Well-posedness of a class of piecewise linear systems

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WELL-POSEDNESS OF A CLASS OF PIECEWISE LINEAR SYSTEMS

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Abstract

The main goal of the paper is to investigate well-posedness issue for a class of piecewise linear systems within the framework of complementarity systems. The class of piecewise linear systems to be considered includes for instance electrical network with piecewise linear resistors, mechanical systems with Coulomb friction. After giving a definition of solution for this class of systems, we present sufficient conditions for uniqueness of solutions.

1 Introduction

Piecewise linear systems have been extensively studied in the literature of circuits and systems theory. While the work on ‘static’ piecewise linear systems has mainly concentrated on representing piecewise linear elements suitably (see e.g. [3, 6, 11]) and solving piecewise linear equations (see e.g. [2, 17]), finding efficient simulation methods is the problem that has received most attention for ‘dynamical’ piecewise linear systems [1, 13].

It seems that earlier work is usually (if not always) based on the well-posedness assumption. The aim of this paper is to investigate well-posedness issue of a class of piecewise linear systems. As a first step, we shall clarify what is meant by ‘solution’. In fact, to define a solution concept for continuous-time systems with piecewise elements is nontrivial especially when ‘relation-type’ piecewise linear elements considered rather than ‘function-type’ ones. Still such a definition is needed for instance to discuss convergence of simulation schemes. After defining the solution, we give sufficient conditions

for uniqueness of solution. The development is based on the complementarity formulation developed in [7, 15, 16].

Throughout the paper, we denote the set of n -tuples of arbitrarily differentiable functions defined on $[0, \infty)$ as $\mathcal{C}_\infty^n$, the set of n -tuples of locally integrable functions defined on $[0, \infty)$ as $\mathcal{L}_{loc}^n$. e is the vector containing 1 in each entry, zero function and constant function which assumes value 1 defined on $[0, \infty)$ are denoted by $0(\cdot)$ and $1(\cdot)$, respectively. Inequalities for vectors must be understood component-wise. For an integer m , $\bar{m}$ denotes the set $\{1, 2, \dots, m\}$. M_{IJ} denotes the submatrix of $M \in \mathbb{R}^{n \times n}$ whose entries lie in the rows of M indexed by $I \subset \bar{n}$ and the columns of M indexed by $J \subset \bar{n}$. If $I = \bar{n}$, we denote M_{IJ} by $M_{\bullet J}$. For a subset S of a topological space, S° denotes the interior of S . $\text{im}(f)$ is the set of image of the map f . For any proposition $P(\sigma)$, we say that ‘ $P(\sigma)$ holds for all sufficiently σ ’ if there exists σ_0 such that $P(\sigma)$ holds for all $\sigma > \sigma_0$. The restriction on (a, b) of a function $x \in \mathcal{C}_\infty$ will be denoted by $x|_{(a, b)}$.

2 Preliminaries

2.1 Complementarity problems

The *Linear Complementarity Problem (LCP)* of mathematical programming and one of its variants, namely (*Extended*) *Horizontal Complementarity Problem (HLCP)*, will play a crucial role in our treatment. In this subsection, we recall formulations and solvability conditions of these two problems.

Problem 2.1 *LCP*(q, M) Given $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$,

find $z \in \mathbb{R}^n$ such that

$$\begin{aligned} z &\geq 0 \\ q + Mz &\geq 0 \\ z^T(q + Mz) &= 0 \end{aligned}$$

For solvability conditions and other results on LCP , we refer to the excellent survey of [4]. We recall a standard result concerning the LCP .

Definition 2.2 A matrix $M \in \mathbb{R}^{n \times n}$ is said to be $\mathcal{P}$ -matrix if all its principal minors are positive.

Theorem 2.3 [4, Theorem 3.3.7] $LCP(q, M)$ has a unique solution for all $q \in \mathbb{R}^n$ if and only if M is a $\mathcal{P}$ -matrix.

There are a number of interesting generalizations of the LCP . Particularly, the (Extended) Horizontal LCP which is closely related to piecewise linear functions has to be mentioned.

