Near-grazing behaviour of oscillators impacting with rigid and compliant obstacles
Molenaar, J.; Weger, de, J.G.; van de Water, W.

Published: 01/01/1999

Citation for published version (APA):
Near-grazing behaviour of oscillators impacting with rigid and compliant obstacles

Jaap Molenaar†, John G. de Weger‡, Willem van de Water‡
† Mathematics Department ‡ Physics Department, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

Abstract. Impacting systems are found in a great variety of mechanical constructions and they are intrinsically nonlinear. In this paper it is shown how near-grazing systems, i.e., systems in which the impacts take place at low speed, can be described by discrete mappings. The derivation of this mapping for a harmonic oscillator with a stop is dealt with in detail. It is found that the resulting mapping for rigid obstacles is somewhat different from those presented earlier in the literature. The derivations are extended to systems with a compliant obstacle. We find that the map for impacts with a compliant obstacle is very similar to the one describing collisions with a rigid obstacle. A notable difference is a change of scale of the bifurcation parameter. We illustrate our findings in the limit of large damping, where the mechanism of period adding can be analyzed exactly. The relevance of our results for experiments on practical impact systems is indicated.

1. Introduction

Impacting behaviour is found in a large number of mechanical systems. Examples are gear rattle, ships colliding against fenders, loosely fitting joints, suspension bridges and ball bearings. In atomic force microscopy, impacts occur on a mesoscopic scale and the interaction forces may have intricate form (Berg and Briggs, 1998). Although the dynamics in between impacts may be linear, the collision introduces an essential nonlinearity. We believe that impacts are the principal source of nonlinearity in discrete mechanical systems.

A special situation arises when the impacts are with zero velocity, so-called grazing impacts. For impacts that are close to grazing, it is possible to condense the mathematical description to a discrete mapping as was first done in Nordmark (1991, 1997); followed by Frederiksson and Nordmark (1997) and Frederiksson (1998). As the bifurcation properties of a mapping are analyzed much more readily than those of a differential equation, the study of these mappings led to the discovery of important organizing principles in the dynamics of impact oscillators.
The Nordmark map (Nordmark, 1991) reads as

\[
\begin{align*}
  x_{n+1} &= \alpha x_n + y_n + \rho \\
  y_{n+1} &= -\gamma x_n
\end{align*}
\]

if \( x_n \leq 0 \),

\[
\begin{align*}
  x_{n+1} &= -\sqrt{x_n} + y_n + \rho \\
  y_{n+1} &= -\gamma r^2 x_n
\end{align*}
\]

if \( x_n > 0 \).  

Here, \((x_n, y_n)\) are the transformed coordinates of the two-dimensional phase space of the original impact oscillator taken at stroboscopic times \( t = n \in \mathbb{Z} \). Equation (1) describes the non-impacting case with \( x_n < 0 \), whereas (2) is applied when the \( x_n \)-coordinate crosses the position of the wall at \( x = 0 \). The parameters \( \alpha \) and \( \gamma \) are related to the parameters of the ordinary differential equation, whereas the restitution coefficient \( r \) gauges the energy loss at impact. The bifurcation parameter \( \rho \) is proportional to the driving amplitude of the impact oscillator. At large negative values of \( \rho \) no impacts occur, and equation (1) applies. When \( \rho \) is increased in a quasi-stationary way from the non-impacting state, impacts first occur at \( \rho = 0 \). These impacts may be with a relatively large velocity and the transition that has taken place may be hysteretic, i.e. impacts may remain when \( \rho \) is subsequently smoothly decreased and may only vanish at a negative value of \( \rho \). The emergence of a square-root in (2) is characteristic for grazing collisions. Because the Jacobian of the map behaves as \( x^{-1/2} \) at \( x = 0 \), we will refer to it as a square-root \emph{singularity}. In the vicinity of the line \( x = 0 \), the map stretches phase space extremely.

Intimately tied to this square-root singularity is the phenomenon of period adding. Depending on the value of the parameters \( \alpha \) and \( \gamma \), this can come either as an infinite series of reverse period addings if \( \rho \) is increased from 0, or as a series of (non-impacting) period-1 to period-\( M \), \( M = 1, 2, \ldots \), transitions at given values of \( \alpha \) and \( \gamma \). In all cases, the \( M \)-periodic orbits reached are \emph{maximal} periodic orbits that have a single impact per period. It is tempting to draw an analogy with period-doubling bifurcations, which are found in systems with a quadratic nonlinearity (Feigenbaum, 1983). In both cases, the order of the nonlinearity determines a characteristic bifurcation scenario.

Figure 1 illustrates these period addings when the bifurcation parameter is reduced from a positive value downwards. As period addings are readily accessible in the experiment (de Weger et al, 1996) we will concentrate on this characteristic feature of impact oscillators when we test the prediction of mappings.

The mapping (1,2) assumes that the wall is perfectly rigid so that impacts take place instantaneously. This idealisation will not be fulfilled in real experiments where collisions occur with a more or less compliant wall which introduces a time delay at impact. The key question then is if a finite compliance will “smooth” the square root singularity, thus essentially altering the typical bifurcation structure of grazing impact oscillators. The prime motivation of this work is a derivation for grazing impact mappings in the case
of collisions with yielding obstacles. To this aim we will first derive in sections 2–3 the mapping for a harmonic oscillator which undergoes grazing impacts with a hard wall. We will use a lucid and straightforward expansion that allows a precise association of the parameters of the resulting map with those of the differential equation. We will argue that (2) needs to be altered slightly but essentially. Having laid out our mathematical tools, the extension to the case of impacts with a non-rigid wall in Sec. 6 appears straightforward. It will also become clear that our derivation transcends the particular realization of a simple harmonic impact oscillator.

