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Mathematical modelling of electrical-optical effects in semiconductor laser operation

by

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Mathematical modelling of electrical-optical effects in semiconductor laser operation

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Abstract. A mathematical model describing the coupling of electrical and optical effects in semiconductor lasers is introduced. Through a multiple-scale asymptotic expansion, the governing equations in the active region are reduced to one parabolic and four first-order hyperbolic partial differential equations. By making further assumptions, partially and fully lumped models for the active region are deduced, which complement (and provide a more systematic derivation of) previous well-established lumped models.

Keywords. semiconductor lasers, mathematical modelling, asymptotics

AMS(MOS) subject classifications. 35B25, 78A60

1 Introduction

Numerous mathematical models have been formulated to simulate electrical and optical effects in semiconductor lasers. The fully space-dependent formulation is generally accepted to comprise an electrical model, consisting of electron and hole continuity equations and Poisson's equation, and an optical model, made up of a wave equation and a photon rate equation [1], [2]. The electrical model, employed in almost all relevant previous papers, relies on Poisson's equation for the electric potential being valid across the whole semiconductor. However, this is inappropriate for the active region in which electromagnetic waves propagate, so the curl of the electric field is non-negligible. These electromagnetic waves are normally represented by the separate optical model. The unsuitably of Poisson's equation has been previously noted in the related context of the mathematical modelling of a semiconductor driven by a current of very high frequency [3]. Furthermore, the high frequency currents generated by the interaction between the large electric field and the charge carriers in the active region are normally neglected in existing models and we address this issue also.

Fully space-dependent models are currently exclusively solved by direct numerical approaches. However, some partially lumped models for the active region have also been reported which comprise one ordinary differential equation and two first-order wave equations [4], [5]. A fully lumped model in terms of two ordinary differential equations (for electron and photon concentrations) is very well-established (see, for example, [6], [7]). In this paper, we attempt to address deficiencies in the modelling of the fully space-dependent case. Using the resulting model as the starting point we then derive, in a systematic fashion, three simplifications which complement these existing lumped models.

A schematic cross-section of a typical semiconductor laser is shown in Figure 1. The passage of current between the metal contact on the substrate and the heat sink causes electrons to be injected into the active layer, where they recombine with holes through both radiative and nonradiative mechanisms. During radiative
recombination the energy released by an electron/hole pair appears in the form of a photon. This can happen through spontaneous emission, in which photons are emitted in random directions, or stimulated emission, in which recombination is initiated by an existing photon. The crucial feature of stimulated emission is that the emitted photon matches the original photon in wavelength, phase and direction. If the end-faces of the semiconductor possess a suitable reflectivity and the current is raised above a certain value, known as the threshold current, then the semiconductor is excited through stimulated emission into laser operation.

The purpose of this paper is to establish a framework for the simulation of coupled electrical and optical effects. We take Maxwell's equations as our starting point and add models for the current and polarization. We neglect temperature variations, assuming that the whole device is maintained at the (constant) ambient temperature, and assume that the gain takes place at a single frequency associated with the band gap of the active region. Strictly, longitudinal modes must satisfy the standing wave condition \( \lambda_m = 2L/m \), where \( m \) is a positive integer, \( \lambda_m \) is the wavelength of the \( m \)th mode and \( L \) is the axial length of the cavity, but in general the photon energy of a number of these modes will be close to the band-gap energy and it is with one such mode that our modelling is concerned. Although several longitudinal modes will experience similar gain, it is reasonable to assume single-mode operation, because this corresponds to the only stable steady state \([8]\). The step in the refractive index and the nearly glancing incidence of the rays ensure that this lasing mode almost completely reflects at the interfaces; it is then confined within the active region, in which we take the band gap to be constant (we note, however, that there are certain devices, including for example cases where the active region is composed of a few quantum wells, in which only a tenth of the lasing mode is confined to the active region—the analysis in this paper will not be valid in such circumstances). We shall thus split the semiconductor into the active region, modelled as lasing, and the surround, modelled as non-lasing, with the region of rapid band-gap change being taken to be part of the surround. The model omits the contribution of spontaneous emission; this effect requires quantization of the laser field, which appears to be possible in our classical framework only by means of an artificial noise current (see \([9, \text{p. 235}]\) and Appendix A). However, our main concern is with the steady state behaviour, which is insensitive to the presence of spontaneous emission.

Based on the above assumptions, a new mathematical model is introduced in Section 2 and non-dimensionalised in Section 3, enabling the dominant balances to be identified. Section 4 describes a multiple-scale asymptotic analysis, whereby the governing equations in the active region are reduced to one parabolic partial differential equation and four first-order wave equations. The two key small parameters are given by the ratio of the wavelength to the longitudinal cavity length and the ratio of the time-scale for an electromagnetic wave to traverse the cavity in the longitudinal direction to the time-scale for variations in carrier concentrations. In Section 5, further assumptions concerning the uniformity of the electron concentration are shown to result in various lumped models for the active region, Appendix B providing an additional example. Finally, Section 6 gives a brief discussion of the results.

2 Mathematical model

2.1 Introduction

This section concerns the formulation of the first of the four mathematical models derived in this paper. The modelling treats lasing and non-lasing regions separately. Both parts of the model are derived from Maxwell's equations (see, for example, \([10]\)), given by

\[ \nabla \cdot \mathbf{D} = \rho, \]  

(1)
where $E$ is the electric field, $D$ is the electric displacement, $H$ is the magnetic field, $J$ is the current density, $\rho$ is the charge density, $t$ is time and the differential operator $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$, where $x, y, z$ are indicated in Figure 1. We take the semiconductor to be a nonmagnetic medium, such that the constitutive equations are given by $D = \varepsilon_0 E + P$ and $B = \mu_0 H$, where $\varepsilon_0$ is the permittivity of free space, $P$ is the polarization and $\mu_0$ is the permeability of free space. The continuity equation, which also applies in both regions, can be derived by taking the divergence of (4) to give

$$\nabla \cdot J = 0.$$  

We now split up the charge density and current density as follows

$$\rho = e(N + p - n), \quad J = J_p + J_n,$$  

where $e$ is the charge on an electron, $n$ and $p$ are the free electron and hole densities, $N$ (which is taken to be time invariant) is the net active impurity concentration, and $J_n$ and $J_p$ are the components of the current density carried by electrons and holes, respectively. Substituting (6) into (5) we obtain

$$e \frac{\partial(p - n)}{\partial t} + \nabla \cdot (J_p + J_n) = 0.$$  

In order to obtain separate continuity equations for electrons and holes we, in the usual way, define $G$ to be the electron-hole generation rate and $R$ to be the electron-hole recombination rate. We can then rewrite (7) as (see [11, p. 11])

$$e \frac{\partial n}{\partial t} = e(G - R) + \nabla \cdot J_n,$$

$$e \frac{\partial p}{\partial t} = e(G - R) - \nabla \cdot J_p,$$  

which will be employed in both the lasing and non-lasing regions.

The electron and hole concentrations and the normal components of the electron and hole current densities are assumed to be continuous at the interface between lasing and non-lasing regions, that is

$$\begin{align*}
[n]_z=0^+ &= [n]_z=0^-, \\
[p]_z=0^+ &= [p]_z=0^-, \\
[n]_{y=0^+} &= [n]_{y=0^-}, \\
[p]_{y=0^+} &= [p]_{y=0^-}, \\
[n]_{y=d^+} &= [n]_{y=d^-}, \\
[p]_{y=d^+} &= [p]_{y=d^-} = 0,
\end{align*}$$

$$\begin{align*}
[J_n \cdot \hat{n}]_{z=0^+} &= [J_n \cdot \hat{n}]_{z=0^-}, \\
[J_p \cdot \hat{n}]_{z=0^+} &= [J_p \cdot \hat{n}]_{z=0^-}, \\
[J_n \cdot \hat{n}]_{y=0^+} &= [J_n \cdot \hat{n}]_{y=0^-} = 0, \\
[J_p \cdot \hat{n}]_{y=0^+} &= [J_p \cdot \hat{n}]_{y=0^-} = 0,
\end{align*}$$

where $\hat{n}$ is the unit normal at the interface. In addition, we have

$$\begin{align*}
[n \times E]_{z=0^+} &= [n \times E]_{z=0^-}, \\
[n \times E]_{y=0^+} &= [n \times E]_{y=0^-} = 0, \\
[B \cdot \hat{n}]_{z=0^+} &= [B \cdot \hat{n}]_{z=0^-} = 0, \\
[B \cdot \hat{n}]_{y=0^+} &= [B \cdot \hat{n}]_{y=0^-} = 0.
\end{align*}$$

We now need separate discussions of the lasing and non-lasing regions. The lasing part of the model applies to the active layer, in which the lasing mode is assumed to be confined. The non-lasing part of the model, which we give first, applies in the remainder of the semiconductor.
2.2 Non-lasing region

In this subsection we are concerned with the parts of the semiconductor which do not contain the lasing mode. This non-lasing part of the model will not feature in the analysis which follows, but it is necessary to include it to complete the first and second models of this paper; moreover, its description will aid the discussion of the (more complicated) lasing region.

