On the integrability of nonlinear partial differential equations

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We investigate the integrability of Nonlinear Partial Differential Equations (NPDEs). The concepts are developed by first discussing the integrability of the KdV equation. We proceed by generalizing the ideas introduced for the KdV equation to other NPDEs. The method is based upon a linearization principle that can be applied on nonlinearities that have a polynomial form. The method is further illustrated by finding solutions of the nonlinear Schrödinger equation and the vector nonlinear Schrödinger equation, which play an important role in optical fiber communication. Finally, it is shown that the method can also be generalized to higher dimensions. © 1999 American Institute of Physics.

I. INTRODUCTION

The conditions under which Nonlinear Partial Differential Equations (NPDEs) can be solved are even in one dimension not well understood. Roughly speaking, the majority of the integrable systems can be classified in three main groups. In the first of these groups are those equations that can be reduced to a quadrature through the existence of an adequate number of integrals of motion. In the second class are those equations that can be mapped into a linear system by applying a number of transformations (hereafter to be called C integrable). The last group consists of differential equations that can be solved by Inverse Scattering Transformations (IST). In the following, we will call equations that can be solved by inverse scattering methods “S integrable.” The discovery of the IST has led to considerable progress in understanding the topic of integrability, since this technique made it possible to investigate the integrability of large classes of NPDEs systematically.

Another important consideration is that most of the work on the integrability of NPDEs has been carried out in one space dimension only. Although the inverse problem of the Schrödinger equation can be generalized to three dimensions, the method is far too complicated to solve higher-dimensional NPDEs. An alternative is the approach, which is also successfully generalized to N dimensions (see, for instance, the book by Ablowitz and Clarkson). Nevertheless, for both these methods the existence of the obtained solutions is difficult to prove. The concept of C integrability, however, has the potential to be generalized to dimensions higher than one. In this paper, we will demonstrate a simple method based upon linearization principles that enables us to compute solutions of large classes of NPDEs by solving a linear algebraic recursion relationship. The result suggests that the method can be generalized to higher-dimensional NPDEs.

In this paper we aim to find integrable differential equations that can be solved by linearization. The basic idea of the method goes back to Stokes, and is used several times to obtain solutions of nonlinear evolution equations. We will apply the method in a slightly different form to find conditions on the integrability of nonlinear evolution equations. Since it is not clear what integrability exactly means, we use in this paper the heuristic definition that a NPDE is integrable if given a sufficiently general initial condition, we can find analytic expressions the time evolution of the solution. For NPDEs that can be solved by inverse scattering techniques, this...
notion is equivalent with the existence of $N$-soliton solutions, since it is implicitly assumed that the obtained solution can be expanded on a Fourier basis. It is shown that the condition of expansion in a Fourier can be replaced by an arbitrary other infinite set of basis functions.

We present the following results. First, we derive a simple method to test NPDEs with a polynomial type of nonlinearity in the presence of $N$-soliton solutions. Necessary conditions that indicate whether a NPDE has $N$-soliton solutions includes that the nonlinearity can be expanded in the same basis functions as the linear part, and second that the dispersion relationship associated with the linearized problem can be solved. The method is demonstrated by first discussing the integrability of the KdV equation in Sec. II. In Sec. III, the concepts derived for the KdV equation are generalized to discuss the integrability of more general NPDEs. Finally, in Sec. IV, the results are applied to investigate the integrability of the coupled nonlinear Schrödinger equation. Moreover, it is indicated that the method can also be used to obtain solutions to higher-dimensional NPDEs. The paper is concluded with a discussion.

II. THE INTEGRABILITY OF THE KdV EQUATION

In order to illustrate the machinery developed throughout this paper, we first discuss the integrability of the KdV equation as an example. The integrability of the KdV equation is a well-studied problem. This makes the KdV equation an ideal object to test the validity of newly developed ideas with respect to the integrability of NPDEs. We will introduce our methods on the integrability of NPDEs by discussing the existence of $N$-soliton solutions for the KdV equation, which is given by

$$u_t + u_{xxx} = 6u_xu.$$  \hspace{1cm} (1)

We try to find solutions of Eq. (1) by substitution of the following Fourier series:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{i n(kx - \omega t)}.$$  \hspace{1cm} (2)