Our conclusion will be that instantaneous and non-instantaneous impacts can be dealt with on an equal footing. Analysis of the resulting formulae shows that the results obtained for the rigid obstacle are robust for deviations of rigidity. The only exception is the case in which the collision velocities are relatively low and the obstacle is a perfectly elastic repeller. Period adding bifurcations are thus expected to be found also in experimental situations which involve compliant obstacles. In section 7 we will
The physical system studied in this paper. The mass $m$ of the oscillator can collide with a yielding wall. We assume that the massless wall is attached with frictionless springs to the fixed world. The resilience of the wall is determined by the stiffness of the second spring. We use normalised units with $m = 1$, the wall position in rest at $u = 0$, and the rest position of the oscillating mass at $u = -1$.

illustrate the predictive power of our mapping when we analyze period addings in the limit of large friction.

2. Harmonic oscillator

A sketch of the physical system is shown in figure 2. We consider a periodically driven harmonic oscillator with mass $m$, friction coefficient $c$, spring constant $k$. The driving force has angular velocity $\omega$ and amplitude $f$. At a distance $\ell > 0$ from the rest position of the oscillator an obstacle is placed. In non-dimensional form the equation of motion reads as

$$\ddot{u} + \nu \dot{u} + \Omega^2 (u + 1) = F \cos(2\pi t + \varphi)$$  \hspace{1cm} (3)

with

$$\nu = 2\pi c / m \omega,$$
$$\Omega^2 = 4\pi^2 k / m \omega^2,$$
$$F = 4\pi^2 f / m \omega^2 \ell.$$  

The position has been scaled such that the rest position of the mass is at $u = -1$ and of the obstacle at $u = 0$. So, the distance $\ell$ is the unit of length. Time has been scaled such that the external forcing has frequency $2\pi$. For later convenience a phase difference $\varphi$ is introduced. With these conventions, the particular solution of (3) has the elegant form

$$p(t) = (1 + \sigma \cos(2\pi t) - 1,$$  \hspace{1cm} (4)

with

$$\sigma = (F - F_g) / F_g.$$
\[ F_g = \sqrt{(\Omega^2 - 4\pi^2)^2 + 4\pi^2p^2}, \]
\[ \varphi = \arccos[(\Omega^2 - 4\pi^2)/F_g], \]

For \( F = F_g \) the particular solution \( p(t) \) grazes the wall at times \( t = n \in \mathbb{Z} \) when it collides with zero velocity. We take the normalized excitation amplitude \( \sigma \) as the bifurcation parameter. One of our tasks will be to derive the relation between \( \sigma \) and the bifurcation parameter \( \rho \) of the mapping. If we denote points in phase space by \( \mathbf{u} = (u, \dot{u})^T \), the solution \( \mathbf{u}_0(t) \) of the undriven oscillator (3) with \( F = 0 \) and with the obstacle ignored, is given by

\[ \mathbf{u}_0(t) = \mathbf{P}(t - t_0)\mathbf{u}_0(t_0) \]

with the evolution matrix \( \mathbf{P}(t) \) of the free system

\[ \mathbf{P}(t) = \frac{1}{s_2 - s_1} \begin{pmatrix} s_2 e^{s_1 t} - s_1 e^{s_2 t} & e^{s_2 t} - e^{s_1 t} \\ s_1 s_2(e^{s_1 t} - e^{s_2 t}) & s_2 e^{s_2 t} - s_1 e^{s_1 t} \end{pmatrix}, \]

where \( s_1 = \frac{1}{2}(\nu + \sqrt{\nu^2 - 4\Omega^2}) \) and \( s_2 = \frac{1}{2}(\nu - \sqrt{\nu^2 - 4\Omega^2}) \). For later convenience we mention that \( P_{11} - \nu P_{12} = P_{22} \). Note that \( \mathbf{P}(0) \) equals the unit matrix. Given the initial conditions \( \mathbf{u}(t_0) \), the general solution of equation (3) is the sum of the solutions in (4) and (6) and can be written as

\[ \mathbf{u}(t) = \mathbf{P}(t - t_0)[\mathbf{u}(t_0) - \mathbf{p}(t_0)] + \mathbf{p}(t) \]

Expression (8) holds as long as no impact takes place, thus if \( u < 0 \). The time–one operator \( \mathbf{P} \equiv \mathbf{P}(1) \) connects states at successive periods of the driving; its eigenvalues are \( \exp(s_1) \) and \( \exp(s_2) \). From this we directly conclude that \( p(t) \) is asymptotically stable in case of damping, i.e., if \( \nu > 0 \). So, the system tends to \( p(t) \) in between impacts. In the overdamped case \( \nu^2 > 4\Omega^2 \) this convergence is fast and without oscillations, whereas in the underdamped case \( \nu^2 < 4\Omega^2 \) this approach is slow and \( u(t) \) will oscillate around \( p(t) \). The difference between under- and overdamped oscillations may lead to essentially different maps, as discussed in section 8. The trace \( \alpha = \text{Tr}(\mathbf{P}) \) and determinant \( \gamma = \text{Det}(\mathbf{P}) \) will play an important role in the following. They are given by the sum and the product of the eigenvalues:

\[ \alpha = e^{s_1} + e^{s_2} = 2e^{-\frac{1}{2}\nu} \cosh\left(\frac{1}{2}\sqrt{\nu^2 - 4\Omega^2}\right) \]
\[ \gamma = e^{s_1} \times e^{s_2} = e^{-\nu}. \]

3. Modeling the impact

To incorporate the effect of the obstacle, an impact model has to be introduced. We denote the velocity and time at collision by \( (v_c, t_c) \) and the velocity and time at recoil by
(v_r, t_r). The duration of the impact, i.e. the time spent in the region u > 0, is denoted by \( \Delta t \). An impact model then must specify the relations

\[
v_r = v_r(v_c, t_c), \quad \text{and} \quad t_r = t_c + \Delta t(v_c, t_c).
\]

In the next sections we first derive expressions for the case of impacts with a perfectly rigid wall when the recoil is instantaneous (\( \Delta t = 0 \)). After that the effect of a compliant wall is studied. For the velocity we assume the simple relation

\[
v_r = -r v_c, \quad 0 \leq r \leq 1.
\]

The case \( r = 1 \) corresponds to an elastic wall. If \( r < 1 \) energy is absorbed during the impact. It is interesting to notice that relation (11) has still a firm physical meaning if \( r < 0 \). Then, the mass does not bounce back, but penetrates the region \( u > 0 \) and instantaneously slows down upon passing the boundary \( u = 0 \). For \( r = -1 \) the obstacle is totally ignored, and we are effectively dealing with an unrestricted oscillator. The latter case can conveniently be used to check the formulae to be derived in the following. The model for non-instantaneous impacts will be dealt with in section 6.

