Nonlinear spin-up in a circular cylinder

* Citation for published version (APA):*

* DOI:*
10.1063/1.868676

* Document status and date:*
Published: 01/01/1995

* Document Version:*
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

* Please check the document version of this publication:*
  * A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
  * The final author version and the galley proof are versions of the publication after peer review.
  * The final published version features the final layout of the paper including the volume, issue and page numbers.

* Link to publication*

* General rights*
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

  * Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
  * You may not further distribute the material or use it for any profit-making activity or commercial gain
  * You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

* Take down policy*
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.

Download date: 22. Apr. 2019
Nonlinear spin-up in a circular cylinder

J. A. van de Konijnenberg and G. J. F. van Heijst
Department of Technical Physics, Fluid Dynamics Laboratory, Eindhoven University of Technology,
P.O. Box 513, 5600 MB Eindhoven, The Netherlands

(Received 10 May 1995; accepted 10 August 1995)

Nonlinear spin-up in a circular cylindrical tank is investigated experimentally and compared with the Wedemeyer model. The experiments were performed with water, using tracer particles floating at the free surface in order to visualize the flow field. The experimentally determined vorticity profiles show differences from the Wedemeyer model that indicate the need for an improved estimation of the Ekman pumping on a finite domain. In particular, the Wedemeyer model appears to be inaccurate in the region close to the sidewall. The vorticity field in a spin-down experiment can be reproduced very well by using numerical data of Rogers and Lance for the Ekman suction of an unbounded rotating fluid over a nonrotating plate. However, a more general use of the data of Rogers and Lance on a bounded domain is shown to be inadequate because this would lead to a violation of mass conservation of the Ekman layer. © 1995 American Institute of Physics.

I. INTRODUCTION

The spin-up of a homogeneous, incompressible fluid in a circular tank rotating about its axis due to an increase in angular velocity of $\Omega - \Delta \Omega$ to $\Omega$ from the container has become a classical problem in rotating fluid dynamics. Most of the work on this problem has been analytical and numerical; accurate experimental results are scarce, and mostly limited to only a few radial positions. During the last few years, however, particle tracking techniques have become available that enable quick and very accurate measurements of the entire time-dependent velocity field. With these tools, the evolution of the vorticity distribution of a number of spin-up experiments has been determined, and the results show aspects of the spin-up process that have not been measured before.

The modern age of spin-up research begins with an article by Greenspan and Howard about the spin-up between two parallel horizontal plates of infinite extent for the case in which the relative increase in angular velocity is very small. This makes the Navier--Stokes equation linear, and therefore mathematically tractable. According to this theory, thin boundary layers of the Ekman type develop at the plates immediately after the increase of the rotation rate of the tank. In these Ekman layers there is a radially outward flow, which is compensated for by a weak inward flow between the Ekman layers. The vertical velocity component at the edge of the Ekman layers, denoted in this paper by $w_{\infty}$, can be considered as the driving force of the flow in the interior, and is known as the Ekman pumping rate. The convergence of the interior fluid leads to an increase in angular velocity, which generally provides spin-up of the fluid on a much shorter time scale than does viscous diffusion. Since the Ekman pumping velocities are very weak, the flow is approximately two dimensional during the spin-up process. Therefore, only the vertical component of the vorticity vector is relevant for the flow in the interior; this component, in a frame corotating with the tank, is denoted by $\omega$.

Nonlinear spin-up in a circular tank is a much more complicated process. The main difficulty is that no analytical solution is known for the velocity field inside the Ekman layer if the nonlinear terms are not equal to zero. In some studies numerical data of Rogers and Lance are used. Rogers and Lance computed the velocity profiles and Ekman pumping rates in the case of an interior flow being in solid-body rotation over an infinite plate; the results for the Ekman pumping velocity as a function of the relative fluid vorticity are presented in Fig. 1. According to these results, the pumping rate depends in a nonlinear way on the ratio between the angular velocity of the interior fluid and the angular velocity of the bottom. Later authors (Goller and Ranov, Weidman, Benton) applied these data locally: they performed numerical computations using the Rogers and Lance data for the pumping rates at each individual fluid element, even though the fluid above the Ekman layer was not in solid-body rotation. Whereas Goller and Ranov concentrated on the effect of the free surface, Weidman and Benton discussed the influence of nonlinear Ekman pumping on azimuthal velocity profiles during the spin-up from rest.

However, the Rogers and Lance data cannot be applied on a finite domain without any problem. A consistent theory has to satisfy conservation of mass in the Ekman layer, i.e.

$$\int_A w_{\infty} dA = 0.$$  \hspace{1cm} (1)

with $A$ the area of the tank, since otherwise there would be no balance between fluid being sucked into the Ekman layer and fluid being ejected out of it. A local application of the Rogers and Lance data leads to a violation of this balance, and therefore to nonconservation of mass and vorticity in the two-dimensional flow above the Ekman layer. As a result of the local use of the Rogers and Lance data, Weidman arrived at the conclusion that there had to be a contracting circular shear layer in the solution. However, assuming a shear layer in which the Rogers and Lance data are not valid may be a satisfactory solution for a selected class of flows only. A singular shear layer would have to account for the finite surplus of the nonlinear Ekman suction in the rest of the domain. Only if this vertical flux is pointed downward can such a shear layer possibly exist, although it will probably be prone to instabilities. However, in the beginning of a spin-up
FIG. 1. Ekman pumping rate \( w_{\omega} / \sqrt{\nu \Omega} \) as a function of the relative fluid vorticity \( \omega/2\Omega \) according to the linear model (---), and to the computed values of Rogers and Lance (○).

experiment the resulting singular vertical flux will be upward, which would cause the shear layer to thicken and thus to lose its singular character.