If we substitute the solution $u(x,t)$ into Eq. (2), we obtain

$$\sum_{n=1}^{\infty} (n\omega + k^3 n^3)A_n e^{i n(kx - \omega t)} = -6k \sum_{n=1}^{\infty} \sum_{l=1}^{n-1} lA_l A_{n-l} e^{i n(kx - \omega t)}.$$  \hspace{1cm} (3)

We can now determine the coefficients $A_n$ by deriving a recursion relationship. This can be achieved by comparing the exponential functions in Eq. (3). If we compare all the terms for which $n = 1$, we find

$$(\omega + k^3)A_1 e^{i kx - \omega t} = 0.$$  \hspace{1cm} (4)

For a nonzero $A_1$, we find that Eq. (4) is satisfied if

$$\omega = -k^3.$$  \hspace{1cm} (5)

If we put $n = 2$ in Eq. (3), we can determine $A_2$ by solving the following relationship:

$$(2\omega + 8k^3)A_2 e^{2i kx - 2\omega t} = -6kA_1 A_1 e^{2i kx - 2\omega t}.$$  \hspace{1cm} (6)

If we use the dispersion relationship (5), we find that $A_2$ is given by

$$A_2 = -\frac{A_1^2}{k^2}.$$  \hspace{1cm} (7)
By repeating this procedure, we can compute all the expansion coefficients $A_n$ of the solutions $u(x,t)$. In general, all the coefficients $A_n$ can be computed by solving the following linear algebraic problem:

$$L^{(n)}(k)A_n = R^{(n)}(k). \tag{8}$$

The operators $L^{(n)}(k)$ and $R^{(n)}(k)$ in Eq. (8) are given by

$$L^{(n)}(k) = n[n^2 - 1]k^3, \quad R^{(n)}(k) = -6k \sum_{l=1}^{n-1} lA_lA_{n-l}. \tag{9}$$

If we compute all the coefficients $A_n$ by using Eq. (9), we then obtain the Fourier expansion of $u(x,t)$, for which the first terms are given by

$$u(x,t) = A_1e^{i(kx-\omega t)} - \frac{A_1^2}{k}e^{2i(kx-\omega t)} + \frac{3A_1^3}{4k^2}e^{-3i(kx-\omega t)} + \ldots. \tag{10}$$

If we substitute $k = 2i\beta$ and $A_1 = 4d\beta$ into Eq. (10), we find

$$u(x,t) = 4d\beta e^{-2(\beta x - 4\beta^3 t)} + 16d^2e^{-4(\beta x - 4\beta^3 t)} + \frac{24d^3}{\beta}e^{-6(\beta x - 4\beta^3 t)} + \ldots. \tag{11}$$

By carrying out the summation in Eq. (11), we can formulate this equation more compactly:

$$u(x,t) = \frac{8d\beta e^{-2(\beta x - 4\beta^3 t)}}{1 + \frac{d}{\beta}e^{-2(\beta x - 4\beta^3 t)}}. \tag{12}$$

Hence, if we put

$$\beta = \frac{1}{2} \sqrt{c}, \quad x_0 = -\frac{1}{\sqrt{c}} \log \left( -\frac{d}{\beta} \right), \quad d < 0, \tag{13}$$

we can simplify Eq. (12) one step further to

$$u(x,t) = -\frac{c}{2} \text{sech}^2 \left( \frac{1}{2} \sqrt{c}(x - ct + x_0) \right). \tag{14}$$

Equation (14) describes the well-known KdV soliton.

What did we learn from this simple exercise? At first, the KdV equation has solutions because of the special structure of the nonlinearity. If we substitute the special solution (2) in the nonlinear part of the KdV equation, we find that we can expand the nonlinearity in the same basis functions as the linear part:

$$6u_xu = \sum_{n=1}^{\infty} D_n e^{i(kx-\omega t)}, \quad D_n = -6k \sum_{l=1}^{n-1} lA_lA_{n-l}. \tag{15}$$