### 4. Instantaneous impacts

Since the dynamics between impacts (via (8)) and during impacts (via (10)) is explicitly known, the orbits can in principle be calculated. The dynamical equations (8,10) can be reduced to a discrete mapping for orbits which undergo near-grazing collisions. The reduction is possible for orbits which remain close to the particular solutions \( p(t) \) in (4) with \( \sigma \approx 0 \). Possible impacts then occur in short time windows around \( t = n \in \mathbb{Z} \). In the literature (Whiston, 1987, 1992; Foale and Bishop, 1992; Budd and Dux, 1994; Budd et al., 1995) one usually studies mappings which connect states of the system at successive impacts. We found it more convenient to focus on the Poincaré map connecting the states \( u_n = (u_n, v_n)^T \) at stroboscopic times \( t = n \in \mathbb{Z} \).

A judicious choice of the state is a crucial step in the present derivation. For \( u_n \) and \( v_n \) we take the position and velocity the system would have at \( t = n \) if the obstacle would have been removed shortly before \( t = n \). This is most conveniently understood from figure 3. If the impact takes place (shortly) after \( t = n \) or if no impact occurs at all around \( t = n \), the state involves the real values of position and velocity. If an impact occurs (shortly) before \( t = n \), \( u_n \) is a virtual state with \( u_n > 0 \). We account for a possible impact near \( t = n \) through the local mapping \( L \):

\[
\hat{u}_n = L(u_n).
\]

The action of \( L \) and the location of the points \( u_n \) and \( \hat{u}_n \) are illustrated in figure 3. The key idea is to calculate the orbit around \( t = n \) with and without accounting for the presence of the obstacle. If an impact occurs before \( t = n \) (figure 3a), \( u_n \) represents
the virtual state at $t = n$ obtained if no obstacle were present, while $\hat{u}_n$ represents the physical state obtained by incorporating the impact. If the impact takes place after $t = n$, the roles of $u_n$ and $\hat{u}_n$ are reversed, as shown in figure 3b. If there is no impact or if the impact is grazing, $\mathcal{L}$ is the identity.

After application of $\mathcal{L}$, the system is propagated from state $\hat{u}_n$ at $t = n$ to state $u_{n+1}$ at $t = n + 1$ without accounting for the obstacle, thus by applying the evolution given by (8). The full mapping is then given by

$$u_{n+1} = \mathbf{P}[\mathcal{L}(u_n) - p(t_n)] + p(t_{n+1}).$$  \hspace{2cm} (13)

From (4) we have $p(t_n) = p(t_{n+1}) = (p(n), \dot{p}(n))^T = \sigma(1, 0)^T \equiv \sigma \mathbf{e}_1$.

Using a parabolic approximation of the orbits around $t = n \in \mathbb{Z}$, one finds that an impact will take place if the quantity

$$\chi_n = v_n^2 - 2\mathcal{A} u_n$$  \hspace{2cm} (14)

is positive. Here, $\mathcal{A}$ is the acceleration of the grazing orbit, i.e. $\ddot{p}(n)$ with $\sigma = 0$:

$$\mathcal{A} = -4\pi^2.$$  \hspace{2cm} (15)

If $\chi_n > 0$ and $u_n > 0$, an impact has occurred shortly before $t = n$, whereas if $\chi_n > 0$ and $u_n < 0$ the oscillator will hit it shortly after $t = n$. Both cases can be dealt with on an equal footing.

We now expand the dynamics of near-grazing orbits in terms of the small parameters $u_n$, $v_n$, and $\sigma$. We are interested in a derivation up to and including the linear order, so terms involving cross products are omitted. \hspace{0.5cm} \dagger To find $\mathcal{L}$ explicitly, we express $\hat{p}_n$ in

\hspace{0.5cm} \dagger We therefore assume that the small parameters $u_n$, $v_n$, and $\sigma$ are all of the same order of magnitude. Other assumptions are possible (de Weger, 1999), but they do not affect our main conclusions.
terms of $u_n$. In the derivation we use the impact characteristics $(v_c, t_c)$ as intermediates. To find approximations for $(v_c, t_c)$ that are accurate up to linear order in $(u_n, v_n, \sigma)$, the orbit $u(t)$ is expanded up to and including third order in time:

$$u(t) = u_n + v_n t + \frac{1}{2} A_n t^2 + \frac{1}{6} B_n t^3.$$  

Here, we conveniently use $t = n$ as the origin of the time axis and introduce the notations $A_n = \ddot{u}(t = n)$ and $B_n = \frac{d^3 u}{dt^3}(t = n)$. Setting $u(t_c) = 0$ we obtain the expression

$$t_c = \frac{1}{A_n} \left( \sqrt{\chi_n} - v_n + \frac{B_n}{3 A_n} u_n \right). \tag{16}$$

Substitution of $t_c$ into the velocity $v \equiv \ddot{u}$ yields

$$v_c = \sqrt{\chi_n} - \frac{2B_n}{3A_n} u_n. \tag{17}$$

From these expressions we see that in lowest order, $t_c$ and $v_c$ scale with $\sqrt{u_n}$. This is due to the parabolic character of the orbit near its maximum. According to (3) the acceleration of the oscillator is given by

$$\ddot{u} = -\nu v - \Omega^2 (u + 1) + (1 + \sigma) F_g \cos(2\pi t + \phi), \tag{18}$$

which is a function of $(u, v, \sigma, t)$. Substituting $t = n$ we find that, in linear approximation, $A_n$ can be written as

$$A_n = \mathcal{A} - \Omega^2 u_n - \nu v_n + (\Omega^2 - 4\pi^2) \sigma \tag{19}$$

with the 'grazing' acceleration $\mathcal{A}$ introduced in (15). In a similar way we obtain

$$\frac{d^3 u}{dt^3} = -\nu \ddot{u} - \Omega^2 v - 2\pi (1 + \sigma) F_g \sin(2\pi t + \phi). \tag{20}$$