Weidman's results were criticized earlier by Benton. Benton doubted the validity of a local application of the Rogers and Lance data and the existence of a shear layer, but he did not mention the imbalance between Ekman suction and blowing. Benton also raised the question whether the angular velocity or the vorticity is the determining factor for the Ekman pumping velocity; indeed, this does not follow from the Rogers and Lance data. However, in a spin-up experiment the angular velocity is negative throughout the domain, so that using this quantity would lead to a uniform downward Ekman pumping. The use of the vorticity as the main quantity determining the Ekman pumping seems much more realistic in this respect.

The imbalance between Ekman suction and blowing does not appear if the pumping velocity is a linear function of the relative vorticity. (In fact, a linear function \( w_{\omega} \) of \( \omega \) is the only local Ekman pumping model with this property. A criterion for a generally applicable pumping model is that it has to satisfy mass conservation of the Ekman layer for any vorticity distribution. Suppose that the vorticity field consists of two distinct regions, one with area \( A_1 \) and vorticity \( \omega_1 \), the other with area \( A_2 \) and vorticity \( \omega_2 \). Since both the integrals of the vorticity and of the Ekman pumping have to be zero, \( A_1 \omega_1 + A_2 \omega_2 = 0 \) and \( A_1 w_{\omega 1} + A_2 w_{\omega 2} = 0 \). These criteria are only satisfied if \( w_{\omega 1}/w_{\omega 1} = \omega_2/\omega_1 \), that is, if \( w_{\omega} \) is a linear function of \( \omega \).) Assuming a linear Ekman pumping provides a very attractive model for the influence of the Ekman layer. If the right proportionality constant is chosen, it is consistent with the linear case, and Fig. 1 shows that this makes a reasonable approximation to the Rogers and Lance data, at least in the range \(-1 < \omega/2\Omega < 1\). This approach does not describe all aspects of the pumping distribution throughout the flow domain, but it captures the most important features reasonably well. Moreover, the simplicity of the linear pumping model makes the theory of the spin-up process more tractable mathematically. The linear model was first applied to nonlinear spin-up by Wedemeyer in the case of spin-up from rest, and extended later by Greenspan and Veneziani to the nonlinear spin-up from a nonzero angular velocity. The Wedemeyer model does not take viscosity in the interior of the flow into account, but the inclusion of interior viscosity is unrelated to Ekman suction, and does not carry conceptual difficulties. The Wedemeyer model leads to a division of the flow into two regions with uniform vorticity, with a contracting circular discontinuity in between.

A consequence of the linear pumping model is that the divergence in a patch of negative relative vorticity is always negative. If the absolute vorticity is also negative, like close to the sidewall in a spin-down experiment, this leads to a further decrease in vorticity, and unless this vortex stretching process is counteracted by viscous diffusion, the patch will collapse into a singularity. This phenomenon was noticed earlier by Maas, who studied separately the effects of nonlinearity and free surface deformation on the spin-up of circular vortices. Maas showed that a region with negative absolute vorticity leads to a shock in the azimuthal velocity profile, which corresponds to a singularity in the vorticity distribution. Experimental evidence of this profile steepening was obtained by Kloosterziel and Van Heijst. These authors studied experimentally the behavior of a decaying isolated cyclonic vortex, in particular the relative steepness of the vorticity profile at the position between the positive-vorticity core and the negative-vorticity outer layer. Although the vorticity profile of the vortex decayed significantly during the experiment, the scaled profiles turned out to become steeper. This steepening is particularly relevant for the evolution of free vortices, since it can lead to their instability. Experiments show that such instabilities may lead to multipolar vortices; in particular, a stable tripoil vortex with a cyclonic core and two anticyclonic satellites can be spontaneously formed from a single isolated cyclonic vortex (Kloosterziel and Van Heijst).

The Wedemeyer model was extended by several authors. In his second paper on spin-up, Veneziani examined the structure of the contracting front, which according to the inviscid Wedemeyer model would be a vorticity discontinuity. Veneziani included viscous effects, and found that the front is in fact a layer of thickness \( E^{1/4} H \), where \( H \) is the distance from the bottom to the free surface, and \( E = \nu / \Omega H^2 \) is the Ekman number.

Watts and Husey used a numerical method to include viscosity in the entire two-dimensional domain for spin-up from rest. These results were compared with experimental data obtained by laser-Doppler velocimetry. The experimental data were in reasonable agreement with the numerical results, but their accuracy is limited, and unfortunately they do not cover the whole flow domain. In their study, the Reynolds number, defined for spin-up from rest as \( a^2 \Omega / \nu \) with \( a \) the radius of the tank, was taken in the range between \( 2 \times 10^5 \) and \( 1.6 \times 10^6 \).