In the non-lasing region, electromagnetic waves are only present to a significant degree in an initial transient during which they propagate out of, or are absorbed in, the semiconductor. The ratio of the time-scale \((Y_2 - Y_2)/c\) for an electromagnetic wave to traverse the laser in the transverse direction to the time-scale \(A_n\) for variations in carrier concentrations is in practice typically of the order

\[
\left( \frac{Y_2 - Y_2}{c} \right) / A_n \approx 10^{-5},
\]

where \(c = 1/\sqrt{\varepsilon \mu_0}\) is the velocity of light, \(\varepsilon\) is the (material dependent) permittivity of the semiconductor, \(Y_2 - Y_2\) is the transverse length-scale (see Figure 1) and \(A_n\) is the reciprocal rate constant associated with recombination traps (discussed below). Therefore, the electric field effectively reacts instantaneously on the time-scale associated with variations in carrier concentrations, justifying the assumption that the time derivative in Faraday’s law can be neglected (that is, we take \(\nabla \times E = 0\)). We assume the polarization to be of the form \(\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}\). The model in the non-lasing region can then be described by three dependent variables: the 'electrostatic' potential \(u(z,t)\), the free electron density \(n(z,t)\) and the free hole density \(p(z,t)\). These satisfy the following equations (see, for example, [12], [11, p. 42] and references therein), which follow from (1)-(8) together with appropriate constitutive assumptions for \(J_n\) and \(J_p\),

\[
\nabla \cdot (\varepsilon \nabla u) = e(n - p - N),
\]

\[
e n = e(G - R) + \nabla \cdot J_n, \quad \text{with} \quad J_n = e(D_n \nabla n - \mu_n n \nabla u + \frac{\alpha n \mu_n}{e} \nabla E_\text{g}),
\]

\[
e p = e(G - R) - \nabla \cdot J_p, \quad \text{with} \quad J_p = e(-D_p \nabla p - \mu_p p \nabla u - \frac{(1 - \alpha) \mu_p}{e} \nabla E_\text{g}),
\]

where \(E = -\nabla u, D_n, D_p\) are the electron and hole diffusivities, \(\mu_n, \mu_p\) are the (material dependent) electron and hole mobilities and the band gap \(E_\text{g}(x)\) is the difference in energy between the bottom of the conduction band and the top of the valence band. In the constitutive assumptions for \(J_n\) and \(J_p\), the first two terms are standard and the third generalises the formulation of [11], substituting equation (2.4-66) of which into (2.3-62)–(2.3-63) gives (15)–(16) with \(\alpha = 1/2\). However, the change in the band gap is in practice known to be distributed unevenly between the electron and hole currents, experimental observations indicating that \(\alpha \approx 2/3\). The rapid band-gap change, which takes place in the non-lasing region adjacent to the lasing region, helps to confine electrons and holes within the lasing region. For the purposes of this study we will adopt constant mobilities for electrons and holes within each layer. The Einstein relations between the diffusivities and mobilities are

\[
D_n = U_T \mu_n, \quad D_p = U_T \mu_p, \quad \text{where the thermal voltage } U_T = \frac{kT}{e},
\]

\(k\) being Boltzmann’s constant and \(T\) the temperature of the device, assumed constant here.

The net recombination rate is given by

\[
R - G = R_{SRH} + R_{spop} + R_{Aug},
\]

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in which (see, for example, [11, p. 103])

\[ R_{SRH} = \frac{(np - n_i^2)}{A_p(n + n_i) + A_n(p + n_i)}, \quad R_{sp} = B(np - n_i^2), \quad R_{Aug} = (C_n n + C_p p)(np - n_i^2), \]

where \( R_{SRH} \) (Shockley-Read-Hall recombination) is due to recombination traps (with reciprocal rate constants \( A_p \) and \( A_n \)), \( R_{sp} \) is due to radiative recombination (with rate constant \( B \)) and \( R_{Aug} \) represents Auger recombination (with rate constants \( C_n \) and \( C_p \)). The intrinsic carrier density (see, for example, [11, p. 29]) is given by

\[ n_i = \sqrt{N_c N_v e^{-E_g/2kT}}, \]

where \( N_c \) and \( N_v \) are the effective densities of states in the conduction and valence bands, given by

\[ N_c = 2 \left( \frac{2\pi kT m_n^*}{\hbar^2} \right)^{3/2}, \quad N_v = 2 \left( \frac{2\pi kT m_p^*}{\hbar^2} \right)^{3/2}, \]

\( m_n^* \) and \( m_p^* \) being the (material dependent) effective masses of electrons and holes and \( h \) being Planck’s constant. The generation of electron-hole pairs due to the mechanisms of impact ionization and absorption of spontaneous emission are negligible (see [13, p. 44] and [7]).

We assume that the normal component of the electric field and the fluxes of electrons and holes are zero on the insulated boundaries, so that (see Figure 1)

- on \( y = Y_2 \) \[ \frac{\partial n}{\partial y} = \frac{\partial p}{\partial y} = 0 \] for \( 0 \leq x \leq w \) and \( 0 \leq z \leq L \);
- on \( y = Y_5 \) \[ \frac{\partial n}{\partial y} = \frac{\partial p}{\partial y} = 0 \] for \( X_0 \leq x \leq X_2 \) and \( 0 \leq z \leq L \);

the silicon dioxide is thus treated as insulating. At the Ohmic contacts, we make the usual assumptions that electron-hole equilibrium holds \((np = n_i^2)\) and that the space-charge vanishes \((n - p - N = 0)\). Therefore, the boundary conditions for \( n \) and \( p \) are

- on \( y = Y_2 \) \[ n = \frac{1}{2} \left( N + \sqrt{N^2 + 4n_i^2} \right), \quad p = \frac{1}{2} \left( -N + \sqrt{N^2 + 4n_i^2} \right) \] for \( 0 \leq z \leq w \) and \( 0 \leq z \leq L \);
- on \( y = Y_5 \) \[ n = \frac{1}{2} \left( N + \sqrt{N^2 + 4n_i^2} \right), \quad p = \frac{1}{2} \left( -N + \sqrt{N^2 + 4n_i^2} \right) \] for \( X_0 \leq x \leq X_2 \) and \( 0 \leq z \leq L \).

The Ohmic boundary conditions for the electrostatic potential are chosen so that the system is in equilibrium when externally applied potentials are zero implying, [12], that

- on \( y = Y_2 \) \[ u = U_T \log \frac{N + \sqrt{N^2 + 4n_i^2}}{2n_i} \] for \( 0 \leq x \leq w \) and \( 0 \leq z \leq L \);
- on \( y = Y_5 \) \[ u = V + U_T \log \frac{N + \sqrt{N^2 + 4n_i^2}}{2n_i} \] for \( X_0 \leq x \leq X_2 \) and \( 0 \leq z \leq L \),

where \( V \) is the voltage applied across the metal contacts. The battery voltage is almost entirely dropped across junctions, where the charge density is non-negligible. The junctions are taken to lie outside the lasing region, so that we assume the electrostatic potential is continuous across this (slender) region. We note that if
and time is taken on the electrical time-scale \((A_n)\) then at leading order the \(y\)-derivatives in (5) are zero and \(J_y\) is independent of \(y\) in the lasing region. We take current to be continuous across the (slender) lasing region. We have

\[
[u]_{y=0}^{y=d^+} = 0, \quad \left[ D_n \frac{\partial n}{\partial y} - D_p \frac{\partial p}{\partial y} - (\mu_n n + \mu_p p) \frac{\partial u}{\partial y} \right]_{y=0}^{y=d^+} = 0, \quad \text{for } 0 \leq z \leq w \text{ and } 0 \leq z \leq L.
\]

We note that if the region of rapid band-gap change had been taken as part of the active region both the potential and its normal derivative would experience jumps [14]. The coupling to the active region is through continuity of the carrier density and current.

### 2.3 Lasing region

In this subsection we are concerned with the part of the semiconductor which contains the lasing mode, namely the active layer. The lasing mode can be viewed as a standing electromagnetic wave, so (despite common practice) the electrostatic approximation is inapplicable in the lasing region.

The two additional physical time-scales in the lasing region are the reciprocal of the frequency of the lasing mode and the time taken for this mode to traverse the laser in the longitudinal direction; the former is given by \(\lambda/c\) and the latter by \(L/c\), where \(\lambda\) is the wavelength of the lasing radiation. The latter is precisely the optical time-scale identified in the isothermal lumped model of [7], the only other relevant time-scale there being an electrical one corresponding here to \(A_n\), the time-scale for carrier variations. In the model discussed here we have thus identified a total of four physical time-scales: the time-scale for carrier variations \((\sim 10^{-8}\text{s})\), that for an electromagnetic wave to traverse the laser in the transverse direction \((\sim 10^{-13}\text{s})\), that for traverse in the longitudinal direction \((\sim 10^{-12}\text{s})\) and the reciprocal of the frequency \((\sim 10^{-15}\text{s})\).

