion

In this paper, a data-driven iterative learning algorithm is developed that incorporates system identification in the input updating process. Analysis of bounded disturbance signals reveals the source of convergence issues in typical pre-existing approaches. The insights that the framework delivers enable the development of a Non-linear Iterative Inversion-based Control approach that overcomes earlier limitations through the incorporation

Fig. 10. (Top) The DFT coefficients of the input for frequencies up to 10 Hz show erratic behavior for pre-existing MF-IIC (—•—), whereas propagation is smooth for the developed NL-IIC (—•—) method. (Bottom) The input propagation factor κ_i , as given by (18), indicates for which iterations the input variations are relatively large.

of learning-gain functions. Application of the developed approach in experiments confirms that superior tracking performance and increased robustness can be achieved with respect to a pre-existing approach. Future research aims to investigate the incorporation of alternative system identification techniques. In particular, those that exploit data from multiple previous trials to update the inverse model.

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A Proof of Lemma 2

PROOF. Substituting (3) in (2) for $i+1$ yields $E_{i+1} = R - GU_{i+1} - D_{i+1}$. Substituting (5) under Assumption 1 to 5 yields $E_{i+1} = R - G \left(U_i + \rho_i \frac{U_i}{Y_i} E_i \right) - D_{i+1}$, where ρ_i denotes $\rho(|Y_i|)$. Using that $E_i = R - GU_i - D_i$ yields $E_{i+1} = \left(1 - \rho_i \frac{GU_i}{Y_i} \right) E_i + D_i - D_{i+1} = \left(\frac{(1-\rho_i)Y_i + \rho_i D_i}{Y_i} \right) E_i + D_i - D_{i+1} = \left(\frac{(1-\rho_i)(R-E_i) + \rho_i D_i}{R-E_i} \right) E_i + D_i - D_{i+1}$. Taking the absolute value and applying the triangle inequality results in $|E_{i+1}| \leq \kappa_i |E_i| + |D_i - D_{i+1}|$, where κ_i is given by (12). Considering the worst sequence of disturbances according to Definition 4 directly results in (10). To show that (11) holds, consider (10). By recursive evaluation, it follows that $|E_i| \leq |E_0| \prod_{j=0}^{i-1} \kappa_j + \Delta\delta \sum_{j=1}^{i-1} \prod_{n=j}^{i-1} \kappa_n + \Delta\delta$, $i \geq 0$, where

κ_i is given by (12). The latter is upper bounded by $|E_i| \leq \bar{\kappa}^i |E_0| + \Delta\delta \sum_{j=0}^{i-1} \bar{\kappa}^j$, $i \geq 0$, where $\bar{\kappa} \in \mathbb{R}_{\geq 0}$ is any number such that $\kappa_i \leq \bar{\kappa} \forall i$. Application of the geometric series $\sum_{j=0}^{i-1} x^j = \frac{x^i - 1}{x - 1}$, $x > 0$, leads to (11).

B Proof of Lemma 3

PROOF. Consider κ_i as given by (12). For a certain $\alpha > 1$, by definition, $\kappa_i \leq \frac{1}{\alpha}$ is equivalent to $\left| \frac{(1-\rho_i)(R-E_i) + \rho_i D_i}{R-E_i} \right| \leq \frac{1}{\alpha}$. If $\bar{\rho} \leq 1$, this is implied if $\alpha(1-\rho_i)|R-E_i| + \alpha\rho_i|D_i| \leq |R-E_i|$ holds. Furthermore, if $1 - \alpha(1-\rho_i) > 0$, which is equivalent to $\rho > \frac{\alpha-1}{\alpha}$, it holds that

$$|R-E_i| \geq \frac{\alpha\rho_i}{1-\alpha(1-\rho_i)} |D_i|. \quad (\text{B.1})$$

Since $\bar{\rho} \leq 1$, it holds that $\frac{\alpha\rho_i}{1-\alpha(1-\rho_i)} \leq \alpha$, such that (B.1) holds if $|R-E_i| \geq \alpha|D_i|$. The latter holds if $|R| - |E_i| \geq \alpha\delta$, which is identical to $|E_i| \leq |R| - \alpha\delta$. Note that this requires that $|R| \geq \alpha\delta$. Consequently, if $|E_i| \leq |R| - \alpha\delta$ holds under the presented conditions, then it holds that $\kappa_i \leq \frac{1}{\alpha}$.