Substitution of (18) and $t = n$ yields

$$B_n = \nu \Omega^2 u_n + (\nu^2 - \Omega^2) v_n - \nu \Omega^2 \sigma. \tag{21}$$

We conclude that in (16) and (17) $A_n$ and $B_n$ can be replaced by their 'grazing' values $\mathcal{A}$ and $\mathcal{B} \equiv 0$, since terms of order higher than linear can be omitted:

$$t_c(u_n, v_n) = \frac{1}{\mathcal{A}} (\sqrt{\chi_n} - v_n), \quad v_c(u_n, v_n) = \sqrt{\chi_n}. \tag{22}$$

Next, we expand $u(t)$ around the (still unknown) quantities $(\hat{u}_n, \hat{v}_n)$:

$$u(t) = \hat{u}_n + \hat{v}_n t + \frac{1}{2} \hat{A}_n t^2 + \frac{1}{6} \hat{B}_n t^3. \tag{23}$$

Substituting $t = t_c$ we obtain the equation

$$u(t_c) = 0 = \hat{u}_n + \hat{v}_n t_c + \frac{1}{2} \hat{A}_n t_c^2 + \frac{1}{6} \hat{B}_n t_c^3. \tag{24}$$

For the velocity we similarly find

$$v(t_c) = -r v_c = \hat{v}_n + \hat{A}_n t_c + \frac{1}{2} \hat{B}_n t_c^2. \tag{25}$$
We remark that $\hat{A}_n$ and $\hat{B}_n$ are given by the expressions (19) and (21) after replacement of $(u_n, v_n)$ by $(\hat{u}_n, \hat{v}_n)$. If we substitute these expressions into (24) and (25) we meet with equations which are linear in $(\hat{u}_n, \hat{v}_n)$. After solving these, we obtain $(\hat{u}_n, \hat{v}_n)$ in terms of $(v_n, u_n)$:

$$\hat{u}_n = -(1 + 2r)u_n - \frac{1}{A}(r + 1)v_n \sqrt{\chi_n},$$  \hspace{1cm} (26)

and

$$\hat{v}_n = v_n - (r + 1)\left(\sqrt{\chi_n} - 2\nu u_n\right).$$ \hspace{1cm} (27)

The latter two expressions describe the action of the map $L$ introduced in (12). Note that the second term in the right hand side of (26) is of linear order, although it contains a product. This is most readily seen in the limit $v_n \gg u_n$, for which we have in good approximation $\hat{u}_n \approx -ru_n$. The map trivially reduces to the identity for the case $r = -1$ (no obstacle). We notice that the expressions yield the correct answer also if the impact occurs after $t = n$. The second term at the right hand side in (26) then assures that $|\hat{u}_n| < |u_n|$, as is to be expected.

It is convenient to write the local map $L$ in matrix notation:

$$L(u_n) = Nu_n + n(u_n),$$ \hspace{1cm} (28)

with

$$N(t) = \begin{pmatrix} -(1 + 2r) & 0 \\ 0 & 1 \end{pmatrix},$$ \hspace{1cm} (29)

and

$$n(u_n) = -(1 + r) \begin{pmatrix} \frac{1}{A} v_n \sqrt{\chi_n} \\ \sqrt{\chi_n} - 2\nu u_n \end{pmatrix}.$$ \hspace{1cm} (30)

The nonlinearity of $L$ is contained in the vector $n$. Substitution of (28) into (13) yields for $\chi_n \geq 0$

$$u_{n+1} = P(Nu_n + n) + \sigma(I - P)e_1,$$ \hspace{1cm} (31)

while for $\chi_n \leq 0$ we have the no-collision case

$$u_{n+1} = Pu_n + \sigma(I - P)e_1.$$ \hspace{1cm} (32)

With equations (31,32) we have brought the dynamics of the harmonic impact oscillator in the form of a mapping. The mapping is approximate in that we have assumed near-grazing orbits. The mapping is not yet very transparent: it still lacks an impact criterion which depends on a single coordinate (such as the map (1,2)). In
the next section we will reshape the mapping via a nonlinear transformation in order to obtain similarity with the map (1,2). It must be realized that this procedure might reduce the accuracy of the map (31,32) which was derived as a linear approximation to the near–grazing dynamics.

5. Natural coordinates

Map (31,32) can be put into a more concise form by realizing that the factor $\chi_n$, which we introduced in (14) and which discriminates between impacting and non-impacting orbits, plays a crucial role. The curve $\chi_n = 0$ in phase space represents the grazing orbit in the vicinity of $t = t_n$. This suggests to choose the family of curves given by $\chi_n = c$ with $c$ constant as level lines of new coordinates:

$$x = l \chi.$$  \hspace{1cm} (33)

Here and in the following equation the constant factor $l > 0$ is introduced for later convenience. For the other new coordinate a convenient choice is the following linear combination of $\chi$ and $v$:

$$y = -l \left[ P_{22} \chi + 2A P_{12} v + 2A \sigma (P_{22} - \gamma) \right].$$  \hspace{1cm} (34)

This combination is chosen such that the second component of the 2-dimensional map (31,32) gets a simple form, as seen in (37,38). In figure 4 the level lines of the new coordinates $(x,y)$ are sketched. Inverting this transformation and retaining only terms linear in $(x,y,\sigma)$, we obtain the expressions

$$u = \frac{x}{l}, \quad v = -\frac{1}{2lA P_{12}} \left[ y + P_{22} x + 2A \sigma (P_{22} - \gamma) \right].$$  \hspace{1cm} (35)

The inverse transformation does not exist if $P_{12} = 0$, but this case corresponds to critical damping and can be trivially accomodated by adjusting the propagator $P(t)$ in equation