This guarantees that we can find an iteration relationship for the expansion coefficients $A_n$. As we will see later, we do not have to restrict to a Fourier expansion of the solution only. In principle, this method works for any set of basis functions as long as we can expand the nonlinearity in the same basis functions as the linear part. In the following, we will show that the structure of the nonlinearity of the KdV equation enables us to construct the Fourier expansion of the $N$ soliton of
If we substitute Eq. (16) into the KdV equation (1), we obtain the following result:

$$\sum_{\mu_1, \mu_2=1}^{\infty} L^{(\mu_1, \mu_2)}(k_1, k_2) C(\mu_1, \mu_2) e^{i(\mu_1 k_1 z_1 + \mu_2 k_2 z_2)}$$

$$= -6 \sum_{\mu_1, \mu_2=1}^{\infty} \sum_{\eta_1, \eta_2=1}^{\mu_1-1, \mu_2-1} M^{(\eta_1, \eta_2)}(k_1, k_2) C(\mu_1 - \eta_1, \mu_2 - \eta_2) C(\eta_1, \eta_2) e^{i(\mu_1 k_1 z_1 + \mu_2 k_2 z_2)},$$

where

$$L^{(\eta_1, \eta_2)}(k_1, k_2) = \sum_{i=1}^{2} n_i [n_i^2 - 1] k_i^3, \quad M^{(\eta_1, \eta_2)}(k_1, k_2) = \sum_{i=1}^{2} n_i k_i.$$ (17)

We solve Eq. (17) by comparing equal exponential powers on both sides. This can be done by defining a parameter $\Gamma = \mu_1 + \mu_2$ and subsequently comparing the powers for $\Gamma = 1, 2, 3, \ldots$. We first discuss the case in which $\Gamma = 1$ in which only the coefficients $C(1,0)$ and $C(0,1)$ contribute:

$$[ \omega_1 + k_1^3] C(1,0) e^{ik_1 z_1} + [ \omega_2 + k_2^3] C(0,1) e^{ik_2 z_2} = 0.$$ (19)

If we put $C(1,0) = A_1$ and $C(0,1) = A_2$, we find that the following linear dispersion relationships must be valid:

$$\omega(k_1) = -k_1^3 \quad \text{and} \quad \omega(k_2) = -k_2^3.$$ (20)

Once the linear dispersion relationships are determined and if the coefficients $C(1,0)$ and $C(0,1)$ have taken their values $A_1$ and $A_2$, we can compute all the other coefficients $C(\mu, \eta)$ by applying the following linear recursion relation:

$$L^{(\mu_1, \mu_2)}(k_1, k_2) C(\mu_1, \mu_2) = R^{(\mu_1, \mu_2)}(k_1, k_2),$$ (21)

where

$$R^{(\mu_1, \mu_2)}(k_1, k_2) = -6 \sum_{\eta_1, \eta_2=1}^{\mu_1-1, \mu_2-1} M^{(\eta_1, \eta_2)}(k_1, k_2) C(\mu_1 - \eta_1, \mu_2 - \eta_2) C(\eta_1, \eta_2).$$ (22)

Equation (21) has a similar structure as Eq. (8). In principle, Eq. (21) provides an efficient tool to compute all the coefficients $C(\mu, \eta)$. We can generalize this result to the $N$-soliton case by assuming that the solution $u(x,t)$ takes the following form:

$$u(x,t) = \sum_{\mu_1, \ldots, \mu_N=1}^{\infty} C(\mu_1 \cdots \mu_N) e^{i(\mu_1 k_1 z_1 + \cdots + \mu_N k_N z_N)} \begin{cases} z_1 = x - \frac{\omega(k_1)}{k_1} \\
\vdots \\
 z_N = x - \frac{\omega(k_N)}{k_N}. \end{cases}$$ (23)
We can determine the nonzero coefficients $C(\mu_1 \cdots \mu_N)$ by substituting Eq. (23) into the KdV equation (1):

$$
\sum_{\mu_1 \cdots \mu_N=1}^{\infty} L^{(\mu_1 \cdots \mu_N)}(k_1 \cdots k_N) C(\mu_1 \cdots \mu_N) e^{i(\mu_1 k_1 z_1 + \cdots + \mu_N k_N t_n)}
= -6 \sum_{\mu_1 \cdots \mu_N=1}^{\infty} \sum_{\eta_1 \cdots \eta_N=1}^{\infty} M^{(\eta_1 \cdots \eta_N)}(k_1 \cdots k_N) C(\mu_1 - \eta_1 \cdots \mu_N - \eta_N) C(\eta_1 \cdots \eta_N)
\times e^{i(\mu_1 k_1 z_1 + \cdots + \mu_N k_N t_n)},
$$