We write Maxwell’s equations in the form

\[
\nabla \cdot D = -\varepsilon(n - p - N^A),
\]

\[
\nabla \times E = -\frac{\partial B}{\partial t},
\]

\[
\nabla \cdot B = 0,
\]

\[
\nabla \times B = \mu_0\varepsilon_0 \frac{\partial E}{\partial t} + \mu_0 \frac{\partial P}{\partial t} + \mu_0 J,
\]

where \(N^A\) is the net impurity density in the active region, which we take to be a constant. We write the polarization as a sum

\[
P = P^{(N)} + P^{(R)}
\]

where \(P^{(N)}\) and \(P^{(R)}\) each represent different material properties. The component \(P^{(N)}\) represents the contribution to the polarization associated with the spatial shift of the electron cloud of a semiconductor atom with respect to its nucleus caused by an imposed electric field (this being assumed to take place instantaneously); we take the standard expression

\[
P^{(N)} = (\varepsilon - \varepsilon_0) E
\]

to be valid for all frequencies and we assume \(\varepsilon\) to be independent of \(n\) and \(p\). The time-periodic component \(P^{(R)}\) represents resonant coupling, which occurs when the frequency of the electric field closely matches a natural frequency of the semiconductor; this coupling involves changes in the number of electron-hole pairs and hence in the number of dipoles contributing to the polarization. Resonance occurs in isolated intervals of
the electromagnetic spectrum, for example at visible frequencies at which gallium arsenide has an electronic
transition. What we are aiming for here is a tractable constitutive law which is guided by the physically
established facts that ([9, p. 242], [8]) the damping is relatively large in practice (that is \( \sigma/\omega = O(1) \), where
\( \omega \) is the relevant electronic resonant angular frequency of the semiconductor (given by \( \omega = 2\pi E_g/h \))
and the positive constant \( \sigma \) is the damping factor) and that the relevant mode of operation is single-mode with
angular frequency \( \Omega = \omega \) (with neighbouring modes having lower gain). We adopt a simple-minded approach
which reproduces these features and is adequate for our purposes, the result (see (30)) being consistent with
the corresponding constitutive assumptions adopted elsewhere (see [9]). We shall follow common practice in
modelling this resonant interaction as a forced, damped harmonic oscillator and write

\[
\frac{\partial^2 P^{(R)}}{\partial t^2} + \frac{\sigma}{\partial t} P^{(R)} + \omega^2 \frac{\partial P^{(R)}}{\partial t} = \sigma \frac{\partial}{\partial t} (\Gamma(n,p)E),
\]

(24)

where \( \Gamma(n,p) \) is the net gain (described below). The 'extra' time derivative in (24) eliminates any time-
derpendent electric fields, the latter playing no role in this context. We assume the laser to be operating in
a single-mode state, with angular frequency \( \omega \), with the electric field taking the form

\[
E = a \cos(\Omega t) + b \sin(\Omega t),
\]

(25)

where \( a \) and \( b \) depend only on \( \pi \) and on time-scales much longer than \( 1/\omega \), as do \( n \) and \( p \). In view of (24)
the polarization is thus taken to be of the form

\[
P^{(R)} = (c \sin(\Omega t) - d \cos(\Omega t))/\Omega,
\]

(26)

where \( c \) and \( d \) again depend only on \( \pi \) and on time-scales much longer than \( 1/\omega \). Neglecting derivatives on
these longer time-scales, we thus obtain

\[
(\omega^2 - \Omega^2)c + \sigma \Omega d = \sigma \Gamma \Omega b,
\]

(27)

\[
(\omega^2 - \Omega^2)d - \sigma \Omega c = -\sigma \Gamma \Omega a.
\]

(28)

In multi-mode models (see, for example, [9, p. 242]) several frequencies are taken to have similar gain,
corresponding to the time-scale \( 1/\sigma \) being not significantly larger than \( 1/\omega \), an assumption we also adopt
here. Semiconductor lasers usually operate at a single frequency (the one corresponding to maximum gain),
because this corresponds to the only stable steady state [8]. As earlier, we take this mode to satisfy \( \Omega = \omega \),
in which case (27)–(28) amounts to

\[
\frac{\partial^2 P^{(R)}}{\partial t^2} = \frac{\partial}{\partial t} (\Gamma(n,p)E).
\]

(29)

Our subsequent analysis will adopt the single-mode assumptions (25) and (26) with \( \Omega = \omega \), implying equation
(29) which can be regarded as an approximation to more sophisticated theories of polarization (see, for
example, [15]). We note that \( P^{(R)} \) is negligible in the non-lasing region, because the dominant part of
\( \Gamma(n,p)E \) varies only on the longer time-scales there.

We now outline a physical description of the function \( \Gamma(n,p) \), which corresponds to the changes in the
number of electron-hole pairs which contribute to the polarization. There are two contributions to the rate
of change of the number of such pairs, one associated with absorption and the other with stimulated gain. In
the former process, a photon produces an electron-hole pair with dipole moment parallel to and in the same
direction as the electric field. The polarization is increased and the electric field decreased as a result of the
loss of a photon (as illustrated in Figure 2a). In the latter process, an electron-hole pair with dipole moment

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We might expect the gain to comprise terms of the form \(np\) (see \[2\]) to represent electron-hole recombination, but the above expression based on the minimum carrier density is the one most frequently adopted (see \[1\]).

The form of the polarization (30) is similar to that in \[9\]. The first term on the right-hand side of (22) essentially accounts for the speed of propagation of electromagnetic radiation in the active region, while the second term expresses the fact that the variations in the electric polarization cannot keep up with changes in the electromagnetic field. The polarization can thus be described as having a memory of the electric field at previous times, as discussed in \[16, p. 265\].

Equations (18)-(21) must be solved in conjunction with

\[
e_e \frac{\partial n}{\partial t} = e(G - R) + \nabla \cdot J_n, \quad \text{with} \quad J_n = e(D_n^e \nabla n + \mu_n^e nE),
\]

\[
e_p \frac{\partial p}{\partial t} = e(G - R) - \nabla \cdot J_p, \quad \text{with} \quad J_p = e(-D_p^e \nabla p + \mu_p^e pE).
\]

We note that the electric field in the active region is of very high frequency and therefore the usual concept of mobility may not strictly be valid; nevertheless, we assume it is legitimate to introduce effective mobilities, namely \(\mu_n^e\) for electrons and \(\mu_p^e\) for holes. These effective mobilities are expected to be smaller than the usual values; otherwise such large currents would be present that lasing would not occur. The Einstein relations are assumed to remain valid, so that \(D_n^e = UT\mu_n^e\) and \(D_p^e = UT\mu_p^e\).

The total net recombination rate in the active region is given by

\[
R - G = R_{SRH} + R_{spon} + R_{Aug} + R_{stim},
\]

in which the net recombination rate due to the emission of stimulated radiation (see \[1\]) is given by

\[
R_{stim} = \frac{1}{E_g^A} E \cdot \frac{\partial P^{(R)}}{\partial t},
\]

where \(E_g^A\) is the (constant) band-gap energy in the active region. This change in number of electrons and holes given by \(R_{stim}\) is what provides the contribution \(P^{(R)}\) to the polarization, the two expressions being consistent in terms of conservation of energy.

Equations (18)-(21), (31)-(32) constitute the lasing part of the model, to be solved in the active layer. A simple consequence of the above model concerns the energy balance in the lasing region \((V)\) of semiconductor, expressed by Poynting’s theorem, which here takes the form

\[
\frac{\partial}{\partial t} \int_V \left( \frac{1}{2} (E \cdot E) + \frac{1}{2\mu_0} (B \cdot B) \right) \, dr = - \int_{\partial V} (E \times H) \cdot dS - \int_V E \cdot (J_n + J_p) \, dr - \int_V E \cdot \frac{\partial P^{(R)}}{\partial t} \, dr,
\]

where the left-hand side represents the rate of change of electromagnetic energy, the first integral on the right-hand side the energy loss through the mirrors, the second the Joule heating and the third the optical energy.
gain. The first and third integrals on the right-hand side of (33) are directly analogous to terms appearing in the well-established single-mode ordinary differential rate equation for the photon density (cf. [7]).

We simplify equations (18)–(21) by the introduction of scalar and vector potentials. We satisfy (20) by writing

\[ B = \nabla \times A, \]

in which \( A \) is the vector potential and (19) by writing

\[ E = -\nabla \phi - \frac{\partial A}{\partial t}, \]

in which \( \phi \) is the scalar potential. We specify \( A \) uniquely via the gauge condition (see [10]) so that \( A \) and \( \phi \) satisfy

\[ \nabla \cdot A + \mu_0 \epsilon \frac{\partial \phi}{\partial t} = 0, \]

so that \( A \) and \( \phi \) satisfy

\[ \nabla^2 A - \mu_0 \epsilon \frac{\partial^2 A}{\partial t^2} = -\mu_0 \frac{\partial P(R)}{\partial t} - \mu_0 J, \]

\[ \nabla^2 \phi - \mu_0 \epsilon \frac{\partial^2 \phi}{\partial t^2} = \frac{1}{\epsilon} \left( \nabla \cdot P(R) - \varepsilon(N^A + p - n) \right), \]

where

\[ \frac{\partial^2 P(R)}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\varepsilon}{\varepsilon} \min(n, p) - n_t \right) \left( \nabla \phi + \frac{\partial A}{\partial t} \right). \]

We now note the boundary conditions for the lasing region. In the limit \( \gamma \to 0 \) (where \( \gamma \) is defined in Table 1), (12) implies

\[ \text{on } z = 0^+, w^- \quad \hat{n} \times E = 0 \quad \text{for } 0 \leq y \leq d \text{ and } 0 \leq z \leq L. \]

The normal fluxes of electrons and holes on the mirrors are assumed zero, so that

\[ \text{on } z = 0^+, L^+ \quad \frac{\partial n}{\partial z} = \frac{\partial p}{\partial z} = 0 \quad \text{for } 0 \leq x \leq w \text{ and } 0 \leq y \leq d. \]

The boundary conditions for the electric and magnetic fields at the mirrors involve the transmitted electromagnetic waves which leave the laser, in \( z < 0 \) and in \( z > L \); these satisfy Maxwell’s equations with suitable radiation conditions. If we were to solve Maxwell’s equations on both sides of the mirrors, we should impose the continuity conditions

\[ [D \cdot \hat{n}]_{z=0^+} = [D \cdot \hat{n}]_{z=L^+} = [B \cdot \hat{n}]_{z=0^-} = [B \cdot \hat{n}]_{z=L^-} = 0, \]

\[ [\hat{n} \times E]_{z=0^+} = [\hat{n} \times E]_{z=L^+} = [\hat{n} \times H]_{z=0^-} = [\hat{n} \times H]_{z=L^-} = 0. \]

In the three models derived below, however, it is possible to employ boundary conditions at the mirrors which approximate the reflected wave as a fraction of the incident wave; this avoids the additional complication of modelling the transmitted electromagnetic wave. We note that in [3], the tangential magnetic field is prescribed on the insulating boundaries, which seems unphysical, since this quantity is a priori unknown.