Hyun et al. published results of the spin-up process from rest obtained from a direct numerical simulation. They used an axisymmetric code, which, due to a nonuniform grid spacing in the vertical direction, was able to solve the structure of the Ekman layers. A comparison was made between
these results and experimental data for a Reynolds number of $0.92 \times 10^4$ obtained by laser-Doppler velocimetry. The agreement between the experimental data and the numerical simulations was excellent. In this paper we present detailed information about the numerically computed flow field, but the experimental results are limited to two radial positions in the tank.

In the present paper, more extensive measurements are presented of the vorticity as a function of the radial position in the tank for three spin-up experiments from rest and one spin-down experiment. The aim is to indicate deviations from the vorticity profile as predicted by the Wedemeyer model, in particular in the region close to the sidewall. The emphasis is put on a physical interpretation of the mechanism that causes these differences. In Sec. II, an overview of the theory of linear spin-up and of the Wedemeyer model is given. The experimental setup is described in Sec. III. In Sec. IV the experimental results are compared with linear pumping models, followed by a discussion of the free surface in Sec. V. In Sec. VI, a spin-down experiment is described. Finally, the main results are summarized and discussed in Sec. VII.

II. THEORY

The main results of the theory of Greenspan and Howard\(^1\) can also be derived by an asymptotic expansion in the kinematic viscosity $\nu$, usually nondimensionalized to the Ekman number $E = \nu \Omega H^2$. However, for this brief review it is not necessary to nondimensionalize every quantity. The equation of motion for linear spin-up in a frame corotating with the tank is given in dimensional quantities by

$$\frac{dU}{dt} = - \frac{1}{\rho} \nabla p - 2 \Omega \times u + \nu \nabla^2 u,$$

with $u$ the velocity of the relative flow, $t$ the time, $\rho$ the density, $p$ the pressure, and $\Omega$ the angular velocity; the fluid is contained in a circular tank with radius $a$, a flat bottom at $z=0$, and a free surface at $z=H$. The viscous term in (2) acts as a singular perturbation. This term is negligible in the major part of the domain, but gives rise to Ekman boundary layers at the bottom of the tank. Moreover, the acceleration term $a_{\Omega} \mathbf{u}$ turns out to be small compared to the Coriolis term. Therefore, the flow in the "interior" region above the Ekman layer (denoted with index $\ast$) is governed to lowest order by the geostrophic balance:

$$\mathbf{u} = - \frac{1}{\rho} \nabla p - 2 \Omega \times \mathbf{u} + \nu \nabla^2 \mathbf{u},$$

which is solved by

$$\omega = - \frac{1}{\rho} \nabla p - 2 \Omega \times \mathbf{u} + \nu \nabla^2 \mathbf{u}.$$

This equation is solved by

$$u = -v r e^{-\xi} \sin \zeta,$$

$$\nu = \nu_1 (1 - e^{-\xi} \cos \zeta),$$

with $u$ and $\nu$ the radial and azimuthal velocity components, $\nu_1$ the azimuthal velocity in the interior, and $\xi = z/\delta$, with $\delta = \sqrt{\nu/\Omega} = H/\sqrt{E}$ the thickness of the Ekman layer.

Equations (5) and (6) describe the structure of the Ekman layer and the interior velocity field to lowest order in the small parameter $E$. However, the spin-up of the interior is caused by a higher-order component of the interior velocity field. This component can be calculated from the lowest-order solution for the Ekman layer. First, the vertical velocity $w$ within the Ekman layer is determined from the continuity equation:

$$w(\zeta) = - \int_0^\zeta \frac{1}{r} \frac{\partial}{\partial r} (ru) \, dr$$

$$- \frac{\delta}{r} \frac{\partial}{\partial r} (\nu_1 \int_0^\zeta e^{-\xi} \sin \zeta' \, d\zeta'),$$

with $r$ the radial position in the tank. Then, by taking the limit $\zeta \to \infty$ and carrying out the integration, one finds the vertical velocity component $w_\infty$ at the top of the Ekman layer:

$$w_\infty = \lim_{\zeta \to \infty} w(\zeta) = \frac{1}{2} \frac{\delta}{\zeta} \omega,$$

where $\omega$ is the relative vorticity in the interior. This Ekman suction causes a two-dimensional divergence in the overlying interior flow given by

$$\nabla \cdot \mathbf{u}_H = \frac{w_\infty}{H} = \frac{1}{2} \sqrt{E} \omega,$$

where $\mathbf{u}_H$ denotes the velocity in the horizontal plane. (In fact, this expression gives the average of $\nabla \cdot \mathbf{u}_H$ over the depth of the tank. However, deviations from this average have no influence on this decay mechanism. It was shown by Greenspan and Howard that such deviations are produced at $t=0$. They consist of an infinite number of oscillatory modes with frequency $2\Omega$, which have small amplitudes, and decay on a time scale $H^2/\xi^2$, with $\tau = \xi / \sqrt{\nu \Omega}$. The decay of these oscillations dominates the last phase of the spin-up process, after the steadily decaying mode has essentially disappeared.) If (10) is inserted in the inviscid two-dimensional vorticity equation in a rotating frame,

$$\frac{\partial \omega}{\partial t} + \mathbf{u}_H \cdot \nabla \omega = -(2 \Omega + \omega) \nabla \cdot \mathbf{u}_H,$$

one finds that for the linear case that

$$\frac{\partial \omega}{\partial t} = -\Omega \sqrt{E} \omega,$$

which is solved by

$$\omega(r,t) = \omega(r,0) e^{-\Omega t},$$

with

$$\tau = \Omega^{-1} (r - 1/2) = \frac{H}{\sqrt{E} \sqrt{\nu \Omega}}.$$


J. A. van de Konijnenberg and G. J. F. van Heijst
Thus, the fluid spins up with a uniform time constant, so that the shape of the vorticity profile, which in the case of spin-up from \( \Omega - \Delta \Omega \) to \( \Omega \) is uniform, remains unchanged.