Figure 4. Level lines of the new $x$–coordinate, defined in (34). The level lines of the new $y$–coordinate (34) have a similar shape, but with respect to a shifted origin.
Applying (33,34) at \( t = n + 1 \) we obtain \((x_{n+1}, y_{n+1})\) as functions of \((u_{n+1}, v_{n+1})\). The latter are expressed in terms of \((u_n, v_n)\) via the map (31,32). Application of the transformation (35) yields the map in transformed coordinates. As long as \( r \neq -1 \), which corresponds to absence of the obstacle, an appropriate choice for the factor \( l \) is

\[
l = \frac{1}{4A^2 (1+r)^2 P_{12}^2}.
\]

After lengthy algebra, in which terms of order higher than linear in \((x, y, \sigma)\) are omitted, we find that the transformed map reads as

\[
\begin{align*}
  x_{n+1} &= \alpha x_n + y_n + \rho \\
  y_{n+1} &= -\gamma x_n \\
  x_{n+1} &= -c_1 \sqrt{x_n} + c_2 x_n + y_n + \rho \\
  y_{n+1} &= c_3 x_n
\end{align*}
\]

if \( x_n \leq 0 \),

if \( x_n \geq 0 \),

The coefficients \( c_1, c_2, \) and \( c_3 \), and the bifurcation parameter \( \rho \), are given by

\[
\begin{align*}
  c_1 &= \text{sign}(P_{12}), \\
  c_2 &= \alpha - 2(1+r)P_{22} + (1+r)^2 P_{22}^2, \\
  c_3 &= (1+2r)\gamma - (1+r)^2 P_{22}^3, \\
  \rho &= \frac{1 - \alpha + \gamma}{2|\lambda| (1+r)^2 P_{12}^2} \sigma.
\end{align*}
\]

If \( r = -1 \), the obstacle is ignored and the dynamics is described by the map in (37) only, irrespective of the sign of \( x_n \). The association of the parameters \( \alpha \) and \( \gamma \) in the collisionless map (37) with the parameters of the ordinary differential equation (3) can also be made by realizing that the eigenvalues of its Jacobian must be the same as those of the linear propagator \( P \), as was also noticed by Chin et al (1994). The form of the remaining parameters, however, can only be learned from a full nonlinear analysis as is done here.

In comparison with the Nordmark map (1,2), our map has several new features. First, we emphasize that the sign of the square root term in equation (38) depends on the linear propagator \( P_{12} \) and is not fixed, as in (2). We will explain in section 8 that a fixed negative sign prohibits period-1 maximal periodic orbits of the underdamped oscillator, in clear contradiction with both experiment and numerical simulation. In a forthcoming publication we will show that the term proportional to \( x_n \) in equation (38) is essential for a quantitative understanding of the destruction of maximal periodic orbits through an additional impact. This term is absent in the Nordmark map (1,2). Finally we point out that the \( c_3 \) parameter differs from the corresponding prefactor in (2). In the next section we will demonstrate that with the groundwork now laid out, the derivation of the map for the case of non-instantaneous impacts can be done readily.
6. Non-instantaneous impacts

If the obstacle is not rigid but compliant, the impacting mass will penetrate the region $u > 0$ for some time interval, $\Delta t$ say. For $u < 0$ the motion of the oscillator is governed by (3). For $u > 0$ we assume the motion to be described by the dimensionless equation

$$\ddot{u} = A - \kappa^2 u.$$  (40)

Since the interval $\Delta t$ is very short, the driving force is represented by the constant acceleration $A$ defined in (15). The wall is assumed to repel the mass harmonically, with spring constant $\kappa^2$. The stiffness of the wall is gauged by the ratio $\Gamma = \kappa^2/\Omega^2$ of the spring constant of the wall to that of the oscillator. An infinitely stiff wall has $\Gamma = \infty$. In equation (40) the wall is assumed to have no friction. To maintain similarity with the instantaneous impact model (11), we assume that energy is instantaneously absorbed by the wall at the moment of collision. The collision velocity $v_c$ is thus reduced by a factor $r$ with $0 \leq r \leq 1$. This approach describing the system with different spring constants in different regions is also followed in the model study of Shaw and Holmes (1983). The solution of (40) with initial conditions $u(t_c) = 0$ and $v(t_c) = r v_c$, is

$$u(t) = \frac{A}{\kappa^2} \left[ 1 - \cos \kappa(t - t_c) \right] + \frac{r v_c}{\kappa} \sin \kappa(t - t_c),$$

which holds in the time window $t_c \leq t \leq t_c + \Delta t$. The length of $\Delta t$ follows from the condition $u(t_c + \Delta t) = 0$:

$$\Delta t = \frac{1}{\kappa} \arccos \frac{1 - \beta^2}{1 + \beta^2}, \quad \text{with} \quad \beta = \frac{r v_c \kappa}{|A|}.$$  (42)

Since the obstacle reacts harmonically, it leaves the region $u > 0$ with the same (reduced) velocity as it had upon entering it. Thus

$$v_r = -r v_c.$$  (43)

In figure 5 $\Delta t$ is drawn as a function of $\beta$. Note that the time delay behaves linearly for low collision velocities, but converges to a constant limiting value for increasing values of $v_c$.

In the following we shall apply the result in (42) for the two limiting cases $\beta \ll 1$ and $\beta \gg 1$. We refer to these as low and high velocity impacts, respectively. Whether the collision velocity is low or high, not only depends on the collision velocity $v_c$, but also on the wall stiffness $\kappa$ and on the restitution coefficient $r$. As the magnitude of the collision velocity depends on the dynamical state of the impact oscillator, it is not possible to give apriori criteria for wall hardness, for example as a criterion on the value of $\Gamma$. From an experimental point of view, an extremely interesting question is how the softness of the wall affects the bifurcation scenario’s which are characteristic for impact oscillators.
Figure 5. General behavior of the residence time $\Delta t$ as a function of $\beta = rv, \kappa / |A|$.