3 Non-dimensionalisation

3.1 Non-lasing region

We define \( u_e = V d/(Y_0 - Y_2) \), which is a representative value of the electrostatic potential, and take \( \mu_e^S \) and \( \mu_p^S \) to be typical values of the electron and hole mobilities in the non-lasing region. We define \( n_e \) to be a
typical value of the electron concentration in the lasing region, which is calculated using the formula (see [7])

\[ n_e = n_t + \frac{c_0}{2aL_{\mu_2}} \log \left( \frac{1}{R_1 R_2} \right), \]

where the various quantities are defined in [7]. Let \( n_t^A \) be the intrinsic carrier density in the lasing region, \( \Delta E \) be the increase in band-gap energy at the edge of the active region and \( w \) be the (lateral) width of the active region (see Figure 1). We introduce a time-scale \( T_e = N C_0 \), where \( C_0 \) is the velocity of light in a vacuum, and transform to dimensionless variables via \( n = n_e \tilde{n}, p = n_e \tilde{p}, u = u_e \tilde{u}, N = n_e \tilde{N}, \mu_n = \mu_{n e} \tilde{\mu}_n, \mu_p = \mu_{p e} \tilde{\mu}_p, D_n = U_2 \tilde{D}_n, D_p = U_2 \tilde{D}_p, n_t = n_{t e} \tilde{n}_t, E_g = E_{g e} + (\Delta E) \tilde{E}_g, x = w \tilde{x}, y = d \tilde{y}, z = \lambda \tilde{z} \) and \( t = \tau \tilde{t} \). The model in the non-lasing region then becomes, introducing \( \tau = \delta \tilde{t} \) since this is the relevant time-scale here,

\[
\begin{align*}
\frac{\partial \tilde{n}}{\partial \tau} &= -\frac{(\hat{\tilde{n}} \hat{\tilde{p}} - N^2 \tilde{n}_t^2)}{A p (\tilde{n} + N \tilde{n}_i) + A_n (\tilde{p} + N \tilde{n}_i)} - B (\tilde{n} \hat{\tilde{p}} - N^2 \tilde{n}_t^2) \quad \text{(36)} \\
&\quad + \hat{\nabla} \cdot \left( \kappa_n \tilde{D}_n \hat{\nabla} \tilde{n} - \tilde{\mu}_n \hat{\nabla} \tilde{u} + \epsilon_n \hat{\nabla} n \hat{\nabla} \tilde{E}_g \right),
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \hat{\tilde{p}}}{\partial \tau} &= -\frac{(\hat{\tilde{n}} \hat{\tilde{p}} - N^2 \tilde{n}_t^2)}{A p (\tilde{n} + N \tilde{n}_i) + A_n (\tilde{p} + N \tilde{n}_i)} - B (\tilde{n} \hat{\tilde{p}} - N^2 \tilde{n}_t^2) \quad \text{(37)} \\
&\quad + \hat{\nabla} \cdot \left( \kappa_p \tilde{D}_p \hat{\nabla} \hat{\tilde{p}} - \tilde{\mu}_p \hat{\nabla} \tilde{u} + \epsilon_p \hat{\nabla} \tilde{p} \hat{\nabla} \tilde{E}_g \right),
\end{align*}
\]

where

\[ \hat{\nabla} = \left( A e \frac{\partial}{\partial \tilde{x}}, A e \frac{\partial}{\partial \tilde{y}}, A e \frac{\partial}{\partial \tilde{z}} \right). \]

The dimensionless constants \( \delta, \nu, A_e, A_p, \delta^{-1} \nu^{-1} A_n, \delta^{-1} \nu^{-1} A_p, \delta \nu B, \delta \nu C_n, \delta \nu C_p, \delta \nu \xi_n, \delta \nu \xi_p, \delta \nu \kappa_n, \delta \nu \kappa_p, N, P, \delta \nu Q_n \) and \( \delta \nu Q_p \) are defined, and typical values given, in Table 1. We note that \( \delta (\ll 1), \nu (\ll 1) \) and \( \delta \nu \) are ratios of time-scales, this being described in Section 3.2.

### 3.2 Lasing region

The model in the lasing region would reduce to that in the non-lasing region were the time-derivative in Faraday’s law neglected, together with stimulated emission and absorption. We define \( A_e \) to be a representative value of the magnitude of the vector potential and calculate its value from the formula (see [7])

\[ A_e^2 = \frac{\epsilon}{2 \mu_e} \frac{J_{ed}}{2 aL_{\mu_2}} \left( \frac{J_{ed} - (A_n + A_p) n_e - B n_t^2 - (C_n + C_p) n_t^2}{c_0} \right) \left[ \frac{1}{R_1 R_2} \right], \]

and introduce \( \phi_e = A_e c_0 \), where the various quantities are defined in [7]; the term in the curly brackets represents an average photon density across the active region. Appropriate definitions of the averaged quantities in [7] in terms of those which appear here will become more apparent in Section 5.2. Making the additional transformation to dimensionless variables \( P^{(R)} = n_e \lambda \delta \nu \hat{\Phi}^{(R)}, \phi = \phi_e \hat{\Phi} \) and \( A = A_e \tilde{A} \), the model in the lasing region becomes

\[
\begin{align*}
\frac{\partial \tilde{n}}{\partial \tilde{t}} &= -\frac{\delta \nu \hat{\tilde{n}} \hat{\tilde{p}} - N^2 \tilde{n}_t^2}{A_p (\tilde{n} + N \tilde{n}_i) + A_n (\tilde{p} + N \tilde{n}_i)} - \delta \nu B (\tilde{n} \hat{\tilde{p}} - N^2 \tilde{n}_t^2) - \delta \nu (C_n \hat{\tilde{n}} + C_p \hat{\tilde{p}}) (\tilde{n} \hat{\tilde{p}} - N^2 \tilde{n}_t^2) \quad \text{(39)} \\
&\quad - \delta \nu F \frac{\partial \hat{\Phi}^{(R)}}{\partial \tilde{t}} \left( \hat{\nabla} \hat{\Phi} + \frac{\partial \tilde{A}}{\partial \tilde{t}} \right) + \delta \nu \hat{\nabla} \cdot \left( D_n \hat{\nabla} \tilde{n} - \tilde{\mu}_n \hat{\nabla} \tilde{u} \right) \left( \hat{\nabla} \hat{\Phi} + \frac{\partial \tilde{A}}{\partial \tilde{t}} \right),
\end{align*}
\]
\[
\begin{align*}
\frac{\partial \phi}{\partial t} &= -\frac{\delta V(n\phi - N^2 \hat{\phi})}{\mathcal{A}_n(n + N \hat{n}) + \mathcal{A}_p(\bar{p} + N \hat{n})} - \delta V B(n\phi - N^2 \hat{\phi}) - \delta V (C_n \hat{n} + C_p \bar{p})(n\phi - N^2 \hat{\phi}) \\
&\quad - \delta V F \frac{\partial \hat{\rho}_{(R)}}{\partial t} \cdot \left( \hat{\nabla} \phi + \frac{\partial \hat{A}}{\partial t} \right) + \delta V \hat{\nabla} \cdot \left( D_n \hat{\nabla} \phi + \bar{p} \mathcal{V}_p \left\{ \hat{\nabla} \phi + \frac{\partial \hat{A}}{\partial t} \right\} \right), \\
\nu \epsilon_R \mathcal{G} \left( \hat{\nabla}^2 \phi - \epsilon_R^2 \frac{\partial^2 \hat{\phi}}{\partial t^2} \right) &= \delta V \hat{\nabla} \cdot \hat{\rho}_{(R)} - (N^* + \bar{p} - \hat{n}), \\
\hat{\nabla}^2 \hat{A} - \epsilon_R^2 \frac{\partial^2 \hat{A}}{\partial t^2} &= -\delta \mathcal{L} \frac{\partial \hat{\rho}_{(R)}}{\partial t} - \delta \mathcal{L} \left( D_n \hat{\nabla} \hat{n} - D_p \hat{\nabla} \bar{p} - \mathcal{V}_n \hat{n} \hat{\nabla} \phi - \mathcal{V}_p \bar{p} \hat{\nabla} \phi - \mathcal{V}_n \hat{n} \frac{\partial \hat{A}}{\partial t} - \mathcal{V}_p \bar{p} \frac{\partial \hat{A}}{\partial t} \right), \\
\frac{\partial^2 \hat{\rho}_{(R)}}{\partial t^2} &= \frac{\partial}{\partial t} \left( \epsilon_R^2 \hat{g}(\hat{n}, \bar{p}) \left( \hat{\nabla} \phi + \frac{\partial \hat{A}}{\partial t} \right) \right),
\end{align*}
\]

with \( \hat{g}(\hat{n}, \bar{p}) = \mathcal{H}(\min(\hat{n}, \bar{p}) - n^*) \), and the gauge condition becomes

\[
\hat{\nabla} \cdot \hat{A} + \epsilon_R^2 \frac{\partial \phi}{\partial t} = 0.
\]