According to the Wedemeyer model, (9) is also valid if the increase in angular velocity is not small. Replacing \( \nabla \cdot \mathbf{u} \) by \( \frac{1}{r} \sqrt{E} \omega \), the inviscid vorticity equation becomes

\[
\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial r} = - \frac{1}{2} \sqrt{E} (2\Omega + \omega) \omega.
\] (15)

At \( t=0 \), the fluid has a uniform relative vorticity \(-2\Delta \Omega \), apart from a shear layer at the outer wall, where the vorticity is singular. According to the linear pumping model, the Ekman blowing in this region is singular as well, compensating for the uniform suction in the rest of the domain. Thus, according to this model, the flow evolves as follows: due to Ekman suction, the fluid having a uniform vorticity at \( t=0 \) contracts in the radial direction, gaining a higher vorticity in the process. While this occurs, the fluid that is sucked into the Ekman layer is transported outward and ejected into the shear region. As a consequence, this region thickens rapidly, so that it fills the gap between the wall and the contracting fluid core. For the core region, the advective term in (15) vanishes identically: the vorticity is uniform at \( t=0 \), and since it can change only by vortex stretching, which, in turn, depends on the vorticity, it will remain uniform. The remaining ordinary differential equation is readily integrated. Applying the initial condition, one finds that in the core,

\[
\omega(r,t) = -2\Omega \frac{\Delta \Omega e^{-\nu t}}{\Delta \Omega e^{-\nu t} + \Omega - \Delta \Omega}.
\] (16)

In this expression the same time constant \( \tau \) appears that was found in the linear case, but now in a more complex way. If the ratio between the initial and final angular velocity is very small, the core fluid keeps its initial vorticity \(-2\Omega \). In that case, the spin-up process will proceed until the core is completely consumed. For the general case the radius \( r_0(t) \) of the core is given by

\[
r_0(t) = a \sqrt{\frac{\Delta \Omega}{\Omega} (1 - e^{-\nu t})},
\] (17)

which implies a contraction from \( a \) to the final value \( a \sqrt{1 - \Delta \Omega / \Omega} \) on the same time scale \( \tau \) (or \( 2\tau \) due to the presence of the square root) that was found for the linear case.

The solution for the flow in the region \( r > r_0 \) was found by Wedemeyer and is given by

\[
\omega(r,t) = 2\Omega \frac{e^{-\nu t}}{1 - e^{-\nu t}}.
\] (18)

The total vorticity field consists therefore of two distinct regions with uniform vorticity. For the core, this implies a solid-body rotation, for the outer region the azimuthal velocity field consists of a solid-body rotation plus the flow due to a potential vortex at the axis of the tank. This combination is such that \( v = 0 \) at the sidewall, and there is no discontinuity in the azimuthal velocity at the front between the two regions.

III. EXPERIMENTAL SETUP

The spin-up experiments were performed by using a tank with radius \( a = 46 \) cm, filled with ordinary tap water to a depth of either 4 or 20 cm, placed on a rotating table. At \( t=0 \), the angular velocity of the table was suddenly changed from 0 to \( \Omega \), and was thereafter kept at this value during the experiment. Other experiments performed with this equipment indicate that fluctuations in the final angular velocity are not bigger than 0.001 rad/s, which is very small compared to a final angular velocity of 0.24 or 1.0 rad/s used in the experiments. The flow was visualized in two different ways. Quantitative results were obtained with small tracer particles floating at the surface of the fluid. Dye was added to the water in order to increase the contrast between the fluid and the particles. First, a video recording of the flow was made with a video camera corotating with the tank. Then, after the experiment, the recording was processed by a PC equipped with a frame grabber. For this purpose an adapted version of the DigImage system designed by Dalziel was used. This is an image processing system including a particle tracking option. Trajectories of individual particles could be determined in a cycle of three stages: (i) a sequence of 16 video images is captured and stored in the memory of the frame grabber; (ii) in each image, particles are located. This procedure is based on a number of user-defined criteria such as brightness and size; (iii) particles in subsequent images corresponding to the same physical particle are identified. In this procedure, matchings at earlier times are used to estimate the positions of the particles in the next video frame. After the particle paths have been found, the positions are stored in a file, and the cycle is repeated. Further processing of these data was enabled by an extension to the DigImage software developed by Van der Plas. This option provides the possibility to extract data files containing the particle velocities from the data file created by the particle tracking routine. Other quantities such as the vorticity were obtained by matching the data with spline functions and manipulating the coefficients of this expansion. However, the streamfunction was calculated from the vorticity by using a Poisson solver instead of integrating the spline functions. In that way one calculates the streamfunction of the solenoidal component of the velocity field, which is more elegant than applying integration techniques if the flow is not exactly divergence free. More detailed information about this method can be found in Nguyen Duc and Sommeria.