Problem 2.4 $HLCP_k^\alpha(q, \{M^i\}_{i=0}^k)$ Given $\alpha > 0$, $q \in \mathbb{R}^n$, and $\{M^i \in \mathbb{R}^{n \times n}\}_{i=0}^k$, find $\{z^i \in \mathbb{R}^n\}_{i=0}^k$ such that

$$\begin{aligned} z^0 &\geq 0, \alpha e \geq z^1 \geq 0, \dots, \alpha e \geq z^{k-1} \geq 0, z^k \geq 0 \\ (z^0)^T z^1 &= (\alpha e - z^1)^T z^2 = \dots = (\alpha e - z^{k-1})^T z^k = 0 \end{aligned}$$

$$M^0 z^0 = q + \sum_{i=1}^k M^i z^i$$

The $HLCP$ was introduced in [9] with $k = 2$ and $M^0 = I$, and further developed in [10] with an eye towards piecewise linear functions. We briefly recall some facts from [14] and state a result on solvability of the problem similar to theorem 2.3.

Definition 2.5 A matrix $R \in \mathbb{R}^{n \times n}$ is called a column representative of $\{M^i\}_{i=0}^k$ if $R_{\bullet i} \in \{M_{\bullet i}^0, M_{\bullet i}^1, \dots, M_{\bullet i}^k\}$ for all $i \in \bar{n}$.

Definition 2.6 We say that $\{M^i\}_{i=0}^k$

- (a) is nondegenerate if the determinants of all column representative matrices are nonzero.
- (b) has the column $\mathcal{W}$ -property if the determinants of all column representative matrices are either positive or negative.

Theorem 2.7 [14] $HLCP_k^\alpha(q, \{M^i\}_{i=0}^k)$ has a unique solution for all $q \in \mathbb{R}^n$ if and only if $\{M^i\}_{i=0}^k$ has the column $\mathcal{W}$ -property.

Remark 2.8 LCP can be regarded as a special case of $HLCP$. Indeed, $LCP(q, M)$ is nothing but $HLCP_1(q, \{I, M\})$. In this case, theorem 2.3 and theorem 2.7 coincide since M is a $\mathcal{P}$ -matrix if and only if the determinants of all column representative matrices of $\{I, M\}$ are positive, i.e., $\{I, M\}$ has the column $\mathcal{W}$ -property. On the other hand, $HLCP_k^\alpha(q, \{M^i\}_{i=0}^k)$ can be written as an LCP whenever M^0 is invertible. For this purpose, we define

$$\begin{aligned} r^\beta(q) &:= (q^T \quad \beta e^T \quad \beta e^T \quad \dots \quad \beta e^T)^T \\ N(\{M^i\}_{i=0}^k) &:= \begin{pmatrix} \tilde{M}^1 & \tilde{M}^2 & \dots & \tilde{M}^{k-1} & \tilde{M}^k \\ -I & 0 & \dots & 0 & 0 \\ 0 & -I & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -I & 0 \end{pmatrix} \end{aligned}$$

where $\tilde{M}^i = (M^0)^{-1} M^i$ for $i \in \bar{k}$. It is easy to see that solution of $HLCP_k^\alpha(q, \{M^i\}_{i=0}^k)$ can be obtained by solving $LCP(r^\alpha((M^0)^{-1}q), N(\{M^i\}_{i=0}^k))$. We shall use this fact in order to utilize previous results concerning with linear complementarity systems.

As we shall see later on, our treatment is based on the formulation of piecewise linear systems as complementarity systems. To be reasonably self-contained, we recall *Rational Complementarity Problem* which is extensively studied in [8].

Problem 2.9 $RCP(q(s), M(s))$ Given $q(s) \in \mathbb{R}^n(s)$ and $M(s) \in \mathbb{R}^{n \times n}(s)$, find $z(s) \in \mathbb{R}^n(s)$ such that

- (a) $z(\sigma) \geq 0$ and $q(\sigma) + M(\sigma)z(\sigma) \geq 0$ for all sufficiently large σ .
- (b) $z^T(s)(q(s) + M(s)z(s)) = 0$ for all $s \in \mathbb{C}$.

Similarly to the generalization of LCP , we can generalize RCP in the following way.