(24)

where

$$
L^{(n_1 \cdots n_N)}(k_1 \cdots k_N) = \sum_{i=1}^{N} n_i [k_i^2 - 1] k_i^3, \quad M^{(n_1 \cdots n_N)}(k_1 \cdots k_N) = \sum_{i=1}^{N} n_i k_i.
$$

(25)

If we use that $\omega(k_i) = -k_i^3$, $(i \in 1 \cdots N)$ and $A_1 = C(1,0,0,\ldots,0), A_2 = C(0,1,0,\ldots,0), \ldots, A_N = C(0,0,1,\ldots,0)$, we find that the expansion coefficients of the $N$-soliton solution for the KdV equation can be computed by solving the following linear relationship:

$$
L^{(\mu_1 \cdots \mu_N)}(k_1 \cdots k_N) C(\mu_1 \cdots \mu_N) = R^{(\mu_1 \cdots \mu_N)}(k_1 \cdots k_N),
$$

(26)

where

$$
R^{(\mu_1 \cdots \mu_N)}(k_1 \cdots k_N) = -6 \sum_{\eta_1 \cdots \eta_N=1}^{\infty} M^{(\eta_1 \cdots \eta_N)}(k_1 \cdots k_N) C(\mu_1 - \eta_1 \cdots \mu_N - \eta_N) C(\eta_1 \cdots \eta_N).
$$

(27)

From the exercise performed in this section, we can conclude that general solutions of the KdV equation can be obtained by solving Eqs. (26). This implies that the KdV equation can be transformed into a simple linear algebraic equation in the coefficient space. We can conclude that the KdV equation has $N$-soliton solutions because the following two conditions are satisfied.

(i) The structure of the nonlinearity of the KdV equation guarantees that the equation has solutions of the form (23). This result implies that the coefficients $R^{(\mu_1 \cdots \mu_N)}(k_1 \cdots k_N)$ exist.

(ii) $L^{(n_1 \cdots n_N)}(k_1 \cdots k_N)$ is not equal to zero if $k_1 \cdots k_N \neq 0$ and $n_1 \cdots n_N \neq 0$. This implies that $L^{(n_1 \cdots n_N)}(k_1 \cdots k_N)$ has an inverse.

In the following section we will show that a similar condition must hold for other NPDEs. In the following section it is shown that the concepts derived for the KdV equation can be generalized to large classes of NPDEs. The results obtained in this section are derived by assuming that the solution of the KdV equation can be expanded in Fourier basis functions. In the following section, it will be shown that similar principles apply for other basis functions.

### III. GENERALIZATIONS

In this section we will present more general results with respect to the integrability of nonlinear evolution equations. This will be done by generalizing the results obtained for the KdV equation. In this section, we focus on NPDEs of the following type:

$$
\mathcal{L}[u(x,t)] = Q[u(x,t)].
$$

(28)

In Eq. (28), the function $u(x,t)$ is an $M$-component vector function having entries $u_i(x,t)$. The operator $\mathcal{L}[\cdot]$ is assumed to take the following form:
The matrices $A^{(n)}$ in Eq. (29) are $M \times M$ matrices and $I$ is the identity matrix. As concluded from the previous section, integrability puts strong constraints on the nonlinearity represented by the operator $Q$. As a necessary condition for the integrability we require that if a solution of Eq. (28) has the following form:

$$u(x,t) = \sum_{\mu_1,\ldots,\mu_N=1}^{\infty} C(\mu_1,\ldots,\mu_N) \exp \left[ i \sum_{r=1}^{N} \sum_{s=1}^{M} \mu_r k_r z_{rs} \right] ; \quad z_{rs} = x - \omega(k_{rs}) t,$$

(30)

then the operator $Q$ must satisfy the following property:

$$Q[u(x,t)] = \sum_{\mu_1,\ldots,\mu_N=1}^{\infty} R(\mu_1,\ldots,\mu_N) \exp \left[ i \sum_{r=1}^{N} \sum_{s=1}^{M} \mu_r k_r z_{rs} \right],$$