For small $\beta$ the time delay grows linearly with $\beta$ and the impacts are soft. For large $\beta$ the time delay is constant and the impacts are hard. In both limits a map with a square root singularity is derived.

6.1. Low velocity impacts

For low velocity impacts, we assume $\beta \ll 1$, i.e.

$$v_c \ll \frac{|A|}{r \kappa}.$$  \hfill (44)

To lowest order in $\beta$, the duration of the impact then is

$$\Delta t = \frac{2r}{|A|} v_c = \frac{2\beta}{\kappa}.$$ \hfill (45)

The stiffness of the wall has now dropped from (45), and the time delay $\Delta t$ could also have been obtained by setting $\kappa$ to zero in (40). The only remaining wall parameter is the restitution coefficient $r$. For purely elastic low-velocity collisions ($r = 1$), therefore, the presence of the wall can be ignored completely. In this case the dynamics is effectively free also in the region $u > 0$ and is described by the regular map (37) everywhere. A caveat, of course, is that this holds true up till the order of our approximations. In higher order approximations the wall may re-emerge in the mapping.

We conclude that in the case of a low velocity collision, the presence of the obstacle is appreciable only if it absorbs energy, thus if $r < 1$. Then, the influence of the obstacle is felt through a sudden reduction of speed if the mass passes $u = 0$. To find the mapping for this case we derive the analogs of (26) and (27) starting from the impact law (10) with (45). To that end the derivation steps (23)–(25) have to be repeated. We eventually find that the local map $\mathcal{L}$ is still of the form (28)–(30), but with the replacement $r \rightarrow -r$. Thus in this case we have

$$\mathcal{L}(u_n) = N u_n + n(u_n),$$ \hfill (46)
with
\[
N(t) = \begin{pmatrix} -(1 - 2r) & 0 \\ 0 & 1 \end{pmatrix}, \quad n(u_n) = -(1 - r) \begin{pmatrix} \frac{1}{A} & v_n \sqrt{x_n} \\ \sqrt{x_n} - 2v u_n \end{pmatrix}.
\] (47)

Since equations (28-30) have the same structure as (46,47), the remainder of the evaluation does not change. In conclusion, if \( r = 1 \) the dynamics is given by the map in (37), irrespective of the sign of \( x_n \). If \( r \neq 1 \), the map is given by equations (37,38) with the coefficients
\[
\begin{align*}
c_1 &= \text{sign}(P_{12}), \\
c_2 &= \alpha - 2(1 - r) P_{22} + (1 - r)^2 P_{22}^2, \\
c_3 &= (1 - 2r) \gamma - (1 - r)^2 P_{22}^3, \\
\rho &= \frac{1 - \alpha + \gamma}{2|A|(1 - r)^2 P_{22}^2} \sigma.
\end{align*}
\] (48)

It is a highly remarkable conclusion that for \( r < 1 \) the impact map is the same as the map for an infinitely stiff wall, but for a sign change of \( r \). The consequence is that soft collisions with a soft wall at \( r < 1 \) should display the same bifurcation phenomena as collisions with a stiff wall, but for a scale change \((1 + r)^2/(1 - r)^2\) of the bifurcation parameter \( \rho \).

6.2. High velocity impacts

High-velocity impacts correspond to the case case \( \beta \gg 1 \), i.e.
\[
\frac{r \frac{k}{v_c} \gg 1}.
\] (49)

From equation (42) we conclude that for large enough \( \beta \) the impact duration \( \Delta t \) is approximately given by
\[
\Delta t \approx \frac{\pi}{K},
\] (50)

and thus independent of the collision velocity \( v_c \). This fact has interesting consequences. Combining equations (49) and (50) we find that for \( \beta \gg 1 \)
\[
\Delta t \ll \frac{r \frac{\pi}{v_c}}{|A|}.
\] (51)

Repeating steps (23-25), but with the collision rule
\[
t_r = t_c + \frac{\pi}{K}, \quad v_r = -r v_c,
\] (52)

we again arrive at (38) with (39), provided that terms which are much smaller than the terms already included in (38) are ignored. Effectively, we are then dealing with the case \( \Delta t = 0 \). This implies that, in leading order, high velocity impacts at a compliant wall are described by the same map as instantaneous impacts at a rigid wall, i.e. by equations (37-39).
The transition from soft to hard collisions can be best be illustrated by concentrating on the remarkable role which is played by the restitution parameter $r$ which governs the energy loss at collision. Let us consider a given dynamical state which is characterized by a given driving amplitude $F$ and frequency $\Omega$, and, consequently, a given impact velocity $v_c$. As long as there is energy loss, $r < 1$, the square-root singularity of impact dynamics will always be felt. For a soft wall, the map parameters and the scaling of the bifurcation parameter are given by equation (48). An increasingly stiffer wall will shift these parameters to those of (39). For perfectly elastic collisions, the wall stiffness needs to be large enough before the effect of impacts is seen in the dynamics. Trivially, for completely inelastic collisions $r = 0$, there is no difference between hard and resilient walls.

These points are illustrated in figure 6 where we show simulations of an impact oscillator described by the ordinary differential equation \( \ddot{u} + \nu \dot{u} + K(u) = F \cos(2\pi t + \phi) \),

\[
K(u) = \begin{cases} 
\Omega^2(u+1) & u \leq 0, \\
\Omega^2(u+1) + \kappa^2 u & u > 0.
\end{cases}
\]

We have chosen the parameters such that the dynamical state is a period-1 to period-3 transition which displays hysteresis. In a quasi–stationary upward scan of $F$ from the non–impacting state, impacts occur first at $F = F_g (\sigma = 0)$. When $\sigma$ is subsequently decreased, the non–impacting state is reached only at a negative value $\sigma = -\sigma_H$.