The figures presenting the experimental results of the vorticity \( \omega/2\Omega \) versus the radial position \( r/a \) in the tank were obtained by plotting the values of \( \omega/2\Omega \) and \( r/a \) for a large number of points on a uniform Cartesian grid covering the whole flow domain. This yields a graph with a certain amount of scatter that depends on the accuracy of the measurements and the degree of axisymmetry. Since in some graphs or parts of graphs the scatter is very small, and the vorticity has been determined in the same way in all experi-
FIG. 2. Vorticity versus radial position for spin-up from 0—0.24 rad/s in a circular tank with radius $a = 46$ cm and depth $H = 4$ cm. The data points correspond to laboratory observations; the horizontal lines represent Wedemeyer’s model (linear pumping model, no interior viscosity); the dashed line was computed by using the linear pumping model and including viscosity.

ments, we may assume that the scatter is caused mainly by deviations from a purely azimuthal flow.

In addition to the experiments with particles, experiments were performed with a small amount of dye added to the otherwise clear tap water. In this way, a qualitative impression of the flow field could be obtained, in particular about the way the flow becomes unstable and turbulent.

Temperature differences in the water in the tank, possibly caused by evaporation cooling at the free surface, the heat produced by the drive mechanism of the table or by temperature differences between the water and the laboratory air, can drive weak convective flows. It was observed in many experiments that at later time levels, the flow field becomes irregular, with a tendency for small, axially aligned vortices to appear. This effect was reduced by allowing the water to assume room temperature over a period of at least 12 h, and by using a (transparent) rigid lid on top of the tank. Between this rigid lid and the fluid surface there was an air layer of several centimeters.

Three different spin-up experiments and one spin-down experiment were performed. The angular velocity was increased from 0 to 0.24 rad/s with a depth of 4, resp. 20 cm, from 0 to 1.0 rad/s with a depth of 20 cm, and decreased from 0.24 to 0 rad/s with a depth of 4 cm, so that the Reynolds number $a^2 \Delta \Omega / \nu$ had values of $5.1 \times 10^4$, $5.1 \times 10^5$, $2.1 \times 10^5$, and $5.1 \times 10^6$, respectively.

IV. RESULTS

Experimental results of the vorticity as a function of the radial position in the tank for the three spin-up experiments are presented in Figs. 2-4. In order to test the validity of the linear pumping model with and without viscosity, the $\omega(r)$ data are compared with several models. The solid curves consisting of two horizontal parts represent the Wedemeyer model. The dashed lines were computed numerically using the vorticity equation in two dimensions, including advection, vortex stretching due to Ekman pumping according to the linear mode, and viscosity:

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial r} = -\frac{1}{2} \sqrt{E} (2 \Omega + \omega) \omega + \nu \left( \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r} \frac{\partial \omega}{\partial r} \right),$$

with $u(r)$ determined from integrating

$$\nabla \cdot \mathbf{u} = \frac{1}{2} \sqrt{E} \omega,$$

over a circular region with radius $r$. The equations were iterated in time with an explicit finite difference method. It was verified that this method is sufficiently accurate by varying the step size $\Delta r$ in the radial direction as well as the time step $\Delta t$. The results presented in this section were computed with both $\Delta r$ and $\Delta t$ small enough to make discretization errors negligible. The curves obtained in this way are more realistic than the discontinuous profiles following from the inviscid Wedemeyer model. In accordance with the results of Venezian, the viscous effects are noticeable in a relatively thin region, only, in particular, if the spin-up time is short.

Figures 2-4 show that the modellings based on linear Ekman pumping capture some essential features of the spin-up process; the size of the core region and the vorticity
profiles at late times are predicted fairly well. However, there are also major differences between the modeling and the experimental data. At early times, the region with uniform positive vorticity predicted by the Wedemeyer model cannot be recognized in the experimental data. In the 0→1.0 rad/s experiment (Fig. 4) the profile is the most strongly smoothed, and the vorticity at r=a is correspondingly low. However, later on in the experiment the vorticity at r=a is higher than...
The \( \omega(r) \) graphs of the 0–0.24 rad/s experiment with \( H = 4 \) cm (Fig. 2) show a particularly remarkable deviation from the Wedemeyer model. At \( t = 90 \) s, a small plateau appears around \( r/a = 0.6 \). From the outer edge of the plateau the vorticity increases to a maximum value at \( r = a \). The plateau gradually becomes more diffuse, but the transition to a higher slope at the sidewall remains to exist during the experiment. A similar plateau or curvature can be recognized in the spin-up experiments with \( H = 20 \) cm (Fig. 3) as well. Although less pronounced, this experiment even shows a region where \( \partial \omega / \partial r \) becomes slightly negative.

One can think of a number of reasons for deviations from the Wedemeyer model. The first one is the influence of free surface deformation. This will be discussed in Sec. V. It appears that for even the 0–1.0 rad/s, the flow is hardly affected by free surface gradually assuming a parabolic curvature.