The dimensionless constants \( \delta V D_n, \delta V D_p, \mathcal{F}, \nu \mathcal{G}, \delta \mathcal{V}_n, \nu^{-1} \mathcal{L}, n^*, N^*, \delta \mathcal{V}_p \) and \( \delta \mathcal{V}_p \) are also defined in Table 1; the constraints \( \nu \ll \delta \ll 1, N \ll 1 \) and \( N^* \ll 1 \) typically hold in practice. The relative permittivity \( \epsilon_R = \epsilon/\epsilon_0 \) is typically of \( O(1) \) and \( \epsilon_R^\Lambda \) is its value in the active region (see Table 1). The small parameters are \( \nu \), representing the ratio of \( L/c \) to the time-scale for electron concentration variations, \( \delta \), the ratio of the wavelength to \( L \), \( N \), the ratio of the intrinsic carrier density to a typical free carrier concentration and \( N^* \), which represents the (low) doping concentration in the active region relative to the free carrier concentration. The model is singularly perturbed in \( \delta \) and \( \nu \) but regular in the other small parameters, so that the terms the latter multiply are henceforth neglected. Thus the key small parameters are \( \delta \) and \( \nu \), and a multiple-scales asymptotic analysis based on these quantities will now be pursued.

### 4 Asymptotic analysis

This section is concerned with simplifying the model in the lasing region. The resulting simplified equations, together with those for the non-lasing region, will form the second of the four models in this paper, which is significantly more tractable than (36)-(44). In particular, physical mechanisms will become apparent in the course of the analysis which are not clear in (39)-(44).

The wavelength of the single-mode radiation \( (\lambda \sim 10^{-4} \text{m}) \) and the cavity length \( (L \sim 250 \times 10^{-6} \text{m}) \) are the two relevant longitudinal length-scales in the laser, so a multiple-scale asymptotic expansion is appropriate. We shall employ a total of seven independent variables. Space is described by four independent variables: the lateral length-scale \( \hat{x} \), the transverse length-scale \( \hat{y} \), the short axial length-scale \( \hat{z} \) and the long axial length-scale \( Z = \delta \hat{z} \); \( \hat{z} = O(1) \) corresponds to a wavelength and \( Z = O(1) \) to the length of the cavity. Time is described by three variables: the shortest time-scale \( \hat{t} \), the intermediate time-scale \( T (T = \delta \hat{t}) \) and the longest time-scale \( \tau (\tau = \delta \nu \hat{t}) \), which has already arisen as the time-scale on which the evolution in the non-lasing region occurs; \( \hat{t} = O(1) \) corresponds to the time-scale for a wave to travel a wavelength, \( T = O(1) \) the time-scale to traverse the cavity and \( \tau = O(1) \) the time-scale for carrier variations. The solution is periodic in \( \hat{t} \) with period of \( 2\pi/\hat{\omega} \) at leading order where \( \hat{\omega} = \omega \tau_e \) is the (known) dimensionless angular frequency (it
Table 1: Dimensionless parameters for a typical GaAs laser diode (where some values are reliable and some best estimates).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Typical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>$\lambda/L$</td>
<td>$4 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$u_e/A_{ce}$</td>
<td>$3 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\nu$</td>
<td>$L \sqrt{e^4 \mu_o/A_n}$</td>
<td>$2 \times 10^{-4}$</td>
</tr>
<tr>
<td>$A_n$</td>
<td>$\lambda/\omega$</td>
<td>0-1</td>
</tr>
<tr>
<td>$A_p$</td>
<td>$\lambda/d$</td>
<td>2</td>
</tr>
<tr>
<td>$\delta^{-1} \nu^{-1} A_n$</td>
<td>$A_n/\tau_e$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\delta^{-1} \nu^{-1} A_p$</td>
<td>$A_p/\tau_e$</td>
<td>$3 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\delta \nu B$</td>
<td>$B_n \tau_e$</td>
<td>$6 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\delta \nu C_n$</td>
<td>$C_n \mu_e^2 \tau_e$</td>
<td>$4 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\delta \nu C_p$</td>
<td>$C_p \mu_h^2 \tau_e$</td>
<td>$4 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\delta \nu D_n$</td>
<td>$D_n^2 \tau_e/\lambda^2$</td>
<td>$3 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\delta \nu D_p$</td>
<td>$D_p^2 \tau_e/\lambda^2$</td>
<td>$2 \times 10^{-9}$</td>
</tr>
<tr>
<td>$\delta \nu E_n$</td>
<td>$2 \tau_e \mu_e^2 \Delta E/3 e \lambda^2$</td>
<td>$1 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\delta \nu E_p$</td>
<td>$\tau_e \mu_h^2 \Delta E/3 e \lambda^2$</td>
<td>$4 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\epsilon^A$</td>
<td>$\epsilon_0 \epsilon_{eh} / \epsilon_{eh}^A$</td>
<td>10</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>$A_e \epsilon_{eh} / \tau_e \epsilon_{eh}^A$</td>
<td>2</td>
</tr>
<tr>
<td>$\nu G$</td>
<td>$e_0 \phi_e / \epsilon_{eh}^A n_e$</td>
<td>$6 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\delta \nu H$</td>
<td>$\alpha_0 A_{eh} / \epsilon_{eh}^A$</td>
<td>$3 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\delta \nu K_n$</td>
<td>$U_T \mu_e^2 \tau_e / \lambda^2$</td>
<td>$3 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\delta \nu K_p$</td>
<td>$U_T \mu_h^2 \tau_e / \lambda^2$</td>
<td>$2 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\nu^{-1} L$</td>
<td>$n_e \mu_e^2 \epsilon_{eh}^A / A_e \tau_e$</td>
<td>$2 \times 10^4$</td>
</tr>
<tr>
<td>$N$</td>
<td>$n_1^A / n_e$</td>
<td>$3 \times 10^{-11}$</td>
</tr>
<tr>
<td>$N^*$</td>
<td>$n_2^A / n_e$</td>
<td>0-4</td>
</tr>
<tr>
<td>$P$</td>
<td>$e_0 u_e / \epsilon_{eh}^A n_e$</td>
<td>$1 \times 10^{-2}$</td>
</tr>
<tr>
<td>$\delta \nu Q_n$</td>
<td>$\mu_n^2 \tau_e u_e / \lambda^2$</td>
<td>$2 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\delta \nu Q_p$</td>
<td>$\mu_p^2 \tau_e u_e / \lambda^2$</td>
<td>$2 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\delta \nu V_n$</td>
<td>$\mu_n^2 A_e / \lambda$</td>
<td>$2 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\delta \nu V_p$</td>
<td>$\mu_p^2 A_e / \lambda$</td>
<td>$2 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

being assumed, as noted above, that one longitudinal mode is dominant). It is also periodic in $T$ with period of $2k/\omega$ at leading order (corresponding to the time taken for a wave to traverse the cavity length twice) where $k = k\lambda$ is the dimensionless wave number. We are concerned with the long-time behaviour of the laser, when the wave has settled down such that on travelling back and forth it recovers its form. We consider only the $y$-component of the lasing mode to be non-zero at leading order, the dominance of the transverse electric mode being observed experimentally [2]. We introduce expansions of the form

$$n \sim n_0 + \delta n_1 + \nu n_2 + \delta^2 n_3 + \delta v n_4, \quad \mathbf{p} \sim p_0 + \delta p_1, \quad \mathbf{A} \sim A_0 + \delta A_1, \quad \mathbf{E}^{(R)} \sim P_0, \quad \phi \sim \phi_0 + \delta \phi_1 \text{ as } \delta, \nu \to 0,$$

where $A_0 = (0, A_{0z}^{(2)}, 0)$ and $A_1 = (0, A_{1z}^{(2)}, 0)$ and seek solutions of the form $\phi_0 = \phi_0(\hat{x}, \hat{y}, \hat{z}, \mathcal{T}, T, \tau)$ and $\phi_1 = \phi_1(\hat{x}, \hat{y}, \hat{z}, \mathcal{T}, T, \tau)$. This restriction on $\phi_0$ implies that to leading order $-\nabla \phi$ describes the component of the electric field resulting from the different diffusion rates for electrons and holes and $-\partial \mathbf{A} / \partial \mathcal{T}$ its high-frequency component. To leading order, equations (39)-(41) imply that the electron and hole densities satisfy
\[ n_0 = n_0(x, y, z, Z, \tau) = p_0. \] 