C Proof of Theorem 2

PROOF. Under the posed conditions, Theorem 1 applies, such that $\lim_{i \rightarrow \infty} |E_i| \leq \Delta\delta \frac{\alpha}{\alpha-1}$. Substituting (2) and (3) in this relation yields

$$\begin{aligned} \lim_{i \rightarrow \infty} |R - GU_i - D_i| &\leq \Delta\delta \frac{\alpha}{\alpha-1}, \\ \lim_{i \rightarrow \infty} |G^{-1}||R - GU_i - D_i| &\leq |G^{-1}| \Delta\delta \frac{\alpha}{\alpha-1}, \end{aligned} \quad (\text{C.1})$$

which implies that

$$\begin{aligned} \lim_{i \rightarrow \infty} |G^{-1}R - U_i| - |G^{-1}|\delta &\leq |G^{-1}| \Delta\delta \frac{\alpha}{\alpha-1}, \\ \lim_{i \rightarrow \infty} |G^{-1}R - U_i| &\leq |G^{-1}| \left(\delta + \Delta\delta \frac{\alpha}{\alpha-1} \right), \\ \lim_{i \rightarrow \infty} |U_\infty - U_i| &\leq r_u, \end{aligned}$$

where $U_\infty = G^{-1}R$ and r_u is given by (14). Furthermore, in case $\Delta\delta = 0$, (C.1) results in $\lim_{i \rightarrow \infty} |G^{-1}(R - D) - U_i| \leq 0$, such that $U_\infty = G^{-1}(R - D)$.

D Proof of Lemma 4

PROOF. Under Assumption 1 to 5, (5) can be written as $U_{i+1} = U_i + \rho_i \frac{U_i}{Y_i} (R - Y_i) = (1 - \rho_i)U_i + \rho_i \frac{RU_i}{Y_i}$, where

ρ_i denotes $\rho(|Y_i|)$. Substituting (3), gives (16). Taking the absolute value yields $|U_{i+1}| = \kappa_i |U_i|$ where κ_i is given by (18).

E Proof of Lemma 5

PROOF. Consider κ_i as given by (18). By definition, $\kappa_i \leq 1$ is equivalent to $\left| \frac{(1-\rho_i)(GU_i+D_i)+\rho_i R}{GU_i+D_i} \right| \leq 1$. If $\bar{\rho} \leq 1$, this is implied if $(1-\rho_i)|GU_i+D_i| + \rho_i|R| \leq |GU_i+D_i|$ holds, which can be written as $\rho_i|GU_i+D_i| - \rho_i|R| \geq 0$. Furthermore, if $\rho(|Y|) > 0$ for $Y \neq 0$, it holds that $|GU_i+D_i| \geq |R_i|$, which is implied if $|U_i| \leq |G^{-1}|(|D_i| - |R|)$. If $|D_i| \geq \beta|R| \forall i$ for a certain $\beta > 1$, then the latter is implied if $|U_i| \leq (\beta - 1)|G^{-1}||R|$.

F Proof of Lemma 6

PROOF. Since $\frac{1}{x}$ becomes arbitrarily large as $x \rightarrow 0$, the open interval $X \triangleq (0, \infty)$ is decomposed as $X = I_1 \cup I_2$, $I_1 = (0, \epsilon]$, $I_2 = (\epsilon, \infty)$, $\epsilon > 0$. Since $\rho(x)$ is bounded, it holds that $\left| \frac{\rho(x)}{x} \right| \leq \sup_{x \in I_2} |\rho(x)| \sup_{x \in I_2} \left| \frac{1}{x} \right| = \frac{\bar{\rho}}{\epsilon} \in \mathbb{R}_{\geq 0}$, such that $f(x)$ is bounded on I_2 . Next, it is shown that $f(x)$ is bounded on I_1 . First note that since $\lim_{x \rightarrow 0} \rho(x) = 0$, $\lim_{x \rightarrow 0^+} \frac{\rho(x)}{x} = \frac{0}{0}$. Since in addition $\frac{d\rho}{dx}|_{x=0} \in \mathbb{R}$, this indeterminate limit exists, and can be explicitly determined by using L'Hôpital's rule [12, p. 400] to yield $\lim_{x \rightarrow 0^+} f(x) = f_0 \in \mathbb{R}$. Second, define the extension

$$\hat{f}(x) = \begin{cases} f(x) & x \in (0, \epsilon], \\ f_0 & x = 0. \end{cases}$$

Clearly, $\hat{f}(x)$ is continuous on the closed interval $[0, \epsilon]$, such that by the theorem of Weierstrass [12, §19.4], $\hat{f}(x)$ is bounded and attains its maximum on the interval. Since $\hat{f}(x)$ contains $f(x)$, the latter is bounded on $I_1 = (0, \epsilon]$.

