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AN OPTIMAL MODEL APPROXIMATION SCHEME FOR LINEAR TIME-INVARIANT BEHAVIORS

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Abstract

This paper considers a problem of optimal model approximation in a behavioral framework. For the class of linear time-invariant ℓ_2 -systems, an angle criterion is introduced to define the distance measure between two systems in this class. Given a system of degree n , a characterization of all optimal approximants of degree $n - 1$ is given. The result is derived using the notion of canonical past-future links.

1 Problem formulation

Following the behavioral framework advocated in [11, 12], a *dynamical system* is a triple $(T, W, \mathcal{B})$ where T is the time set, W the signal space and $\mathcal{B} \subseteq W^T$ a family of mappings $w : T \rightarrow W$, called the *behavior* of the system. The behavior $\mathcal{B}$ represents those time series that are compatible with the laws which describe the system. In this paper we consider *discrete time systems* with time set $T = \mathbb{Z}$ and finite dimensional real signal space $W = \mathbb{R}^q$, where q is a positive integer. An ℓ_2 -system is a system whose behavior $\mathcal{B}$ is a closed subset of ℓ_2 . Notions such as linearity and time-invariance carry over in a natural way to ℓ_2 -systems and are assumed to be known (See [11, 12] for details). An ℓ_2 -system is said to be *complete* if a trajectory w belongs to $\mathcal{B}$ whenever $w \in \ell_2$ and its restrictions $w_F \in \mathcal{B}_F$ for all (finite) intervals $F \subset \mathbb{Z}$. The set of all linear, time-invariant and complete ℓ_2 -systems will be the model class of interest in this paper. This class of systems is denoted by $\mathbb{L}^q$, or by $\mathbb{L}$ if the dimension of the signal space is clear from the context.

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Definition 1.1 (Complexity) The *complexity* of a system $\mathcal{B} \in \mathbb{L}^q$ is the pair of integers (m, n) that satisfy

$$\dim(\mathcal{B}|_{[1, N]}) = mN + n$$

for all $N > n$.

These numbers are well defined [7–9], m is called the *rank* and n the *degree* of $\mathcal{B}$. The rank and degree of a system have simple system theoretic interpretations: the rank m corresponds to the number of inputs and the degree n to the dimension of the state space in any (minimal) input-state-output representation of the system $\mathcal{B}$. In fact, $0 \leq m \leq q$ and one can show that $m = 0$ implies $n = 0$ and $\mathcal{B} = \{0\}$. Similarly, $m = q$ implies $n = 0$ and $\mathcal{B} = \ell_2^q$.

As an approximation criterion for dynamical systems we consider the *angle* between two systems. This is defined as follows. Let $w, w' \in \ell_2$ be two time series and define

$$\theta(w, w') := \begin{cases} \pi/2 & \text{if } w = 0 \text{ or } w' = 0 \\ \arccos\left(\frac{|\langle w, w' \rangle|}{\|w\| \|w'\|}\right) & \text{otherwise} \end{cases}.$$

The angle between a time series and a closed linear subspace $\mathcal{B}$ of ℓ_2 is defined (with some abuse of notation) as

$$\theta(w, \mathcal{B}) := \min_{w' \in \mathcal{B}} \theta(w, w').$$

This minimum exists and it is easy to see that it is attained for the orthogonal projection of w onto $\mathcal{B}'$, i.e., $\theta(w, \mathcal{B}) = \theta(w, w')$ with w' the orthogonal projection of w on $\mathcal{B}$.

Definition 1.2 (Angle) The angle between two non-zero systems $\mathcal{B}, \mathcal{B}' \in \mathbb{L}$ is defined as

$$\theta(\mathcal{B}, \mathcal{B}') := \max \left\{ \sup_{w \in \mathcal{B}, w \neq 0} \theta(w, \mathcal{B}'), \sup_{w' \in \mathcal{B}', w' \neq 0} \theta(w', \mathcal{B}) \right\}.$$

Define $\theta(\{0\}, \{0\}) := 0$. The angle is called *flat* if $\theta(\mathcal{B}, \mathcal{B}') = \theta(w, \mathcal{B}') = \theta(w', \mathcal{B})$ for all nonzero $w \in \mathcal{B}$ and $w' \in \mathcal{B}'$.