(31)

where $C(\mu_1,\ldots,\mu_N)$ and $R(\mu_1,\ldots,\mu_N)$ are $M$-dimensional vector functions. Similarly as for the KdV equation, the vector function $R(\mu_1,\ldots,\mu_N)$ is specified by the nonlinearity. In other words, we require that given a solution of the form (30), the nonlinear operator $Q[u(x,t)]$ can be expanded in the same set of basis functions as $L[u(x,t)]$. In the previous section, we have shown that the nonlinearity of the KdV equation satisfies this condition. In general, large classes of nonlinear operators will have the property (31) and among them we are especially interested in the subclass $\hat{P}$, which plays an important role in nonlinear optics:

$$\hat{P}[u(x,t)] = P_N \left( u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^p u}{\partial x^q} \right),$$

(32)

where $P_N$ are polynomials of order $N$. If we let act the linear operator $L$ onto the solution (30), we obtain the following relationship:

$$L[u(x,t)] = \left( \sum_{r=1}^{N} \sum_{s=1}^{M} \mu_r \omega(k_{rs}) + \sum_{n=1}^{K} A^{(n)} \left[ i \sum_{r=1}^{N} \sum_{s=1}^{M} \mu_r k_r z_{rs} \right]^n \right) u(x,t).$$

(33)

From this result, we can identify a matrix $L^{(\mu_1,\ldots,\mu_N)}(k_{ij})$, which is given by

$$L^{(\mu_1,\ldots,\mu_N)}(k_{ij}) = \sum_{r=1}^{N} \sum_{s=1}^{M} \mu_r \omega(k_{rs}) + \sum_{n=1}^{K} A^{(n)} \left[ i \sum_{r=1}^{N} \sum_{s=1}^{M} \mu_r k_r z_{rs} \right]^n.$$

(34)

This result implies that the coefficients $C(\mu_1,\ldots,\mu_N)$ that determine the solution (30) can be determined by solving

$$L^{(\mu_1,\ldots,\mu_N)}(k_{ij}) C(\mu_1,\ldots,\mu_N) = R(\mu_1,\ldots,\mu_N).$$

(35)

The coefficients $C(1,0,0,\ldots,0), C(0,1,0,\ldots,0), \ldots, C(0,\ldots,0,1)$ are determined by the initial condition.

In principle, we expand the solution $u(x,t)$ in an arbitrary set of basis functions. Suppose as an example a function $\tilde{u}(x,t)$ that can be expanded in the set of basis functions $f^{(n)}(x,t|k,\omega)$:

$$\tilde{u}(x,t) = \sum_{n=1}^{\infty} \alpha_n f^{(n)}(x,t|k,\omega).$$

(36)

We define the set $S$ as the basis function:

$$S = \{ f^{(1)}(x,t|k,\omega), f^{(2)}(x,t|k,\omega), f^{(3)}(x,t|k,\omega), \ldots \},$$

(37)
which have the following properties:

I: if \( f^n(x,t|k,\omega) \in \mathcal{S} \Rightarrow \frac{\partial}{\partial t} f^n(x,t|k,\omega) = \hat{\alpha}_n(k,\omega) f^m(x,t|k,\omega) \) \((f^m(x,t) \in \mathcal{S})\),

II: if \( f^n(x,t|k,\omega) \in \mathcal{S} \Rightarrow \frac{\partial}{\partial x} f^n(x,t|k,\omega) = \hat{\beta}_n(k,\omega) f^m(x,t|k,\omega) \) \((f^m(x,t) \in \mathcal{S})\),

III: if \( f^n(x,t) \in \mathcal{S} \) and \( f^m(x,t) \in \mathcal{S} \Rightarrow f^n(x,t) \cdot f^m(x,t) \in \mathcal{S} \). \((38)\)

The properties I and II guarantee that \( \mathcal{L}[\hat{u}(x,t)] \) can be expanded in basis functions \( f^n(x,t|k,\omega) \):

\[
\mathcal{L}[\hat{u}(x,t)] = \sum_{n=1}^{\infty} \hat{L}^{(n)} \alpha_n f^n(x,t|k,\omega),
\]

where the precise structure of the operator \( \hat{L}^{(n)} \) is determined by the linear differential operator \( \mathcal{L} \).