Problem 2.10 $HRCP(q(s), \{M^i(s)\}_{i=0}^k)$ Given $q(s) \in \mathbb{R}^n(s)$ and $\{M^i(s) \in \mathbb{R}^{n \times n}(s)\}_{i=0}^k$, find $\{z^i(s) \in \mathbb{R}^n(s)\}_{i=0}^k$ such that

- (a) $z^0(\sigma) \geq 0$, $\sigma^{-1} \geq z^1(\sigma) \geq 0, \dots$,
 $\sigma^{-1} \geq z^{k-1}(\sigma) \geq 0$, $z^k(\sigma) \geq 0$ for all sufficiently large σ .
- (b) $(z^0(s))^T z^1(s) = (s^{-1}e - z^1(s))^T z^2(s) = \dots = (s^{-1}e - z^{k-1}(s))^T z^k(s) = 0$ for all $s \in \mathbb{C}$.

2.2 Piecewise linear curves

In this subsection, we shall introduce the class of piecewise linear elements to be considered in the sequel and underline the relation between this class and $HLCP$. Firstly, a series of definitions are in order.

Figure 1: Two piecewise linear curves

Definition 2.11 A collection of closed intervals, $\mathcal{S} = \{S_i\}_{i=1}^k$ is said to be a *subdivision of $\mathbb{R}$* if $S_i^\circ \cap S_j^\circ = \emptyset$ whenever $i \neq j$ and $\cup_{i=1}^k S_i = \mathbb{R}$.

Now, we can define the class of piecewise linear elements considered in the paper.

Definition 2.12 A function $f : \mathbb{R} \rightarrow \mathbb{R}^2$ is said to be a *piecewise linear curve with k pieces* if there exists a subdivision $\mathcal{S} = \{S_i\}_{i=1}^k$ of $\mathbb{R}$ such that f is affine on each S_i .

Notice that the two characteristics depicted in figure 1 are piecewise linear curves and also that they are not piecewise linear functions defined from $\mathbb{R}$ to $\mathbb{R}$. Next, we give a representation of piecewise linear element in terms of *HLC*P.

Definition 2.13 $(q_y, \{Y^i\}_{i=0}^k, q_u, \{U^i\}_{i=0}^k)$ is said to be a *complementarity representation of $\{f_i\}_{i=1}^m$* if

$$(y_i, u_i) \in \text{im}(f_i) \quad \forall i \in \bar{m}$$

$$\Updownarrow$$

$$HLC P_k^1(q_y - y, \{Y^i\}_{i=0}^k) \& HLC P_k^1(q_u - u, \{U^i\}_{i=0}^k)$$

has a common unique solution

$$\text{where } u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} \text{ and } y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}.$$

The following lemma reveals that there exists a correspondence between the piecewise linear curves and *HLC*Ps.

Lemma 2.14 Let $\{f_i\}_{i=1}^m$ be piecewise linear curves with $k_i + 1$ pieces and $k = \max_i k_i$. $\{f_i\}_{i=1}^m$ admits a complementarity representation $(q_y, \{Y^i\}_{i=0}^k, q_u, \{U^i\}_{i=0}^k)$.

Proof see [10].

Example 2.15 Consider the piecewise linear curves depicted in figure 1. f_i s can be given by

$$f_1(\theta) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \theta + \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \text{if } \theta \in (-\infty, 0], \\ \begin{pmatrix} 0 \\ -2 \end{pmatrix} \theta + \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{if } \theta \in [0, 1], \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \theta + \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \text{if } \theta \in [1, \infty). \end{cases}$$

$$f_2(\theta) = \begin{cases} \begin{pmatrix} 0 \\ -1 \end{pmatrix} \theta & \text{if } \theta \in (-\infty, 0], \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \theta & \text{if } \theta \in [0, \infty]. \end{cases}$$

Clearly, f_1 and f_2 are piecewise linear curves with 3 and 2 pieces, respectively. It can be checked that a complementarity representation $(q_y, \{Y^i\}_{i=0}^2, q_u, \{U^i\}_{i=0}^2)$ can be given by

$$q_y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Y^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$q_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad U^0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad U^1 = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}, \quad U^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

However, this is not the unique representation for $\{f_1, f_2\}$. In fact,

$$q_y = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad Y^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$q_u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad U^0 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad U^1 = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}, \quad U^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

form another representation. For any set of piecewise linear curves, there are multiple representations in general. However, it can be shown that if for one representation $\{S U^i + T Y^i\}_{i=0}^k$ has column $\mathcal{W}$ -property where $S, T \in \mathbb{R}^{m \times m}$ then the same holds for all other representations.