In figure 6a we show the case of inelastic impacts ($r = 0.1$) with stiff ($\Gamma = \infty$) and compliant ($\Gamma = 10$) walls. Obviously, in both cases the same bifurcation structure is observed, apart form an apparent rescaling of $\sigma_H$. We recall that both cases are described by the same map, but for a sign reversal of $r$ in the coefficients. If we assume that the principal effect of wall compliance is a scale change of $\rho$ with a factor $(1 + r)^2/(1 - r)^2$, the hysteresis should change with a factor $(1 + r)^2/(1 - r)^2 \approx 1.5$. This number roughly agrees with the observed ratio $\sigma_H(\Gamma = \infty)/\sigma_H(\Gamma = 10) \approx 2.0$. We notice that this factor should not depend on $\Gamma$ as long as we are in the low collision velocity regime. Figure 6b illustrates that for perfectly elastic collisions ($r = 1$) wall compliance profoundly influences the bifurcation structure. The period-1 to period-3 transition

\[\text{An efficient numerical scheme for solving Eq. (53) in the presence of grazing impacts uses the}\]

\[\text{analytical solutions for the motion between impacts. The (exact) positions } u(t) \text{ are computed in a small}\]

\[\text{number of discrete points } t_1, \ldots, t_k \text{ in each drive period. It is crucial not to miss boundary crossings}\]

\[\text{of } u(t). \text{ These crossings are detected both directly by checking } u(t_1), \ldots, u(t_k) \text{ and by computing}\]

\[\text{the position } u(t) \text{ at the turning points of the velocity } \dot{u}. \text{ The number } k \text{ of discrete points in the regions}\]

\[u < 0 \text{ and } u > 0 \text{ is taken proportional to } \Omega \text{ and } \kappa, \text{ respectively.}\]
Figure 6. Period-1 to period-3 transitions for an underdamped oscillator \((\nu, \Omega) = (0.196, 7.368)\) found through integration of the differential equation (53). The grazing excitation amplitude \(F_g\) is indicated by vertical dashed lines. The bifurcation parameter \(\sigma\) is plotted along the upper horizontal axis. (a) Curves 1: restitution coefficient \(r = 0.1\) and stiffness ratio \(\Gamma = \infty\). Curves 2: same as 1 but for a compliant wall with \(\Gamma = 10\). A compliant wall leads to an apparent scale reduction of the bifurcation parameter. (b) Restitution coefficient \(r = 1\) and stiffness ratio \(\Gamma = 10\).

has now apparently changed to a forward bifurcation. From the numerical simulations it appears that for small \(\sigma\), \(\Delta t\) depends approximately linearly on the collision velocity
we therefore are in the low collision velocity regime. For increasing $\sigma$, $v_\text{c}$ will also increase and we will move in figure 5 from the low velocity asymptote $\Delta t = 2\beta/\kappa$ through intermediate values of $\beta$ towards the high velocity case $\Delta t = \pi/\kappa$. As expected, the numerical results (not shown here) now display a more complicated bifurcation structure.

Of course, a completely different behavior results when the velocity reduction occurs after the impacting mass leaves a soft wall. The time delay is then independent of the restitution coefficient $r$. However, we believe that the restitution rule considered in this paper is the more physical one.

7. Period adding bifurcations

As mentioned in the Introduction, one of the characteristic features of maps with a square root singularity is the phenomenon of period adding bifurcations. The emergence of this type of bifurcation can be understood in an extremely simple way from the map in the limit of large damping, $\nu \gg 1$. However, it will appear that this simplicity comes at a price, namely that the number of addings is not as large as those in figure 1 which was computed from the mapping (37,38,48) for different parameter values. The analysis also allows us to highlight the difference between our map and the Nordmark map (1,2) and to stress the importance of wall-compliance.

In the case of a strongly damped oscillator we have in (37,38,48) $s_1 \approx 0$, $s_2 \approx -\nu$, and accordingly $\alpha \approx 1$, $\gamma \approx 0$, $P_{11} \approx 1$, $P_{12} \approx 1/\nu$, and $P_{22} \approx 0$. The two-dimensional map then tends to the one-dimensional map

$$x_{n+1} = x_n + \rho \quad \text{if } x_n \leq 0. \quad (54)$$

$$x_{n+1} = -\sqrt{x_n} + x_n + \rho \quad \text{if } x_n \geq 0, \quad (55)$$

with $\rho$ given by

$$\rho \approx \frac{\nu \Omega^2}{2 |A|(1 - r)^2} \sigma. \quad (56)$$

The influence of the restitution coefficient $r$ is now only felt through the scaling of the bifurcation parameter. The reduced map resembles the ones analyzed by Nusse and Yorke (1992), Nusse, Ott and Yorke (1994) and Lamba and Budd (1994).

The nature of maximal periodic orbits is illustrated in figure 7. Starting at some $x_1 < 0$, the orbit creeps in the direction of the origin with steps of size $\rho$ according to (54) until it collides and is thrown back onto the negative $x$-axis according to (55). Maximal $M$-periodic orbits are stable only in a given interval of the bifurcation parameter $\rho$. For too large $\rho$, a given $M$-periodic orbit is destroyed by an additional impact at time $M - 1$, whereas for too small $\rho$, the cycle looses stability as the orbit is not sufficiently thrown back at impact. We will now compute the stability interval $[\rho_{M}^{L}, \rho_{M}^{U}]$. 
The cycle elements $x_1 \ldots x_M$ of a maximal periodic orbit satisfy
\begin{equation}
x_1 < x_2 < \ldots < x_{M-1} < 0 < x_M,
\end{equation}
with $x_M$ mapped onto $x_1$. Since $x_{M+1}$ is found by applying (54) $M - 1$ times and (55) once, the orbit satisfies
\begin{align}
x_{M+1} &= -\sqrt{x_M + x + \rho} \\
&= -\sqrt{x_1 + (M - 1)\rho + x_1 + (M - 1)\rho + \rho}.
\end{align}
At the upper boundary $\rho_M^U$ of its existence interval, the $M$-periodic orbit grazes the wall at $t = M - 1$, as is illustrated in figure 7. Then we have
\begin{equation}
x_1 = -(M - 2)\rho_M^U, \ x_{M-1} = 0, \text{ and } x_M = \rho_M^U.
\end{equation}
From the condition $x_{M+1} = x_1$ it then follows that
\begin{equation}
\rho_M^U = \frac{1}{M^2}.
\end{equation}
It can be seen that maximal periodic orbits with increasing periods $M$ live at decreasing values of the bifurcation parameter $\rho$.