The second reason is the appearance of three-dimensional turbulence. Several types of instability of the Ekman layer are known; some references on this subject are Faller, Tatro and Mollo-Christensen, and Greenspan. However, these instabilities were observed only in the spin-down experiment. Instead, we found a type of instability described earlier by Weidman, which is caused by the strong Ekman blowing at the sidewall. This instability occurs first at the bottom, where the upward Ekman flux enters the interior, but rapidly spreads upward to the free surface. As a result, in a short time a thick annular region next to the sidewall becomes turbulent. As the positive vorticity region becomes more diffuse, the turbulence gradually subsides. Dye visualizations indicate that in all three spin-up experiments the flow relaminarizes after one or two minutes. Except for very shallow tanks, one would expect the occurrence of this type of instability to depend on the value of \( \omega \) close to the wall, and therefore to be independent of the height of the tank. For spin-up from rest this means that this type of turbulence occurs if a critical Reynolds number is exceeded. Experiments by the present authors indicate that this critical Reynolds number is about \( 2 \times 10^3 \). Due to the turbulence, the vorticity profile in the outer region is more strongly smoothed than would have been the case in a laminar flow. This is particularly clear in the experiment from 0 to 1.0 rad/s, in which the turbulence is notably vigorous. After the turbulence has subsided, the size of the core region in this experiment is about twice as small as the Wedemeyer model predicts. The difference between the experimental flow and the Wedemeyer model caused by the initial turbulence remains to exist during the rest of the experiment.

The third reason is that the linear pumping model is insufficiently accurate, and should be replaced by a nonlinear model. Unfortunately, a general expression for the Ekman pumping in the nonlinear case is not available. Nevertheless it is possible to draw some preliminary conclusions from qualitative arguments concentrating on the vorticity at the sidewall. For this purpose it is convenient to distinguish between a "general" nonlinear Ekman suction in the sense of a Rogers and Lance-like model that is somehow corrected for the net flux into the Ekman layer, and a deviation that is specifically connected with the presence of the sidewall. Starting with a weak point of the Wedemeyer model, the Rogers and Lance data indicate that the linear pumping law overestimates Ekman pumping if the relative vorticity is very high. Thus, one would expect to find higher vorticity levels at the sidewall than according to the linear pumping model. However, a correction to the linear pumping model inevitably leads to a net flow into the Ekman layer, which has to be compensated for by an upward flux, distributed somehow over the domain. The smallest Ekman blowing at the sidewall is found when the radial distribution of this mass flux is zero at the sidewall; the result would be that the Ekman suction at \( r = a \) is given by the value of Rogers and Lance. However, in some experiments the vorticity at the sidewall appears to be higher than would follow from applying the Rogers and Lance data at \( r = a \). This means that the Ekman suction at the sidewall is lower than is to be expected on the basis of a "general" nonlinear pumping.

This argument suggests that the sidewall itself plays a role in determining the Ekman suction profile. It was noticed earlier by Benton that Ekman pumping inhibition due to the no-slip condition at the sidewall might influence the evolution of the vorticity profile. Since the Ekman suction has to vanish at \( r = a \), the spin-down of the high vorticity at the sidewall is slower than it would have been if the Ekman suction profile were to depend on the vorticity profile alone. Like the introduction of a nonlinear Ekman suction, the inhibition of the upward Ekman flux affects the balance between Ekman suction and blowing, and must be compensated for elsewhere in the domain. The vorticity graphs, in particular in Fig. 2(c), suggest that the additional flux leaves the Ekman layer in the region \( 0.6 < r/a < 0.8 \). Apparently, the flow for \( r/a < 0.6 \) is affected little by the presence of the sidewall, and produces an outward flux within the Ekman layer. This outward flux is deflected by the no-slip condition at the sidewall, and emerges in an annular region at some distance from the sidewall. In this annular region the vorticity is lower than according to the Wedemeyer model. Depending on the vorticity profile resulting from the initial turbulence and the smoothing effect of interior viscosity, the increased Ekman pumping may lead to a horizontal part in the vorticity graph.

The fourth reason is that the measurements may be afflicted with a slight inaccuracy. From the experimental results it seems as if the vorticity in the very center of the tank becomes smaller than \( -2 \Omega \) at the moment the uniform vorticity core has shrunk to zero radius, which means that the absolute vorticity would become negative. However, it is difficult to think of a physical mechanism leading to a negative absolute vorticity. The effect is probably caused by an insufficient resolution of the measurements. In a typical experiment there are about 200 particles at the surface, so on the average there are only two particles in the region \( r/a = 0.1 \). Obviously this leads to a limited accuracy, especially if the actual \( \omega(r) \) function is highly curved. Similarly, the region \( r/a > 0.9 \) is poorly resolved. In experimental practice, the radial contraction and the tendency for the particles to stick to the sidewall make it difficult to have a large number of particles within a close distance from the sidewall. Instead, the
measured particle paths were completed with zero vectors along the sidewall. Thus, for $r/a > 0.9$ the integral of the vorticity rather than the vorticity itself is calculated correctly. This means that a very strong curvature in the $w(r)$ graph very close to the sidewall will not be resolved, but replaced by a straight line having the same integral over $r$.