3 Main results

We consider piecewise linear systems of the type

$$\begin{aligned} \dot{x}(t) &= A x(t) + B u(t) \\ y(t) &= C x(t) + D u(t) \end{aligned} \tag{1}$$

$$(y_i(t), u_i(t)) \in \text{im}(f_i) \quad \forall t \in [0, \infty) \quad \forall i \in \bar{m}$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$, and $\{f_i\}_{i=1}^m$ are piecewise linear curves.

Inevitably, one encounters nonsmooth state trajectories in the context of piecewise linear systems. This means that the differential equation in (1) has to be reinterpreted in a suitable way in order to be mathematically precise. We shall use the notion of Carathéodory solution of the differential equation to define a solution concept for piecewise linear systems.

Definition 3.1 $(u, x, y) \in \mathcal{L}_{loc}^{m+n+m}$ is said to be a *solution of (1) for initial state x_0* if

$$(a) \quad x(t) = x_0 + \int_0^t A x(s) + B u(s) ds,$$

Figure 2: Relay with deadzone characteristic

(b) $y(t) = Cx(t) + Du(t),$

(c) $(y_i(t), u_i(t)) \in \text{im}(f_i) \quad \forall i \in \bar{m} \text{ and for almost all } t \in [0, \infty).$

In the case of a relay characteristic, this definition automatically includes the ‘sliding mode’ as defined in [5]. Next, we define the concept of initial solution which enables us to formulate uniqueness problem as a complementarity problem.

Definition 3.2 $(u, x, y) \in \mathcal{C}_{\infty}^{m+n+m}$ is said to be *an initial solution of (1) for initial state x_0* if there exists $\epsilon > 0$ such that

(a) $\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,$

(b) $y(t) = Cx(t) + Du(t),$

(c) $(y_i(t), u_i(t)) \in \text{im}(f_i) \quad \forall i \in \bar{m} \text{ and for all } t \in [0, \epsilon].$

Note that the definition requires differentiability of state trajectories and is weaker than the previous one. We can look at piecewise linear systems as hybrid systems. In this context, ‘event’ times can be defined as the time instants on which (y_i, u_i) moves to another piece of the $\text{im}(f_i)$ for some i , or equivalently on which the state trajectory is not differentiable. Using this terminology, the relationship between global and initial solution concepts can be made clear. In fact, the restrictions of the initial solution and global solution for the same initial state x_0 on $[0, T]$ must be the same if there is no event on $[0, T]$.

To emphasize the importance of well-posedness, two illustrative examples are in order.

Example 3.3 Consider the linear system

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = -x_1 - u \quad y = -x_2$$

and a ‘relay with deadzone’-type piecewise linear characteristic between u and y as illustrated in figure 2 (a). One

can easily verify that

$$\begin{pmatrix} 0(t) & [\mp \sin t & \mp \cos t]^T & \pm \cos t \\ \pm t & [\mp t & \mp 1(t)]^T & \pm 1(t) \\ \pm 1(t) & \begin{bmatrix} \pm \cos t \mp \sin t \mp 1(t) \\ \mp \cos t \mp \sin t \end{bmatrix} & \pm \sin t \pm \cos t \end{pmatrix}$$

are all initial solutions for initial state $[0 \quad \mp 1]^T$.

It can be argued that the nonuniqueness is a result of ‘relation-type’ characteristic between u and y . In fact, it is well known from the theory of ordinary differential equations that if u is a piecewise linear function of y and $D = 0$ then there exists a unique solution for each initial state. However, it is also well-known that nonuniqueness may emerge whenever $D \neq 0$ as depicted in the following example.

Example 3.4 Consider the linear system

$$\begin{aligned} \dot{x}_1 &= u_1 & y_1 &= x_1 + u_2 \\ \dot{x}_2 &= u_2 & y_2 &= x_2 + u_1 \end{aligned}$$

and a ‘saturation’-type characteristic between u_i and y_i as illustrated in figure 2 (b).