Since periodicity in \( \hat{\tau} \) requires
\[
\int_{\hat{\tau}=0}^{2\pi/\omega} \frac{\partial n_0}{\partial \hat{\tau}} \, d\hat{\tau} = 0, \tag{45}
\]
from (39) we have
\[
\frac{\partial n_0}{\partial \tau} + \frac{\partial n_2}{\partial t} = -\frac{n_0}{A_p + A_n} - Bn_0^2 - (C_n + C_p)n_0^3 - \epsilon^A \mathcal{E} g(n_0) \frac{\omega}{2\pi} \int_{\hat{\tau}=0}^{2\pi/\omega} \left( \frac{\partial A_0^{(2)}}{\partial \hat{\tau}} \right)^2 \, d\hat{\tau} + D_n \nabla^2 n_0
\]
\[-\nabla_n \nabla \cdot \left( n_0 \nabla \phi_0 \right), \tag{46}\]
where \( g(n_0) = \mathcal{H}(n_0 - n^*) \) and we have assumed that \( \partial P_0/\partial \hat{\tau} \) is harmonic (see (26)). Periodicity in \( T \) demands that
\[
\int_{T=0}^{2\pi/\omega} \frac{\partial n_2}{\partial T} \, dT = 0 \tag{47}
\]
so
\[
\frac{\partial n_0}{\partial \tau} = -\frac{n_0}{A_p + A_n} - Bn_0^2 - (C_n + C_p)n_0^3 - \epsilon^A \mathcal{E} g(n_0) \frac{\omega}{4\pi k} \int_{T=0}^{2\pi/\omega} \left( \frac{\partial A_0^{(2)}}{\partial \hat{\tau}} \right)^2 \, d\hat{\tau} + D_n \nabla^2 n_0
\]
\[-\nabla_n \nabla \cdot \left( n_0 \nabla \phi_0 \right). \tag{48}\]

We use (40) to obtain \( \partial p_0/\partial \tau \) by a similar manner, and subtracting this from (48) yields
\[
\nabla \cdot \left( n_0 \nabla \phi_0 \right) = \frac{(D_n - D_p)}{(\nabla_n + \nabla_p)} \nabla^2 n_0; \tag{49}\]
using this to eliminate \( \phi_0 \) from (48) gives
\[
\frac{\partial n_0}{\partial \tau} = -\frac{n_0}{A_p + A_n} - Bn_0^2 - (C_n + C_p)n_0^3 - \epsilon^A \mathcal{E} g(n_0) \frac{\omega}{4\pi k} \int_{T=0}^{2\pi/\omega} \left( \frac{\partial A_0^{(2)}}{\partial \hat{\tau}} \right)^2 \, d\hat{\tau} + \frac{D_n \nabla_p + D_p \nabla_n \nabla^2 n_0}{\nabla_n + \nabla_p}. \tag{50}\]
The local reduction of electrons and holes (the integral term on the right-hand side of (50)) due to spatial-hole burning thus competes with the smoothing action of diffusion (the final term on the right-hand side of (50)); spatial-hole burning is the terminology used (see [9, p. 249]) for the local reduction of electron and hole concentrations caused by the spatial variations in stimulated emission that result from the standing waves. The drift mechanism (due to the electric field) is not present in (46), (48) or (50) as it has an integral of zero over the period of oscillation \( 2\pi/\omega \).

Defining the d'Alembertian operator by
\[
\Box = \epsilon^A \frac{\partial^2}{\partial \hat{\tau}^2} - A_n \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \hat{\tau}^2},
\]
we also have from (42) and (44) that
\[
\Box A_0^{(2)} = 0; \tag{51}\]
it follows from (34) that
\[
A_0^{(3)} = 0 \text{ on } \hat{x} = 0, 1. \tag{52}\]
Using the earlier assumption that the electric field is harmonic (see (25)), we apply the method of separation of variables to write the required solution to (51) in the form

\[ A_0^{(2)} = \sum_{r=1}^{\infty} \left( A_2^r \cos(\omega t - k_r z) + B_2^r \sin(\omega t - k_r z) + A_2^- \cos(\omega t + k_r z) + B_2^- \sin(\omega t + k_r z) \right) \sin(r\pi x), \quad (53) \]

in which \( r \) is an integer representing the different lateral modes of the waveguide, with (in view of (51)) the dispersion relation being \( k_r^2 = \omega^2 \epsilon_0^2 - (r\pi A_r)^2 \). The corresponding dimensionless wavelength is given by \( \lambda_r = 2\pi/k_r \) where \( \lambda_r = \lambda_r/\lambda \) and \( \lambda_r \) is the wavelength of the \( r \)th lateral mode. Henceforth we assume the primary lateral modes \( (r = 1 \text{ and } \hat{k} = \hat{k}_1) \) are dominant, taking

\[ A_0^{(2)} = \left( A_1^+ \cos(\omega t - \hat{k} z) + B_1^+ \sin(\omega t - \hat{k} z) + A_2^- \cos(\omega t + \hat{k} z) + B_2^- \sin(\omega t + \hat{k} z) \right) \sin(\pi x), \quad (54) \]

where \( A_1^\pm(Z,T,\tau), B_1^\pm(Z,T,\tau), A_2^- \) and \( B_2^- \) are amplitude envelopes, which are to be determined from secularity conditions on \( A_1^{(2)} \). Equations (42) and (44) imply that

\[ \Box A_1^{(2)} = 2 \frac{\partial^2 A_0^{(2)}}{\partial z^2} \frac{\partial^2 A_0^{(2)}}{\partial T^2} - 2\epsilon_0^2 \frac{\partial^2 A_0^{(2)}}{\partial T^2} + \epsilon_0^2 \frac{\partial A_0^{(2)}}{\partial T} \left\{ (D_n - D_p)A_y \frac{\partial n_0}{\partial y} - (\nu_n + \nu_p) n_0 A_y \frac{\partial \phi_0}{\partial y} - (\nu_n + \nu_p) n_0 \frac{\partial A_0^{(2)}}{\partial T} \right\}, \]

where we have assumed that \( \partial P_0/\partial \tilde{t} \) is harmonic (see (26)). We require \( A_1^{(2)} \) to satisfy the same boundary conditions as \( A_0^{(2)}, \) that is (52). We thus consider functions \( \chi(\tilde{x}, \tilde{z}, \tilde{\tau}) \) which satisfy \( \Box \chi = 0 \) with the boundary conditions \( \chi(0, \tilde{z}, \tilde{\tau}) = 0, \chi(1, \tilde{z}, \tilde{\tau}) = 0, \chi(\tilde{x}, \tilde{z}, 0) = \chi(\tilde{x}, \tilde{z}, 2\pi/\tilde{k}) \) and \( \chi(\tilde{x}, \tilde{z}, \tilde{\tau}) = \chi(\tilde{x}, \tilde{z}, \tilde{\tau} + 2\pi/\tilde{\omega}) \). The Fredholm alternative then requires that

\[ \int_{\tilde{\tau}=0}^{1} \int_{\tilde{z}=0}^{2\pi/\tilde{\omega}} \int_{\tilde{\tau}=0}^{2\pi/\tilde{\omega}} \chi \sum_{r=1}^{\infty} \partial A_0^{(2)} \partial \tilde{t} d\tilde{z} d\tilde{\tau} = 0. \quad (55) \]

for each of the four linearly independent \( \chi \) given by \( \sin(\pi \tilde{x}) \cos(\omega t - \tilde{k} z), \sin(\pi \tilde{x}) \sin(\omega t - \tilde{k} z), \sin(\pi \tilde{x}) \cos(\omega t + \tilde{k} z) \) and \( \sin(\pi \tilde{x}) \sin(\omega t + \tilde{k} z) \). Therefore we obtain four secularity conditions. By symmetry in the direction of propagation, the equations for \( A_2^- \) and \( B_2^- \) may be derived from the equations for \( A_1^+ \) and \( B_1^+ \) by replacing the quantities \( \tilde{k}, A_1^+, B_1^+, A_2^-, B_2^- \) by \( -\tilde{k}, A_2^-, B_2^-, A_1^+, B_1^+, \) respectively. Therefore, it is only necessary to state two secularity conditions in order to specify all four. The first-order wave equations associated with \( A_1^+ \) and \( B_1^+ \) are given by

\[ \begin{align*}
A_1^+ \int_{\tilde{z}=0}^{2\pi/\tilde{K}} (\Psi_1 - \Psi_2) d\tilde{z} + A_2^- \int_{\tilde{z}=0}^{2\pi/\tilde{K}} (\Psi_1 - \Psi_2) \cos(2\tilde{k} \tilde{z}) d\tilde{z} + B_2^- \int_{\tilde{z}=0}^{2\pi/\tilde{K}} (\Psi_1 - \Psi_2) \sin(2\tilde{k} \tilde{z}) d\tilde{z},
\end{align*} \]

\[ \begin{align*}
B_1^+ \int_{\tilde{z}=0}^{2\pi/\tilde{K}} (\Psi_1 - \Psi_2) d\tilde{z} + A_2^- \int_{\tilde{z}=0}^{2\pi/\tilde{K}} (\Psi_1 - \Psi_2) \cos(2\tilde{k} \tilde{z}) d\tilde{z} + B_2^- \int_{\tilde{z}=0}^{2\pi/\tilde{K}} (\Psi_1 - \Psi_2) \sin(2\tilde{k} \tilde{z}) d\tilde{z},
\end{align*} \]

where

\[ \Psi_1(\tilde{x}, Z, \tau) = \int_{\tilde{z}=0}^{1} \epsilon_r^2 Lg(n_0) \sin^2(\pi \tilde{x}) d\tilde{z}, \quad \Psi_2(\tilde{x}, Z, \tau) = \int_{\tilde{z}=0}^{1} L(\nu_n + \nu_p) n_0 \omega \sin^2(\pi \tilde{x}) d\tilde{z}. \]
The characteristics of the first-order wave equations are thus given by

\[
\frac{dZ}{dT} = \pm \frac{k}{\epsilon \omega} = \pm \frac{d\omega}{dk} = \pm \epsilon_\omega,
\]

where $\epsilon_\omega$ is the dimensionless group velocity and $\omega^2 = (k^2 + (\pi A^2)) / \epsilon_\omega$.