Obviously, $0 \leq \theta(\mathcal{B}, \mathcal{B}') \leq \pi/2$. Furthermore, one can show that $\theta(\mathcal{B}, \mathcal{B}') = 0$ if and only if $\mathcal{B} = \mathcal{B}'$; $\theta(\mathcal{B}, \mathcal{B}') = \theta(\mathcal{B}^\perp, \mathcal{B}'^\perp)$ and $\theta(\mathcal{B}, \mathcal{B}') = \pi/2$ whenever $m \neq m'$.

The angle between two systems is closely related to the ℓ_2 -gap introduced in [1] and further analyzed in [2–4]. Actually, one can show that the sinus of the angle between two ℓ_2 -systems corresponds to the usual ℓ_2 -gap between the system behaviors.

Now consider the following approximation problem.

Definition 1.3 Given a system $\mathcal{B} \in \mathbb{L}$ with rank m and degree n , find a system $\mathcal{B}' \in \mathbb{L}$ with the same rank m and degree $n' < n$, such that the angle $\theta(\mathcal{B}, \mathcal{B}')$ is minimized.

We therefore wish to find systems of lower degree which minimize the angle criterion. The class of all systems $\mathcal{B}'$ that meet the specifications of Definition 1.3 is denoted by $\mathbb{L}(\mathcal{B}, n')$. Precisely, if $\mathbb{L}_{m,n}$ denotes the set of systems in $\mathbb{L}$ with rank m and degree at most n then, with $\mathcal{B} \in \mathbb{L}_{m,n}$ and $n' < n$, we define

$$\mathbb{L}(\mathcal{B}, n') := \{\mathcal{B}' \in \mathbb{L}_{m,n'} \mid \theta(\mathcal{B}, \mathcal{B}') \leq \theta(\mathcal{B}, \mathcal{B}'') \text{ for all } \mathcal{B}'' \in \mathbb{L}_{m,n'}\}. \quad (1.1)$$

Systems in $\mathbb{L}(\mathcal{B}, n')$ are called *optimal approximants* of $\mathcal{B}$ of degree n' . Note that in the problem formulated in Definition 1.3, the rank of the approximant is required to be the same as the rank of the to-be-approximated system. This requirement is motivated by the observation that an approximant $\mathcal{B}'$ of $\mathcal{B}$ with different rank will necessarily have maximum angle. In view of the angle criterion it is therefore of little interest to also reduce the rank m of $\mathcal{B}$.

2 Past-future links

In this section we introduce the system structures that are relevant for the model approximation problem formulated in Definition 1.3. Let $\mathcal{B} \in \mathbb{L}$ be a given system and define its past and future behavior as

$$\begin{aligned} \mathcal{B}^- &:= \mathcal{B}_{\mathbb{Z}_-} \\ \mathcal{B}^+ &:= \mathcal{B}_{\mathbb{Z}_+}. \end{aligned} \quad (2.1)$$

Obviously, for every concatenated¹ trajectory $p \wedge f$ belonging to $\mathcal{B}$, its past p belongs to $\mathcal{B}^-$ and its future f to $\mathcal{B}^+$. The converse, however, is not true: $p \in \mathcal{B}^-$, $f \in \mathcal{B}^+$ does not imply that the concatenation $p \wedge f$ belongs to $\mathcal{B}$. Trajectories $p \in \mathcal{B}^-$ and $f \in \mathcal{B}^+$ are said to be *compatible* (or *linked*) if their concatenation $p \wedge f \in \mathcal{B}$. For any compatible pair, f is said to be a *minimal future* of p if $\|f\|$, is minimal among all compatible futures of p . Similarly, p is said to be a *minimal past* of f if $\|p\|$, is minimal among all compatible pasts of f .

¹The concatenation of $p \in \mathcal{B}^-$ and $f \in \mathcal{B}^+$ is the trajectory $w : \mathbb{Z} \rightarrow W$ whose restriction $w_{\mathbb{Z}_-} = p$ and $w_{\mathbb{Z}_+} = f$.