For initial state $[0 \quad 0]^T$, there are three different ‘global’ solutions:

$$\begin{pmatrix} \mp 2 \begin{bmatrix} 1(t) \\ 1(t) \end{bmatrix} & \mp 2 \begin{bmatrix} t \\ t \end{bmatrix} & \mp 2 \begin{bmatrix} t+1 \\ t+1 \end{bmatrix} \\ \begin{bmatrix} 0(t) \\ 0(t) \end{bmatrix} & \begin{bmatrix} 0(t) \\ 0(t) \end{bmatrix} & \begin{bmatrix} 0(t) \\ 0(t) \end{bmatrix} \end{pmatrix}$$

Now, we state the following lemma which can be regarded as the generalization of the similar results obtained for complementarity systems to piecewise linear systems.

Lemma 3.5 Let $(q_y, \{Y^i\}_{i=0}^k, q_u, \{U^i\}_{i=0}^k)$ be a complementarity representation of piecewise linear curves $\{f_i\}_{i=1}^m$ and $G(s) = D + C(sI - A)^{-1}B$. Let

$$q_{x_0}(s) := s^{-1}q_y - s^{-1}G(s)q_u - C(sI - A)^{-1}x_0$$

and $M^i(s) := Y^i - G(s)U^i$ for $i \in \bar{m}$. The followings hold:

(a) If $\text{HRCP}(q_{x_0}(s), \{M^i(s)\}_{i=0}^k)$ has a unique strictly proper solution then for any pair of initial solutions (u, x, y) and (u', x', y') of (1) for initial state x_0 there exists $\epsilon > 0$ such that $(u, x, y)|_{(0, \epsilon)} = (u', x', y')|_{(0, \epsilon)}$.

(b) If $\{M^0(\infty), M^k(\infty)\}$ is nondegenerate and $\{M^i(\sigma)\}_{i=0}^k$ has the column $\mathcal{W}$ -property for all sufficiently large σ then $\text{HRCP}(q_{x_0}(s), \{M^i(s)\}_{i=0}^k)$ has a unique strictly proper solution for all $x_0 \in \mathbb{R}^n$.

Proof (a) We omit the proof of this part because of space limitations. However, it is not difficult to see that it can be proven by using theorem 5.14 of [8] and keeping in the mind the equivalence between *HLCP* and *LCP* explained in remark 2.8.

(b) Since $\{M^0(\infty), M^k(\infty)\}$ is nondegenerate, $M^0(\infty)$ is invertible, and hence $M^0(s)$ is invertible as a rational matrix. Then, $HRC P(q_{x_0}(s), \{M^i(s)\}_{i=0}^k)$ is equivalent to $RCP(r^{s^{-1}}((M^0(s))^{-1}q_{x_0}(s)), N(\{M^i(s)\}_{i=0}^k))$ as explained in remark 2.8. On the other hand, the column $\mathcal{W}$ -property of $\{M^i(s)\}_{i=0}^k$ for all sufficiently large σ implies that for all $x_0 \in \mathbb{R}^n$, $HLCP_k^1(q_{x_0}(\sigma), \{M^i(\sigma)\}_{i=0}^k)$ has a unique solution for all sufficiently large σ and hence, by remark 2.8, $LCP(r^{\sigma^{-1}}((M^0(\sigma))^{-1}q_{x_0}(\sigma)), N(\{M^i(\sigma)\}_{i=0}^k))$ has a unique solution for all sufficiently large σ . According to theorem 4.1 of [8], $RCP(r^{s^{-1}}((M^0(s))^{-1}q_{x_0}(s)), N(\{M^i(s)\}_{i=0}^k))$ has a solution for all $x_0 \in \mathbb{R}^n$. It follows that $HRC P(q_{x_0}(s), \{M^i(s)\}_{i=0}^k)$ has a solution. The uniqueness of this solution can be deduced from corollary 4.10 of [8] by again using remark 2.8. Now, suppose that $\{z^i(s)\}_{i=0}^k$ is the unique solution of $HRC P(q_{x_0}(s), \{M^i(s)\}_{i=0}^k)$. It is clear that $z^1(s), z^2(s), \dots, z^{k-1}(s)$ are all strictly proper. Then, we have