A general consideration with respect to unresolved “strong curvatures” at the sidewall is that regardless of the modeling used to represent the Ekman suction, the two-dimensional vorticity equation plus the zero velocity condition provides a boundary condition to the vorticity. By integrating the two-dimensional vorticity equation, given by

$$\frac{\partial \omega}{\partial t} + \mathbf{u}_H \cdot \nabla \omega = - (\omega + 2 \Omega) \mathbf{v}_H \cdot \mathbf{u}_H + \nu \nabla^2 \omega,$$

over the whole fluid domain, one finds that

$$\frac{\partial}{\partial t} \int \omega \, dA + \int \mathbf{u}_H \cdot \nabla \omega \, dA$$

$$= - \int \nabla \cdot \mathbf{u}_H \, dA - 2 \Omega \int \mathbf{v}_H \cdot \mathbf{u}_H \, dA$$

$$+ \nu \int \nabla^2 \omega \, dA. \quad (22)$$

In this equation, the vortex stretching term has been splitted into two. The first term on the left-hand side of the equation is the temporal derivative of the circulation of the fluid at the sidewall, and must therefore be identically zero. The second term on the left-hand side and the first term on the right-hand side of (22) can be written as one single integral of $\nabla \cdot \mathbf{u}_H$; by using Gauss' theorem in two dimensions and the zero velocity condition at the sidewall, one finds that the result is zero. For a consistent model, the integral over $\nabla \cdot \mathbf{u}_H$ must vanish too, since otherwise there would be an outflow through the sidewall. For spin-up with a uniform depth, this condition corresponds with (1). Thus, since all other terms drop out of the equation, the integral over the viscous term must be equal to zero as well. In case of the circular tank, this leads to

$$\left. \frac{\partial \omega}{\partial r} \right|_{r=a} = 0. \quad (23)$$

In fact, this conclusion has no specific connection with spin-up; the result is valid for any two-dimensional flow with circular symmetry. With the experimental method used here, a region close to the sidewall where $\partial \omega/\partial r$ becomes small could not be resolved. Whether this is caused by the two-dimensional vorticity equation being inadequate in this respect, or by a strong curvature in the $w(r)$ graph, is not clear. It is conceivable that the layer in which $\partial w/\partial r$ goes to zero is kept very thin by the radial dependence of the Ekman suction close to the sidewall. Such an effect is also observed in the numerical results in Sec. V for spin-up with a strong surface deformation.

Nevertheless, there can be little doubt about the high vorticity values at $r=a$, and the plateau in the experiment in the tank with 4 cm depth. The experiments described here were selected from a more extensive series of experiments for various values of angular velocity and depth. All these experiments showed the differences from the Wedemeyer model described in this section. For spin-up from 0 to 0.24 rad/s, the plateau was most clearly visible for a depth of about 4 cm. In view of the rather peculiar appearance of the vorticity profile this experiment has been repeated, with practically the same results.

V. INFLUENCE OF FREE SURFACE DEFORMATION

The effect of the free surface on spin-up was described by several authors. The first contribution was given by Greenspan and Howard. They gave an expression for the influence of a slightly parabolically curved free surface (small Froude number $4\Omega^2 r^2 / g H$). This work was extended by O'Donnell and Linden to cases in which the surface elevation is not small compared to the depth of the fluid [Froude number $\mathcal{F}$]. Numerical results for the influence of the free surface on nonlinear spin-up in a circular tank were presented by Goller and Ranov. They reported a considerable effect of the free surface on the spin-up, but a useful comparison between spin-up with and without a deformable free surface was not made. Therefore, we performed similar computations to estimate the influence of the free surface for the experiment with $0 \rightarrow 1.0$ rad/s, for which the effect of the free surface is strongest. These computations were performed with the linear pumping model, extended to include interior viscosity and surface deformation. It was shown in Sec. IV that the linear pumping model does not account for all aspects of the observed vorticity profiles, but it still provides a reasonable model for the Ekman suction, so that this method may be used to estimate the magnitude of the effect of the free surface.

The fluid depth, denoted by $h(r,t)$, enters the vorticity equation through the two-dimensional divergence. Conservation of mass requires that

$$w_x = \frac{\partial h}{\partial t} + \nabla \cdot (h \mathbf{u}_H) \quad (24)$$

or

$$\nabla \cdot \mathbf{u}_H = \frac{1}{h} \left( w_x - \frac{\partial h}{\partial t} - \frac{h}{\partial r} \right). \quad (25)$$

Although the relation between $w_x$ and $\nabla \cdot \mathbf{u}_H$ is now more complicated than in the case of a uniform depth, the criteria for zero flux into the Ekman layer and no radial outflow at the sidewall still coincide: integrating (24) we find that

$$\int_A \nabla \cdot (h \mathbf{u}_H) dA = 0, \quad (26)$$

so that the radial velocity at $r=a$ is zero.

The depth $h(r)$ was computed by assuming a balance between the centripetal force and the pressure gradient:

$$\frac{v^2}{r} = \frac{1}{\rho} \frac{\partial p}{\partial r} = g \frac{\partial h}{\partial r}. \quad (27)$$

Time-dependent, advective or viscous effects on this balance were not taken into account. As the boundary condition at $r=a$, the gradient of the vorticity was taken to be zero. The
The curves in Fig. 5 for a rigid surface correspond to the same results as the dashed curves in Fig. 4. In order to obtain an impression of the effect of surface deformation yet, the computations were also performed with a gravity acceleration of 0.98 m/s² instead of 9.8 m/s². In this way the Froude number is decreased without affecting the angular velocity and the dimensions of the tank. According to the results in Fig. 5, the vorticity at the sidewall is higher than without a deformable surface; this is caused by vortex stretching because of the rising surface, and to the weaker influence of Ekman pumping due to a greater depth. The increased vorticity values at the wall are compensated for by lower values elsewhere in the domain, so the vorticity integrated over the flow domain is conserved. However, due to the absolute vorticity being zero, the vorticity in the core is independent of vortex stretching and is therefore unchanged. It can hardly be seen from Fig. 5, but there is indeed a boundary layer at r = a where $\partial \omega / \partial r$ goes to zero. This boundary layer is very thin, since it is caused by the presence of interior viscosity, which is only a small effect.