Property III in Eq. (38) guarantees nonlinearities of the type \( \hat{\beta} \) can be expanded in the same basis functions \( f^n(x,t|k,\omega) \). If the nonlinearity represented by the operator \( \hat{\beta} \) can also be expanded in the same basis functions \( f^n(x,t|k,\omega) \):

\[
Q[\hat{u}(x,t)] = \sum_{n=1}^{\infty} \hat{R}_n f^n(x,t),
\]

then, we can compute the expansion coefficients \( \alpha_n \) by solving the relationship

\[
\alpha_n = [\hat{L}^{(n)}]^{-1} \hat{R}_n,
\]

where \( \alpha_n \) is determined by the initial condition. Of course, we can generalize this result further by replacing Eq. (30) by

\[
u(x,t) = \sum_{\mu_1 \cdots \mu_N=1}^{\infty} \hat{C}(\mu_1 \cdots \mu_N) \prod_{i=1}^{N} \prod_{j=1}^{M} f^{(i)}(x,t|\hat{k}_{ij},\hat{\omega}_{ij}).\]

(42)

The structure of the solutions proposed in Eq. (42) is, in fact, a generalization of Eq. (30). If we replace \( f^{(i)}(x,t|\hat{k}_{ij},\hat{\omega}_{ij}) \) by \( \exp[i\mu_k \hat{x}_{ij}] \), the form (30) is retained. Following a similar approach as in the case of Fourier basis functions, we find that if the conditions (38) hold for the solution (42), the linear part of the differential equation acts on the solution (42) like

\[
\mathcal{L}[\nu(x,t)] = \left( i \sum_{i=1}^{N} \sum_{j=1}^{M} \hat{\omega}_{ij} + \sum_{n=1}^{N} \hat{A}^{(n)} \sum_{i=1}^{N} \sum_{j=1}^{M} \hat{k}_{ij} \right) \mu_n \nu(x,t),
\]

(43)

where it is assumed that \( \partial_t f^{(i)}(x,t|\hat{k}_{ij},\hat{\omega}_{ij}) = \hat{\omega}_{ij} f^{(i)}(x,t|\hat{k}_{ij},\hat{\omega}_{ij}) \) and \( \partial_x f^{(i)}(x,t) = \hat{k}_{ij} f^{(i)}(x,t|\hat{k}_{ij},\hat{\omega}_{ij}) \). This relationship enables us to identify an operator \( \hat{\mathcal{L}}^{(ij)} \times (\hat{\omega}_{ij},\hat{k}_{ij}) \) according to

\[
\hat{\mathcal{L}}^{(ij)}(\hat{\omega}_{ij},\hat{k}_{ij}) = \left( i \sum_{i=1}^{N} \sum_{j=1}^{M} \hat{\omega}_{ij} + \sum_{n=1}^{N} \hat{A}^{(n)} \sum_{i=1}^{N} \sum_{j=1}^{M} \hat{k}_{ij} \right)^n.
\]

(44)

If we, moreover, assume that the operator \( Q \) is of the class \( \hat{\beta} \) so that

\[
Q[\nu(x,t)] = \sum_{\mu_1 \cdots \mu_N=1}^{\infty} \hat{R}(\mu_1 \cdots \mu_N) \prod_{i=1}^{N} \prod_{j=1}^{M} f^{(i)}(x,t|\hat{k}_{ij},\hat{\omega}_{ij}),
\]

(45)
then the expansion coefficients are determined by the following linear iteration series:

\[
\mathbf{L}^{(ij)}(\hat{\omega}_{ij}, \hat{k}_{ij}) \hat{C}(\mu_1 \cdots \mu_N) = \hat{R}(\mu_1 \cdots \mu_N). \tag{46}
\]

From this result, we can conclude that we can transform Eq. (28) into Eq. (46). We can conclude that a NPDE of the form of Eq. (28) is integrable if the following two conditions are satisfied.

(i) The nonlinearity must have such a structure that it can be expanded in the same basis functions as the linear part. In other words, the nonlinearity must guarantee that Eq. (45) is satisfied.

(ii) The inverse matrix \( \mathbf{L}^{(ij)}(\hat{\omega}_{ij}, \hat{k}_{ij}) \) must exist.