Definition 2.1 (Past-future links) Let $\mathcal{B} \in \mathbb{L}$. A *past-future link* of $\mathcal{B}$ is a system trajectory $p \wedge f \in \mathcal{B}$ in which p is a minimal past of f and (at the same time) f a minimal future of p . The set of all past-future links of $\mathcal{B}$ is denoted by $\mathcal{B}^\leftrightarrow$. The set of all minimal futures of trajectories in $\mathcal{B}^-$ is denoted by $\mathcal{B}^\rightarrow$. Similarly, $\mathcal{B}^\leftarrow$ denotes the set of all minimal pasts of trajectories in $\mathcal{B}^+$.

It is easily shown that $\mathcal{B}^\rightarrow = [\mathcal{B}^\leftrightarrow]^+$ and $\mathcal{B}^\leftarrow = [\mathcal{B}^\leftrightarrow]^-$. The *weakest forward* and *weakest backward gain* of a system $\mathcal{B} \in \mathbb{L}$ are defined as, respectively,

$$\begin{aligned} \rho_f &:= \min\left\{\frac{\|f\|}{\|p\|} \mid 0 \neq (p \wedge f) \in \mathcal{B}^\leftrightarrow\right\} \\ \rho_p &:= \min\left\{\frac{\|p\|}{\|f\|} \mid 0 \neq (p \wedge f) \in \mathcal{B}^\leftrightarrow\right\} \end{aligned}$$

Weakest forward and backward links in $\mathcal{B}$ are past-future links that achieve the ratios ρ_f and ρ_p , respectively. The *weakest gain*, ρ , of $\mathcal{B}$ is the minimum of ρ_f and ρ_p , and *weakest links* are weakest forward or backward links that achieve this ratio.

Next, we refine the notion of weakest forward and weakest backward gain of a system to a finite set of increasing past-future ratios

$$\rho_p =: \rho_1 \leq \dots \leq \rho_k \leq \dots \rho_n := \rho_f^{-1}.$$

Here, n is the degree of the system $\mathcal{B}$ and the positive real numbers ρ_k , the *canonical past-future ratios* of $\mathcal{B}$, are defined recursively by setting $\rho_1 = \rho_p$ and

$$\begin{aligned} \rho_k &:= \min\left\{\frac{\|p\|}{\|f\|} \mid 0 \neq (p \wedge f) \in \mathcal{B}^\leftrightarrow \text{ and } (p \wedge f) \perp (p_{(j)} \wedge f_{(j)}), \text{ for } j = 1, \dots, k-1\right\}, \end{aligned}$$

where, $(p_{(j)} \wedge f_{(j)})$, the j -th *canonical past-future link*, is the element in $\mathcal{B}^\leftrightarrow$ that achieves the j -th past-future ratio $\frac{\|p_{(j)}\|}{\|f_{(j)}\|} = \rho_j$.

The following proposition summarizes some basic properties of past-future links and canonical past-future ratios.

Proposition 2.2 Let $\mathcal{B} \in \mathbb{L}$ have degree n and let $\tilde{\mathcal{B}} = \mathcal{B}^\perp$ denote its orthogonal complement. Then

1. $\mathcal{B}^\leftrightarrow \perp \tilde{\mathcal{B}}^\leftrightarrow$.
2. $\dim(\mathcal{B}^\leftrightarrow) = \dim(\mathcal{B}^\leftarrow) = \dim(\mathcal{B}^\rightarrow) = \dim(\tilde{\mathcal{B}}^\leftrightarrow) = n$.
3. $\mathcal{B}^\leftarrow = \tilde{\mathcal{B}}^\leftarrow$ and $\mathcal{B}^\rightarrow = \tilde{\mathcal{B}}^\rightarrow$.
4. $\mathcal{B}^\leftarrow = \mathcal{B}^- \cap \tilde{\mathcal{B}}^-$ and $\mathcal{B}^\rightarrow = \mathcal{B}^+ \cap \tilde{\mathcal{B}}^+$
5. If $\{\rho_k\}_{k=1}^n$ are the canonical past-future ratios of $\mathcal{B}$ and $\{\tilde{\rho}_k\}_{k=1}^n$ the ones of $\tilde{\mathcal{B}}$ then $\rho_k = \tilde{\rho}_{n-k+1}^{-1}$ for $k = 1, \dots, n$.