G Proof of Theorem 5

PROOF. Under the condition presented in Lemma 7, the sequences $\{U_i\}$ are described by (20a) or (20b). A proof of each statement is as follows. If $|R| > |D|$, (20a) results in $\lim_{i \rightarrow \infty} U_i = G^{-1}(R - D)$ and $\lim_{i \rightarrow \infty} E_i = 0$, such that P1 is satisfied with $r_e = 0$, which proves statement (i). Relation (12) for $\rho(|Y|) = 1$ results in

$\kappa_i = \frac{|D|}{|Y_i|}$. Requiring that $\kappa_i < 1 \forall i$ results in $|D| < |Y_i| = |GU_i + D|$. In combination with (20a) this yields

$$\begin{aligned} |D| &< \left| \frac{U_0(R-D)}{(G^{-1}(R-D)-U_0)\left(\frac{D}{R}\right)^i + U_0} + D \right| \Leftrightarrow \\ &|D| \left| (G^{-1}(R-D) - U_0)\left(\frac{D}{R}\right)^i + U_0 \right| \\ &< \left| U_0 R + (G^{-1}(R-D) - U_0)\left(\frac{D}{R}\right)^i D \right|. \end{aligned}$$

The latter holds if $2|G^{-1}(R-D) - U_0| \left| \frac{D}{R} \right|^i < |U_0| \left(\left| \frac{R}{D} \right| - 1 \right)$. By assuming that $|R| > 2|D|$, this is satisfied if $2|G^{-1}(R-D)| < |U_0| \left(\left| \frac{R}{D} \right| - 2 \right)$, which proves statement (ii). If $|R| < |D|$, (20a) results in $\lim_{i \rightarrow \infty} U_i = 0$. If $R = D$, (20b) results in $\lim_{i \rightarrow \infty} U_i = 0$. If $|R| = |D|$, $R \neq D$, (20a) results in bounded periodic cycles or aperiodic sequences [13]. Consequently, U_i converges to either 0 or $G^{-1}(R-D)$, with exception of the case that $|R| = |D| \wedge R \neq D$.

H Proof of Proposition 1

PROOF. Consider (21). Since $\bar{\rho} = 1$, $\rho(|Y|) \neq 0$ for $Y \neq 0$, $\rho(|Y|)$ satisfies the criteria of Theorem 3. Moreover, since $\lim_{|Y| \rightarrow 0} \rho(|Y|) = 0$, and $\frac{d\rho}{d|Y|}|_{|Y|=0} = 0$, $\rho(|Y|)$ satisfies the criteria of Theorem 4. Recalling (19) in Theorem 4 shows that $\lim_{|Y| \rightarrow 0} \kappa \leq 1 + \rho(0) + |R| \lim_{|Y| \rightarrow 0} \frac{\rho(|Y|)}{|Y|} = 1$, where the indeterminate limit in the third term is evaluated using L'Hôpital's rule. Furthermore, $\tau = \sup_{|Y| \in \mathbb{R}_{\geq 0}} \frac{\rho(|Y|)}{|Y|}$ is obtained by computing the stationary point on $[0, \gamma]$ from $\frac{d}{d|Y|} \left(\frac{\rho(|Y|)}{|Y|} \right) = \cos(\mu) + \mu \sin(\mu) - 1 = 0$, where $\mu = \frac{\pi}{\gamma}|Y|$, $0 < \mu \leq \pi$. This results in a unique solution that is numerically approximated as 2.3311. This is the maximizer of $\frac{\rho(|Y|)}{|Y|}$ of on $(0, \gamma]$, where τ the global maximum on $(0, \infty)$ since $\tau\gamma > 1$.

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