Finally, substituting (54) into (50) and evaluating the integral in $t$, we obtain

\[
\frac{\partial n_0}{\partial T} = -\frac{n_0}{A_p + A_n} - B n_0^2 - (C_n + C_p)n_0^3 - \epsilon_\omega^2 \mathcal{F}(n_0) \int_0^{2k/\omega} \left\{ (A^+_n)^2 + (B^+_n)^2 + (A^-_n)^2 + (B^-_n)^2 \\
+ 2 (A^+_n A^-_n + B^+_n B^-_n) \cos(2k \hat{z}) + 2 (A^+_n B^-_n - B^+_n A^-_n) \sin(2k \hat{z}) \right\} \pi \omega^2 \sin^2(\pi \hat{z}) dT + \frac{D_n V_p + D_p V_n}{V_n + V_p} \dot{\Phi} n_0.
\]

(58)

We now outline the boundary conditions and periodicity conditions required for the simplified equations (56)-(57) and (58). Rewriting the boundary conditions (9)-(11) and (35) in dimensionless form gives

\[
\left[ \frac{\partial n_0}{\partial \hat{x}} \right]_{\hat{x} = 0} = \left[ \frac{\partial n_0}{\partial \hat{y}} \right]_{\hat{y} = 0} = \left[ \frac{\partial n_0}{\partial \hat{z}} \right]_{\hat{z} = 0} = 0,
\]

\[
\left[ \frac{\partial n_0}{\partial \hat{x}} \right]_{\hat{x} = 1} = \left[ \frac{\partial n_0}{\partial \hat{y}} \right]_{\hat{y} = 0} = \left[ \frac{\partial n_0}{\partial \hat{z}} \right]_{\hat{z} = 1} = 0,
\]

As indicated in Section 2.3, we shall approximate the imperfect mirrors in terms of the reflectivities $R(1)$ at $Z = 0$ and $R(2)$ at $Z = 1$. The boundary conditions for the amplitude envelopes are then given by the expressions

\[
(A^+_0 (0, T, \tau))^2 = R(1) (B^+_0 (0, T, \tau))^2, \quad (B^+_0 (0, T, \tau))^2 = R(1) (B^-_0 (0, T, \tau))^2,
\]

\[
(A^-_1 (1, T, \tau))^2 = R(2) (A^+_1 (1, T, \tau))^2, \quad (B^-_1 (1, T, \tau))^2 = R(2) (B^+_1 (1, T, \tau))^2.
\]

(59)

(60)

These boundary conditions avoid the additional complication of modelling the transmitted electromagnetic wave, as discussed in Section 2.3. The amplitude envelopes must also be periodic in $T$, such that

\[
A^+_Z (Z, T, \tau) = A^+_Z (Z, T + 2k/\omega, \tau), \quad B^+_Z (Z, T, \tau) = B^+_Z (Z, T + 2k/\omega, \tau),
\]

\[
A^-_Z (Z, T, \tau) = A^-_Z (Z, T + 2k/\omega, \tau), \quad B^-_Z (Z, T, \tau) = B^-_Z (Z, T + 2k/\omega, \tau).
\]

(61)

(62)

The four secularity conditions form a system of equations which describes the physical mechanisms associated with the variation of the amplitude envelope. The integral $\Psi_1$ on the right-hand side of (56)-(57) is associated with gain and absorption (corresponding to the constitutive law for $P^{(R)}$) while the integral $\Psi_2$ describes the interaction between the electric field and charge carriers (corresponding to the constitutive law for $J$). All the cross-coupling terms sample the Fourier coefficients of $n_0$ in at least one of $\sin(2k \hat{x})$ and $\cos(2k \hat{x})$, this sampling corresponding to the interaction between waves travelling in opposite directions in $\hat{z}$. We observe that $2k$ is the wave number associated with longitudinal spatial-hole burning; longitudinal spatial-hole burning describes the longitudinal variation of electron and hole concentrations. Therefore, we have identified the spatial-hole burning mechanism as being responsible for the main interaction between waves travelling in opposite directions. Spatial-hole burning is known to inhibit the growth of the lasing mode as the current increases [9, p. 249].

In summary, the model in the lasing region has been simplified such that the unknowns $n_0$, $A^+_Z$ and $B^+_Z$ are solutions to the system given by the second-order diffusion equation (58) and the four first-order wave equations represented by (56)-(57).
5 Lumped models

5.1 Partially lumped

Unlike the preceding analysis, the two lumped models which follow require ad hoc averaging approaches; nevertheless, they are instructive in terms of the interpretation and generalisation of existing models of the same class. In this subsection, we deduce the third of the four models introduced in this paper. The non-lasing region is not explicitly included in this model, being approximated by a supply current for the lasing region. Firstly, we integrate (58) with respect to \(x, y\) and \(z\) and apply the boundary conditions on \(n_0\), giving

\[
\frac{k}{2\pi} \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{2\pi/k} \left( -\frac{\partial n_0}{\partial r} - \frac{n_0}{A_p + A_n} - Bn_0^2 - (C_n + C_p)n_0^3 \right) d\tilde{y}d\tilde{z}
\]

\[-\frac{\omega^2 \epsilon^2}{8\pi} \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{2\pi/k} \int_{T=0}^{2k/\omega} g(n_0) \left\{ (A_1^+)^2 + (B_1^+)^2 + (A_2^-)^2 + (B_2^-)^2 \right\}
\]

\[+ 2(A_1^+ A_2^- + B_1^+ B_2^-) \cos(2k\tilde{z}) + 2(A_1^+ B_2^- - B_1^+ A_2^-) \sin(2k\tilde{z}) \}\sin^2(\pi\tilde{x})dTd\tilde{y}d\tilde{z} + \tilde{J}_y = 0,
\]

where

\[
\tilde{J}_y = \frac{k}{2\pi} \int_{z=0}^{1} \int_{y=0}^{1} \int_{x=0}^{2\pi/k} \left[ \epsilon^2 D_n V_p + D_p V_n \frac{\partial n_0}{\partial y} \right] \frac{\partial y}{\partial z} d\tilde{y}d\tilde{z},
\]

represents the supply of current through the boundary of the lasing region and is henceforth treated as known; the current source in the lateral direction is assumed negligible. We now expand \(n_0\) in a Fourier expansion for the physically significant modes dependent on \(\tilde{x}, \tilde{y}\) and \(\tilde{z}\), as follows

\[
n_0 = N_0(Z, \tau) + N_1(Z, \tau) \sin(\tilde{k}\tilde{x}) + N_2(Z, \tau) \cos(\tilde{k}\tilde{x}) + N_3(Z, \tau) \sin(2\tilde{k}\tilde{x}) + N_4(Z, \tau) \cos(2\tilde{k}\tilde{x}) + \ldots,
\]

(63)

terms in \(\tilde{x}\) and \(\tilde{y}\) not being listed here. While the previous expressions are exact deductions from (39)–(44), we will now satisfy the equations only in an averaged sense, effectively making the simplifying assumption that electron density is independent of \(x, y, \) and \(z\). This assumption neglects the diffusion of carriers and spatial-hole burning effects in \(x, y, \) and \(z\). To achieve this aim, we simply truncate the Fourier series (63) after one term. Writing \(n_0 = N_0(Z, \tau)\), we then have

\[
\frac{\partial N_0}{\partial \tau} = \tilde{J}_y - \frac{N_0}{A_p + A_n} - Bn_0^2 - (C_n + C_p)N_0^3 - \epsilon^2 F g(N_0) \frac{\omega}{2k} \int_{T=0}^{2k/\omega} \left( \hat{I}^+ + \hat{I}^- \right) dT,
\]

(64)

where we define the intensities of the electric field to be

\[
\hat{I}^\pm = \frac{\omega^2}{4} \left\{ (A_1^\pm)^2 + (B_1^\pm)^2 \right\},
\]

where the superscript + (−) again denotes the wave travelling in the direction of increasing (decreasing) \(\tilde{z}\). The intensities satisfy

\[
\frac{\epsilon^2 \omega}{k} \frac{\partial \hat{I}^\pm}{\partial T} \pm \frac{\partial \hat{I}^\pm}{\partial Z} = \frac{\omega}{k} \left( \epsilon^2 L g(N_0) - L(V_n + V_p)N_0 \right) \hat{I}^\pm,
\]

(65)

with parametric dependence on \(\tau\) through \(N_0\); the boundary conditions are \(\hat{I}^+(0, T, \tau) = R^{(1)} \hat{I}^-(0, T, \tau)\) and \(\hat{I}^-(1, T, \tau) = R^{(2)} \hat{I}^+(1, T, \tau)\). The intensities must also satisfy the periodicity condition given by \(\hat{I}^+(Z, T, \tau) = \hat{I}^+(Z + 2k/\omega, \tau)\) and \(\hat{I}^-(Z, T, \tau) = \hat{I}^-(Z + 2k/\omega, \tau)\). A schematic of light intensities in a laser is shown in Figure 3. The model in the lasing region has thus been simplified such that \(N_0\) and \(\hat{I}^\pm\) are solutions to an ordinary differential equation (64) and two first-order wave equations (65).
The equations (64)-(65) complement a previous, well-established, model which comprises one ordinary differential equation and two first-order wave equations [4]. The derivation of these equations in the literature is phenomenological. In this subsection we have not only provided a systematic derivation of these equations (based on specified assumptions), but a general framework from which other models can be derived (by adopting other assumptions, Appendix B providing an example).