6. there exist orthonormal bases $\{p_{(1)}, \dots, p_{(n)}\}$ for $\mathcal{B}^{\Leftarrow}$ and $\{f_{(1)}, \dots, f_{(n)}\}$ for $\mathcal{B}^{\Rightarrow}$ such that $\{x_{(1)}, \dots, x_{(n)}\}$ and $\{\tilde{x}_{(1)}, \dots, \tilde{x}_{(n)}\}$ with

$$x_{(k)} := \tilde{\gamma}_k p_{(k)} \wedge \gamma_k f_{(k)} \quad (2.2)$$

$$\tilde{x}_{(k)} := \gamma_k p_{(k)} \wedge -\tilde{\gamma}_k f_{(k)} \quad (2.3)$$

where

$$\gamma_k^2 := \frac{1}{1 + \rho_k^2}; \quad \tilde{\gamma}_k^2 := \frac{\rho_k^2}{1 + \rho_k^2}$$

define an orthonormal basis for $\mathcal{B}^{\Leftarrow}$ and $\tilde{\mathcal{B}}^{\Leftarrow}$, respectively. $\{x_{(k)}\}_{k=1}^n$ and $\{\tilde{x}_{(k)}\}_{k=1}^n$ are the canonical past-future links corresponding to the canonical ratios $\{\rho_k\}_{k=1}^n$ and $\{\tilde{\rho}_k\}_{k=1}^n$ and whenever all canonical past-future ratios are distinct, these bases are unique modulo n sign changes in one of the four bases.

Before returning to the model approximation problem, we discuss how dynamical systems can be generated from time series. This is achieved by a process called *completion*. For a subspace $\mathcal{S} \subseteq \ell_2^q$ its completion is defined as

$$\text{comp}(\mathcal{S}) := \{w \in \ell_2^q \mid w_F \in \mathcal{S}_F \text{ for all intervals } F \subset \mathbb{Z}\}.$$

It follows that $\text{comp}(\mathcal{S})$ is the smallest ℓ_2 -complete set that contains $\mathcal{S}$.

Definition 2.3 The system generated by a time series $w \in \ell_2^q$ is

$$\mathcal{B}(w) := \text{comp}(\text{span}[\text{shifts}(w)]).$$

Here, $\text{span}(\cdot)$ produces the linear span of its arguments and $\text{shifts}(w)$ is the collection of all shifts of the time series w , i.e., $\text{shifts}(w) := \{\sigma^k w \mid k \in \mathbb{Z}\}$ where $\sigma^k w(t) := w(t+k)$ is the k -shift. For any $w \in \ell_2^q$, the generated system $\mathcal{B}(w)$ actually belongs to $\mathbb{L}^q$, $w \in \mathcal{B}(w)$ for any $w \in \ell_2$ and it is easily seen that $\mathcal{B}(w)$ is the smallest (in the sense of set inclusions) dynamical system in $\mathbb{L}$ containing w . $\mathcal{B}(w)$ is also referred to as the *most powerful unfalsified model* [11] of w in the model class $\mathbb{L}^q$. There exist an obvious generalization of Definition 2.3 where systems are generated by finite sets W of time series in ℓ_2 . Interestingly, every system $\mathcal{B} \in \mathbb{L}^q$ can be generated by a *finite number* of time series:

Proposition 2.4 Every system in $\mathbb{L}$ can be generated by a finite set W of trajectories in ℓ_2 . $\mathcal{B}(W)$ has rank at most m if the generating set W consists of m time series of finite degree, that is, if the dimensions of the sets $\{(\sigma^k w)_{\mathbb{Z}_+}\}_{k \geq 0}$ and $\{(\sigma^k w)_{\mathbb{Z}_-}\}_{k < 0}$ are both finite for all $w \in W$.

3 Optimal reductions

Let $\mathcal{B} \in \mathbb{L}$ have degree n and let $x_{(k)}$ and $\tilde{x}_{(k)}$ be an orthonormal basis of $\mathcal{B}^{\Leftarrow}$ as defined in (2.2) and (2.3). Let $p_{(k)} \wedge f_{(k)}$ denote the k -th canonical past-future link, as introduced in the previous section. Consider for $k = 1, \dots, n$ the sets