From this result we can conclude that provided a solution (30) exists, the integrability of the NPDE is completely determined by the linear part of the evolution equation. These are also the conditions that guarantee the integrability of Eq. (28). In the following section, we apply these concepts to examine the integrability of some NPDEs.

IV. EXAMPLES

In this section, we will apply the machinery developed in the previous sections to investigate the integrability of various NPDEs. As a first example, we consider the nonlinear Schrödinger equation:

\[
i \partial_t u = \partial_x u + 2uu^* u. \tag{47}
\]

If we substitute

\[
u(x,t) = e^{iax} e^{i(x^2 - b^2)/4t} e^{i\phi} \sum_{n=1}^{\infty} A_n e^{-n(bx - 2abt)}, \tag{48}
\]

into Eq. (47), we obtain

\[
\sum_{n=1}^{\infty} [(1-n^2)b^2] A_n e^{-n(bx - 2abt)} = 2 \sum_{l=1}^{n-1} \sum_{m=1}^{n-l-1} A_l A_m A_{n-m} e^{-n(bx - 2abt)}. \tag{49}
\]

It can be verified that for \( n = 1 \) the linear dispersion relationship \( \omega = -k^2 \) \( (k = a + bi) \) is satisfied. Since both the left-hand side and the right-hand side can be expanded in the same Fourier basis functions, we can determine the expansion coefficients by the following recursion relationship:

\[
L^{(n)}(k) A_n = R^{(n)}; \quad k = a + bi, \tag{50}
\]

where

\[
L^{(n)} = [1-n^2]b^2; \quad R^{(n)} = 2 \sum_{l=1}^{n-2} \sum_{m=1}^{n-l-1} A_l A_m A_{n-m-l}. \tag{51}
\]

If we assume that \( A_1 = A \), then by computing all the coefficients \( A_n \), and carrying out the summation, similarly as in Eq. (11), we obtain the NLS soliton:

\[
u(x,t) = Ae^{iax} e^{i(x^2 - b^2)/4t} e^{i\phi} e^{\xi_0} \text{sech}(bx - 2abt + \xi_0), \quad \xi_0 = -\frac{1}{2} \log \left( \frac{A^2}{4b^2} \right). \tag{52}
\]

Similarly as for the KdV equation, the two-soliton solution of the nonlinear Schrödinger equation can be computed by considering solutions:
By generalizing this procedure, as presented in Sec. III, the \(N\)-soliton solution of the nonlinear Schrödinger equation can be computed.

As a second example we consider the coupled nonlinear Schrödinger equation:

\[
\begin{align*}
iu_{1t} &= u_{1xx} + (|u_1|^2 + |u_2|^2)u_1 = 0, \\
iu_{2t} &= u_{2xx} + (|u_2|^2 + |u_1|^2)u_2 = 0.
\end{align*}
\] (54)

If we make the following substitution for the solution \(u(x,t) = [u_1(x,t), u_2(x,t)]^T\):

\[
u(x,t) = e^{iax}e^{i(a^2-b^2)t} \sum_{n=1}^{\infty} A_ne^{-n(bx-2at)}, \quad \mathbf{A}^{(n)} = (A_1^{(n)}, A_2^{(n)})^T,
\] (55)

into Eq. (54), it can be verified that both the left-hand side and the right-hand side of Eq. (54) can be expanded in the same basis functions. This is due to the fact that both \(u_1(x,t)\) and \(u_2(x,t)\) have the same dispersion relation \(\omega(k) = -k^2\). As a result, we can determine the expansion coefficients \(A^{(n)}\) by solving the following recursion relation:

\[
\mathbf{L}^{(n)}(k)\mathbf{A}^{(n)} = \mathbf{R}^{(n)}, \quad k = a + bi,
\] (56)

where

\[
\mathbf{L}^{(n)} = i[1-n^2]b^2; \quad \mathbf{R}^{(n)} = \sum_{l=1}^{n-2} \sum_{m=1}^{n-l-1} \left( A_1^{(l)} A_2^{(m)} A_1^{(n-m-l)} + A_2^{(l)} A_1^{(m)} A_2^{(n-m-l)} \right).
\] (57)

As a last example, we consider the three-dimensional nonlinear Schrödinger equation:

\[
i\partial_t u = \sum_{n=1}^{3} \partial_{x_n}^2 u + 2uu^* u.
\] (58)