$$M^0(s)z^0(s) = w(s) + M^k(s)z^k(s) \quad (2)$$

Note that the complementarity conditions 2.10(b) yield that $(z^0(s))^T z^k(s) = 0$ for all $s \in \mathbb{C}$. Indeed, if $z_i^k(s) \neq 0$ for some $i \in \bar{m}$ then $z_i^{k-1}(s) = z_i^{k-2}(s) = \dots = z_i^1(s) = s^{-1}$ and hence $z_i^0(s) \equiv 0$. Let $J = \{j | z_j^0(s) \neq 0\}$. Obviously, $z_j^k(s) \equiv 0$. We can rewrite (2) as

$$\underbrace{\begin{pmatrix} M_{JJ}^0(s) & M_{JJ^c}^k(s) \\ M_{J^cJ}^0(s) & M_{J^cJ^c}^k(s) \end{pmatrix}}_{R(s)} \begin{pmatrix} z_J^0(s) \\ -z_{J^c}^k(s) \end{pmatrix} = \begin{pmatrix} w_J(s) \\ w_{J^c}(s) \end{pmatrix}$$

Note that $R(\infty)$ can be obtained by rearranging columns of one of the column representative matrices of $\{M^0(\infty), M^k(\infty)\}$. Thus, it is invertible. Clearly, this yields that $R(s)$ is invertible as a rational matrix. Furthermore, its inverse is proper. Consequently, $z^0(s)$ and $z^k(s)$ are strictly proper due to the fact that $w(s)$ is strictly proper.

Finally, we state our main result.

Theorem 3.6 *Let $(q_y, \{Y^i\}_{i=0}^k, q_u, \{U^i\}_{i=0}^k)$ be a complementarity representation of piecewise linear curves $\{f_i\}_{i=1}^m$ and $G(s) = D + C(sI - A)^{-1}B$. Suppose that*

$\{Y^0 - DU^0, Y^k - DU^k\}$ is nondegenerate and B is of full column rank. If $\{Y^i - G(\sigma)U^i\}_{i=0}^k$ has the column $\mathcal{W}$ -property for all sufficiently large σ then for each $x_0 \in \mathbb{R}^n$ there exists at most one solution of (1).

Proof Suppose that for some initial state $x_0 \in \mathbb{R}^n$, there exists two different solutions of (1), say (u, x, y) and (u', x', y') . From definition 3.1 (a), it is clear that x and x' are continuous. It follows from the fact that B is of full column rank that $x = x'$ implies $u = u'$ and $y = y'$ for almost everywhere. Therefore, $x \neq x'$, i.e., there exist $T > 0$ and $\epsilon > 0$ such that $x(T) = x'(T)$ and $x(t+T) \neq x'(t+T)$ for $t \in (0, \epsilon)$. This means that there exist two different initial solutions of (1) for initial state $x(T)$. According to lemma 3.5 (a), $HRC P(q_{x(T)}, \{Y^i - G(s)U^i\}_{i=0}^k)$ cannot have a unique strictly proper solution. However, this contradicts the result we get from lemma 3.5 (b).

Note that, since $G(s)$ is rational, the condition in the theorem can be verified effectively by looking at the sign of the leading coefficients of the involved determinants.

The relay systems as they are introduced in [12] are in the scope of the above theorem. It can be checked that $(q_y, \{Y^i\}_{i=0}^2, q_u, \{U^i\}_{i=0}^2)$ is a complementarity representation for a linear relay system with ‘normalized’ relays where

$$\begin{aligned} q_y &= 0, & Y^0 &= I, & Y^1 &= 0, & Y^2 &= I \\ q_u &= e, & U^0 &= 0, & U^1 &= -2I, & U^2 &= 0. \end{aligned}$$

Then, according to theorem 3.6, the column $\mathcal{W}$ -property of $\{I, 2G(\sigma), I\}$ for sufficiently large σ is sufficient for the uniqueness of solutions. It is obvious that $\{I, 2G(\sigma), I\}$ has the column $\mathcal{W}$ -property for sufficiently large σ if and only if $G(\sigma)$ is a $\mathcal{P}$ -matrix for sufficiently large σ . The latter condition is exactly what is presented as a sufficient condition for uniqueness of solutions of linear relay systems in [12].

4 Conclusions

For a class of piecewise linear systems, sufficient conditions for uniqueness of solutions are given within the framework of linear complementarity systems. Further topics for future work include obtaining necessary conditions, extending results to larger classes of piecewise linear systems and incorporating the results in simulation methods.

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