$$\mathcal{B}(x_{(k)}) \quad (3.1a)$$

$$\mathcal{B}(\tilde{x}_{(k)}) \quad (3.1b)$$

$$\mathcal{B}(p_{(k)} \wedge 0) \quad (3.1c)$$

$$\mathcal{B}(0 \wedge f_{(k)}) \quad (3.1d)$$

The sets (3.1c) and (3.1d) are therefore the most powerful unfalsified models of a trajectory that consists of a truncation of the k -th canonical past-future link. It follows from Proposition 2.4 that the sets (3.1) have all rank 1. By construction,

$$\mathcal{B}(x_{(k)}) \subseteq \mathcal{B} \quad (3.2a)$$

$$\mathcal{B}(\tilde{x}_{(k)}) \subseteq \tilde{\mathcal{B}}. \quad (3.2b)$$

It can be shown that the time series $x_{(k)}$, $\tilde{x}_{(k)}$, $p_{(k)} \wedge 0$ and $0 \wedge f_{(k)}$ are all of finite degree, so that, by Proposition 2.4 the inclusions (3.2) are equalities whenever the system $\mathcal{B}$ has rank $m = 1$. The following proposition provides a main tool in the construction of optimal approximate systems.

Proposition 3.1 Let $\mathcal{B} \in \mathbb{L}$ have degree n and let $1 \leq k \leq n$. Then

1. the systems (3.1) have rank $m = 1$.
2. the systems (3.1c), (3.1d) have degree at most $n - 1$.
3. $\theta(\mathcal{B}(x_{(k)}), \mathcal{B}(p_{(k)} \wedge 0)) = \arcsin(\gamma_k)$
4. $\theta(\mathcal{B}(\tilde{x}_{(k)}), \mathcal{B}(0 \wedge f_{(k)})) = \arcsin(\tilde{\gamma}_k)$
5. $\theta(\mathcal{B}(\tilde{x}_{(k)}), \mathcal{B}(p_{(k)} \wedge 0)) = \arcsin(\tilde{\gamma}_k)$
6. $\theta(\mathcal{B}(\tilde{x}_{(k)}), \mathcal{B}(0 \wedge f_{(k)})) = \arcsin(\gamma_k)$

The systems (3.1c), (3.1d) are candidate systems for approximate models as their degree is strictly smaller than the degree of $\mathcal{B}$. In fact, the weakest links of $\mathcal{B}$ precisely define optimal approximants in the sense of Definition 1.3. This is stated in the following theorem, which is the main result of this paper and an immediate consequence of the above proposition.

Theorem 3.2 Let $\mathcal{B} \in \mathbb{L}$ be a system of rank $m = 1$ and degree n , with weakest gain ρ . Let $(p \wedge f) \in \mathcal{B}^{\Leftarrow}$ denote a weakest link in $\mathcal{B}$. Define

$$\mathcal{B}' := \begin{cases} \mathcal{B}(p \wedge 0) & \text{if } \|p\| > \|f\| \\ \mathcal{B}(0 \wedge f) & \text{if } \|p\| \leq \|f\| \end{cases} \quad (3.3)$$

Then $\mathcal{B}'$ is an optimal reduced order approximation of $\mathcal{B}$ of rank m and degree $n - 1$. Furthermore, the angle $\theta(\mathcal{B}, \mathcal{B}') = \arctan(\rho)$. Conversely, all systems $\mathcal{B}' \in \mathbb{L}$ with degree at most $n - 1$ that achieve the angle $\arctan(\rho)$ are of this form, i.e., they are given by (3.3) where $(p \wedge f) \in \mathcal{B}^{\Leftrightarrow}$ is a weakest link in $\mathcal{B}$.

Theorem 3.2 therefore implies that for any $\mathcal{B} \in \mathbb{L}$ of rank $m = 1$, the optimal approximants

$$\mathbb{B}(\mathcal{B}, n - 1) = \{\mathcal{B}' \mid \mathcal{B}' \text{ satisfies (3.3) with } (p \wedge f) \in \mathcal{B}^{\Leftrightarrow} \text{ a weakest link in } \mathcal{B}\}.$$

4 Conclusions

We formalized an optimal model approximation problem in the behavioral setting and provided a complete solution for systems of rank one and reductions of the degree of the to-be-approximated system with one. Generalizations to higher rank systems are straightforward. The algorithm for the calculations of optimal reductions makes use of isometric state space representations and will be detailed elsewhere [10].

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