The lower bound of $\rho_M^L$ of the $\rho-$interval of existence of the maximal period $M$ orbit follows from the stability criterion
\begin{equation}
\left| \frac{\partial x_{M+1}}{\partial x_1} \right| \leq 1.
\end{equation}
Using equation (58), the requirement becomes
\begin{equation}
\left| - \frac{1}{2(x_1 + (M - 1)\rho)^{1/2}} + 1 \right| < 1.
\end{equation}
With the period condition $x_{M+1} = x_1$ we finally have
\[ \rho'_M = \frac{1}{4M}. \] (63)
In conclusion, for the strongly damped oscillator, maximal period-$M$ orbits exist in parameter intervals
\[ \rho_M \in \left[ \frac{1}{4M}, \frac{1}{M^2} \right]. \] (64)
Clearly, as $M$ increases, the lower bound overtakes the upper bound and we conclude that maximal periodic orbits only exist up till a maximum period $M_{\text{max}} = 3$.

Our conclusion is beautifully illustrated in figure 8, where we compare iterations of the map with numerical simulations of the differential equation (53) for collisions with a yielding wall with stiffness ratio $\Gamma = 10$. In both cases a truncated period adding sequence is seen with stability intervals which agree well with the prediction (64). A similar comparison was done for collisions with an infinitely hard wall. The appropriate map is the same as (54, 55), but for a sign reversal of $r$, which only affects the scale of the bifurcation parameter $\rho$ (56). At the used restitution coefficient $r = 0.9$ in figure 8, the scale change is $O(10^2)$. In the differential equation this change moves the windows of maximal periodic orbits to values of $\sigma$ which are so large that the map is no longer a valid approximation. Accordingly, no windows of periodic orbits were observed in the simulations of equation (53). The striking agreement between the simple mapping and the simulation results for the differential equation illustrates that allowance for a yielding wall is an essential ingredient of impact maps.

The map (37, 38) reduces to a one-dimensional map if $\gamma = 0$, in all other cases the mapping is essentially two-dimensional, which makes a bifurcation analysis much more technical. The frictionless oscillator is studied in Lamba and Budd (1994). For the Nordmark map (1, 2) an exhaustive analysis has been done by Chin et al (1994). We expect that many results of their work will carry over to our mapping. The characteristics of maximal periodic orbits in underdamped oscillators will be compared to experimental results in a forthcoming paper.

8. Discussion

In this paper we have devised a mapping for orbits of a harmonic impact oscillator which are close to grazing. We have assumed that this occurs at small values of the control parameter $\sigma$ and small impact velocities. For period adding bifurcations these quantities may not both be small at the same time. When in a quasi-stationary upward scan of $\sigma$ an orbit first starts impacting at $\sigma = 0$, it may do so at a rather high velocity. The impacts may remain at $\sigma < 0$ when $\sigma$ is subsequently lowered, only to vanish at some negative value $\sigma = -\sigma_H$. The transition therefore displays hysteresis. A quantitatively
Figure 8. Truncated series of inverse period addings for the strongly damped oscillator with $(\alpha, \gamma) = (0.99, 0.01)$ and $r = 0.9$. (a) Iterations of the map (37,38) with (48) which is derived for low-velocity impacts with a yielding wall. The vertical lines indicate the stability intervals given by equation (64). (b) Integration of the differential equation (53) using a stiffness ratio $\Gamma = \kappa^2/\Omega^2 = 10$. The sign and scale of the position is determined by the stroboscopic phase. In both cases the horizontal axis is the physical bifurcation parameter $\sigma = F/F_g - 1$. 
correct description of this phenomenon is a challenge for the derived mapping. An additional complication is that the collision velocity depends on the dynamical state of the impact oscillator, and it is difficult to decide apriori whether collisions are hard or soft. These challenges will be met in a forthcoming paper.

A noticeable difference between the Nordmark map (1,2) and the present one (37,38) is the presence of the sign factor in front of the square root. In the overdamped case \( \nu^2 - 4\Omega^2 > 0 \) the sign factor has constant positive sign, but it may change sign for underdamped oscillations since then

\[
\gamma_{12} = \gamma_{12}^{1/2} \frac{\sin \psi}{\psi}, \quad \text{with} \quad \psi = \frac{1}{2} \sqrt{4\Omega^2 - \nu^2}.
\]

We recall that the case \( \psi = 2\pi \) corresponds to the situation where the oscillator is driven with a frequency that equals its own freely swinging frequency. It may readily be appreciated from figure 7 that maximal \( M \)-periodic orbits need to be thrown back sufficiently at impact, and therefore in general need a square-root term with negative sign. The condition for the existence of maximal periodic orbits of period \( M > 1 \) then is \( 2\pi < \psi < 3\pi \) (modulo \( 2\pi \)). It can also be appreciated that a map with a positive square root term can only support period-1 impacting orbits. Further, it appears that the combination of a negative square root sign with a negative \( \alpha \) supports period-2 maximal periodic orbits. Since

\[
\alpha = 2 \gamma_{12}^{1/2} \cos \psi,
\]

period-2 orbits occur in \( 5\pi/2 < \psi < 3\pi \) (modulo \( 2\pi \)). This simple observation agrees with both experiment and numerical simulation of the differential equation. We notice that both the period-1 and the period-2 maximal periodic orbits were missed by Chin et al (1994) who considered only negative square root terms and the case \( \alpha > 0 \).

A remarkable conclusion of the present work is that soft impacts with a yielding wall give rise to an impact map which differs from the hard wall case, roughly through a rescaling of the bifurcation parameter \( \rho \). For the compliant wall we have considered an admittedly simple collision model, but we believe that our main conclusion, namely the persistence of the square root singularity in impact mappings, is robust.

References