The evolution of the free surface elevation as a function of the radial position in the tank is given in Fig. 6. At $t = 120$ s, a large part of the fluid is still at rest, and only close to the sidewall is the fluid level higher than in the rest of the domain. During the experiment the surface at larger radii gradually rises; the mass required for this elevation is drained away fluid from the central region. Finally, the free surface...

FIG. 5. Numerical results for the influence of a deformable free surface on the vorticity profiles of the $0\rightarrow1.0$ rad/s experiment. All results were obtained by using the viscous linear model. The solid curve corresponds to a rigid surface, the dotted curve to a gravity acceleration $g$ of 9.8 m/s², and the dashed curve to $g = 0.98$ m/s².

FIG. 6. Numerical results for the elevation of the free surface as a function of the radial position in the tank for the $0\rightarrow1.0$ rad/s experiment.
FIG. 7. Vorticity versus radial position for spin-down from 0.24—0 rad/s in a tank with radius \( a=46 \) cm and depth \( H=4 \) cm. The dashed line represents spin-down according to (32), with \( \tau^* \) given by (33) with \( k=1.369 \pm 1 \). This value of \( \tau^* \) has also been used to nondimensionalize the time \( t \).

Surface assumes a parabolic profile, given by

\[
h(r) = H + \frac{\omega^2}{2g} \left( r^2 - \frac{1}{2} a^2 \right).
\]  

(28)

VI. SPIN-DOWN

Spin-down flows are usually more unstable than spin-up flows; even if the final angular velocity is nonzero, the flow close to the sidewall and in the Ekman layer may become three-dimensionally turbulent. In the case of spin-up from rest, the flow does not approach a final state of rotation, so that motions in the vertical direction are not suppressed, and the turbulence is particularly persistent.

In the particular experiment discussed in this section, the results of which are presented in Fig. 7, the flow is turbulent in a thick annular region extending radially inward from the sidewall to approximately halfway to the center. The instability may be caused by the azimuthal velocity not satisfying Rayleigh’s stability criterion, by the high vertical velocities due to Ekman pumping, or by the high horizontal velocities in the Ekman layer. In most spin-down experiments, the flow becomes turbulent so quickly that it is difficult to distinguish between these different types of instability.

If the flow is linear, the Ekman pumping velocity is given by

\[
w_m = \frac{1}{2} \omega \delta \frac{1}{2} \sqrt{\frac{\nu}{\Omega}}.
\]  

(29)

However, for a fluid rotating with uniform angular velocity \( \omega \gg \Omega \), the Ekman layer thickness is determined by the angular velocity of the fluid rather than by the angular velocity of the container. Therefore, the Ekman suction of a rigidly rotating fluid over a nonrotating plate can be written as

\[
w_m = k \sqrt{\nu \omega} / 2,
\]  

(30)

with \( k \) a constant of order unity, calculated by Rogers and Lance (1960) as 1.369 ± 1. Equation (30) is a nonlinear relation between \( w_m \) and \( \omega \), so care should be taken about mass conservation. However, in the case of spin-down from rest, (30) leads to a consistent picture. Points with \( \omega<0 \) are concentrated in a shear layer at \( r=a \). By assuming that this shear layer compensates for the uniform Ekman pumping in the rest of the domain, the local character of (30) can be maintained. Since the Ekman pumping in the shear layer is directed downward, the two-dimensional divergence in the shear layer is negative, so that the shear layer absorbs the outward flow from the uniform vorticity region and does not move away from the sidewall.

Inserting (30) in the vorticity equation leads to

\[
\frac{\partial \omega}{\partial t} = -k \sqrt{\frac{\nu}{2H}} \omega^{3/2},
\]  

(31)

which is solved by

\[
\omega(r,t) = \frac{-2 \Delta \Omega}{(t/\tau^*+1)^2},
\]  

(32)

with

\[
\tau^* = \frac{2H}{k \sqrt{\nu \Delta \Omega}}.
\]  

(33)
essential improvement would be to take the influence of the sidewall into account. Due to the no-slip condition at the side wall, the Ekman pumping at the sidewall is inhibited, which is compensated for by an increased pumping at some distance from the sidewall. A representation of Ekman and Stewartson boundary layers in terms of a divergence in the overlying two-dimensional flow is desirable, since this acts as a closure of the two-dimensional vorticity equation. In this way, three-dimensional computations can be avoided in many rotating flow problems. However, it is unclear whether a simple model can be found that is suitable for a wide class of rotating flows. Such a model requires knowledge of nonlinear Ekman layers on a finite domain, which is not available at present.

ACKNOWLEDGMENTS

The authors wish to thank Gert van der Plas for his assistance with the processing of the data from the DigImage system. One of us (JvK) gratefully acknowledges financial support from the Dutch Foundation of Fundamental Research (FOM).