If we substitute

\[
u(x,t) = e^{ia \cdot x} e^{i(a \cdot a - b \cdot b)t} e^{i\phi} \sum_{n=1}^{\infty} A_n e^{-n(b \cdot x - 2a \cdot a)},
\] (59)

into Eq. (47), we obtain

\[
\sum_{n=1}^{\infty} [(1-n^2)b \cdot b] A_n e^{-n(b \cdot x - 2a \cdot a)} = 2 \sum_{n=1}^{\infty} \sum_{l=1}^{n-2} \sum_{m=1}^{n-l-1} A_l A_m A_{n-m-l} e^{-n(b \cdot x - 2a \cdot a)}.
\] (60)

In Eq. (59) and Eq. (60), it is used that \(x = (x_1, x_2, x_3)^T\), \(a = (a_1, a_2, a_3)^T\), and \(b = (b_1, b_2, b_3)^T\). It can be verified that for \(n = 1\) the linear dispersion relationship \(\omega^2 = -k \cdot k (k = a + bi)\) is satisfied. Since both the left-hand side and the right-hand side can be expanded in the same Fourier basis functions, we can determine the expansion coefficients by the following recursion relationship:

\[
\mathbf{L}^{(n)}(k)\mathbf{A}_n = \mathbf{R}^{(n)}, \quad k = a + bi,
\] (61)

where
Similarly as in the one-dimensional case, explicit solutions of the three-dimensional nonlinear Schrödinger equation can be obtained by carrying out the summation of the expansion coefficients. The discussion can be made more general by using other expansion functions, similarly as in Eq. (53).

V. DISCUSSION AND CONCLUSIONS

We have presented a method to investigate the integrability for NPDEs having a polynomial type of nonlinearity. It has to be remarked that we have assumed throughout this paper that integrability is equivalent with the existence of \( N \) solitons. It is shown that two conditions play an important role. The first condition is that the nonlinearity can be expanded in the same basis functions as the linear part. The second condition is that the linearized part of the NPDE has nontrivial solutions. The method is presented by investigating the integrability of the KdV equation as an example. In Sec. III the method is first generalized for NPDEs having solutions that can be expanded in an infinite set of Fourier basis functions. Later on, it is shown that we do not have to restrict ourselves to Fourier basis functions only. Moreover, it is likely that the method also works for nonpolynomial types of nonlinearity, at least if the nonlinearity can be expanded in polynomial form. The paper is concluded by applying the method on the nonlinear Schrödinger equation, the coupled nonlinear Schrödinger equation, and a three-dimensional example. It is shown that we can derive special solutions of the three-dimensional nonlinear Schrödinger equation.

There is an interesting link between the work carried out in this paper and Hirota’s method \(^5\) in which it is shown for the KdV equation that by applying the transformation

\[
 u = 2(\log F)_{xx},
\]

the solution \( F \) can be written as

\[
 F(x,t) = \det[M],
\]

where the \( N \times N \) matrix \( M \) has the entries

\[
 M_{ij}(x,t) = \delta_{ij} + \frac{2(P_i P_j)^{1/2}}{P_i + P_j} e^{(1/2)(\xi_i + \xi_j)}; \quad \xi_i = P_i x - P_i^3 t - \xi_i^0,
\]

and \( P_i \) and \( \xi_i^0 \) are arbitrary constants. The result presented above was obtained by assuming that the solution \( F(x,t) \) can be expanded in a similar series that formed the starting point in this paper:

\[
 F(x,t) = 1 + F^{(1)}(x,t) + F^{(2)}(x,t) + \cdots.
\]

The major difference between the method presented in this paper and Hirota’s method is that the latter succeeded to formulate solutions of the KdV equation by using a finite number of functions \( F^{(N)}(x,t) \), whereas our method needs an infinite number of basis functions. It is also interesting to mention that the solutions obtained by Hirota have a similar structure as inverse scattering solutions for rational reflection coefficients as obtained by Sabatier.\(^10\) Moreover, in Ref. 11 it is shown for the KdV equation that Fourier expansion of the inverse scattering solutions as derived by Sabatier is equal to the series (11). The solutions derived in this paper can therefore be regarded as a Fourier expansion of Hirota’s solution.
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