A notable omission in (65) is the contribution of the spontaneous emission to the lasing mode, which is indispensable for the correct treatment of the transient behaviour, but does not have a significant effect on the large-time behaviour (see [4]). As previously observed in [9, p. 235], rigorous treatment of the spontaneous emission requires quantization of the laser field which is not possible in a classical framework. Indeed, the terms corresponding to spontaneous emission are too small to appear in the leading order problem. However, it is possible to add terms to take into account the rate at which spontaneously emitted photons are added to the energy in the lasing mode, this being outlined in Appendix A.

In this subsection we have considered the situation in which the electron density is treated as being independent of \( \hat{x}, \hat{y} \) and \( \hat{z} \) (or, more precisely, in which the equations are satisfied only in an averaged sense with respect to \( \hat{x}, \hat{y} \) and \( \hat{z} \)). If we had integrated (58) with respect to \( \hat{x} \) and \( \hat{y} \) and included three terms in the Fourier series (63) (\( N_0, N_3 \) and \( N_4 \)) instead of only one (appropriately averaging in \( \hat{z} \)), then we would have retained effects due to longitudinal spatial-hole burning and the diffusion of carriers. The resulting model comprises three ordinary differential equations and four first-order wave equations, this being outlined in Appendix B. Previous attempts to model spatial-hole burning have taken the form of gain saturation (see, for example, [6] and references therein), which is a far less systematic approach.

### 5.2 Fully lumped

In this subsection, we derive the last of the four models. As in the third model, the non-lasing region is approximated by a uniform supply current to the lasing region. We average the electron and photon concentrations over the length of the laser to simplify the third model further. This additional modification averages out the variations in gain along the laser in the longitudinal direction. We define

\[
N^\dagger(\tau) = \int_{Z=0}^{1} N_0(Z, \tau) dZ, \quad S^\dagger(T, \tau) = \int_{Z=0}^{1} \left( \hat{I}^+ + \hat{I}^- \right) dZ,
\]

the second of which can be viewed as a photon density. Integrating with respect to \( Z \) and approximating the integrals of various products as the products of the corresponding integrals (now satisfying the equations only in an averaged sense in terms of \( Z \) also), we have

\[
\frac{dN^\dagger}{d\tau} = \int_{Z=0}^{1} \hat{J}_p dZ - \frac{N^\dagger}{A_p + A_n} - BN^{12} - (C_p + C_n) N^{13} - \epsilon^2 \mathcal{F} g(N^\dagger) \frac{\hat{\omega}}{2k} \int_{T=0}^{2k/\hat{\omega}} S^\dagger dT, \quad (66)
\]

\[
\frac{\epsilon^2 \hat{\omega}}{k} \frac{\partial S^\dagger}{\partial T} = - \left\{ \hat{I}^+(1, T, \tau) - \hat{I}^-(1, T, \tau) \right\} - \left\{ \hat{I}^-(0, T, \tau) - \hat{I}^+(0, T, \tau) \right\} + \frac{\hat{\omega}}{k} \left( \epsilon^2 \mathcal{L} g(N^\dagger) - \mathcal{L}(V_n + V_p) N^\dagger \right) S^\dagger. \quad (67)
\]

The averaged intensities must also satisfy the periodicity condition \( S^\dagger(T, \tau) = S^\dagger(T + 2k/\hat{\omega}, \tau) \). The first four terms on the right-hand side of (67) represent the mirror losses. In order to average the effect of the mirror loss over the whole device it is standard to approximate these terms by an additional absorption constant [9, p. 234]. The closure of the equations in this manner is a reasonable approach if the absorption constant is chosen to be consistent with the partially lumped model [9, p. 33]. The first term on the right-hand side of (66) is now regarded as a specified constant current density.
The model in the lasing region has now been simplified such that $N$ and $S$ are solutions to two ordinary differential equations (66)-(67) which complement the well-established single-mode rate equations [6]. As in the partially lumped case, the contribution of the spontaneous emission to the lasing mode is omitted from (67); terms which take the spontaneously emitted photons into account are outlined in Appendix A.

6 Summary

This paper has established a framework for the variety of classes of model that can be employed to simulate electrical and optical effects in semiconductor lasers. The first of the four models incorporates Maxwell's equations with expressions for the current and polarization. The semiconductor is split into the active region, modelled as lasing, and the surround, modelled as non-lasing. The lasing part of the model is composed of two drift-diffusion equations, four second-order wave equations and a gauge condition. The non-lasing part of the model consists of two drift-diffusion equations and Poisson's equation. The polarization in the lasing part of the model is treated as dependent on the electric field, not only at the present time but also at previous times. The fully space-dependent mathematical model is generally accepted in the literature to comprise an electrical model, consisting of electron and hole continuity equations and Poisson's equation, and an optical model, made up of a wave equation and a photon rate equation [1], [2].

The second of the four models is derived from the first by means of a multiple-scale asymptotic expansion applied to the equations in the lasing region; the equations in the non-lasing region only respond on the longest (electrical) time-scale. To leading order, the linear equations for the lasing mode can be viewed as representing a wave guide. At next order, we obtain nonlinear secularity conditions for the lasing mode envelope. This asymptotic procedure leads to the major simplification of removing the shortest time-scale, which is the reciprocal of the frequency $(10^{-15} s)$. The equations in the lasing region are reduced to four first-order wave equations for the envelope of the guided wave and a diffusion equation for the electron concentration. This second model is more amenable to the application of numerical methods and to gaining insight into the physical mechanisms, it being clear in the electron concentration equation that there is a competition between the spatial-hole burning process of gain (local reduction of electrons and holes) and the smoothing action of diffusion of electrons and holes. It is noteworthy that the drift mechanism (due to the electric field) is not present as it has an average of zero over the period of oscillation. It is also interesting to note that there is an interaction between waves travelling in opposite directions. The effect of longitudinal spatial-hole burning (local variations of electron and hole concentrations in the longitudinal direction) in coupling these waves has not been observed in any previous model, despite being physically apparent.

The third (partially lumped) model is derived from the equations in the lasing part of the second model by averaging the electron concentration over the lateral, transverse and short longitudinal length-scales. The equations in the non-lasing region are approximated by a supply current to the lasing region. This model consists of an ordinary differential equation and two first-order wave equations. The fourth (fully lumped) model relies on the further simplification resulting from averaging the electron and photon concentrations over the longitudinal length of the laser. This final modification results in two ordinary differential equations. Both of these models complement previous well-established models determined from even more ad hoc arguments [4], [6] and include expressions for the various lumped parameters in terms of physical quantities. Both the partially and fully lumped models in the literature [4], [6] contain the contribution of spontaneous emission to the lasing mode and the addition of terms accounting for spontaneously emitted photons is outlined in the current context in Appendix A. For the fourth model it is necessary to average the effect of the mirrors over the whole device in order to close the system of equations.
The third and the fourth models in this paper have been shown to exhibit a transcritical bifurcation for the intensity as a function of current; see [17] and [7], respectively. The intensity in the third and fourth models is the square of the amplitude of the electric field, so the corresponding bifurcations in the first and second models (for the electric field amplitude as a function of current) will be of pitchfork type.

In summary, four mathematical models have been derived for electrical-optical effects in semiconductor laser operation, as follows

**Model I**

- **Lasing Region**: Four second-order wave equations, two drift-diffusion equations and a gauge condition.
- **Non-lasing Region**: Two drift-diffusion equations and Poisson's equation.

**Model II**

- **Lasing Region**: Four first-order wave equations and a diffusion equation.
- **Non-lasing Region**: Two drift-diffusion equations and Poisson's equation.

**Model III** (Lasing region only)

- Two first-order wave equations and an ordinary differential equation.

**Model IV** (Lasing region only)

- Two ordinary differential equations.

The model in the lasing region is in some senses more tractable asymptotically than that in the non-lasing region due to the abundance of small parameters. The second model is deduced from the first by a systematic asymptotic analysis, whilst the third and fourth arise from ad hoc averaging procedures. The first, third and fourth models have counterparts in the existing literature, whereas the second has none and probably represents the most important aspect of the current paper. We conclude by noting that these models provide an appropriate framework for analysing thermal effects. The third model coupled to two one-dimensional heat equations (one for each of the lasing and non-lasing regions) provides a very suitable basis for the study of localised hot-spots and is the subject of current work [17].

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**References**


Appendix A. The travelling-wave and single-mode rate equations

This appendix briefly summarises the dimensional formulation of the two lumped models derived in this paper with additional spontaneous emission terms.

Let $J$ be the specified current density per unit axial length, $\beta$ be the fraction of spontaneous emission (assumed constant) which enters the lasing mode, $N^t(z, t)$ be the electron concentration, $I^+(z, t)$ be the light intensity of the wave travelling in the positive $z$ direction and $I^-(z, t)$ be the light intensity of the wave travelling in the negative $z$ direction. For the partially lumped model (or travelling-wave rate equations; cf. equations (64)-(65)), we have

$$\frac{\partial N^t}{\partial t} = \frac{J}{ed} - \frac{N^t}{2A_n} - BN^{12} - (C_n + C_p)N^{13} - a(N^t - n_t) \frac{\omega}{2Lk} \int_{t'=t}^{t'=t+2Lk/\omega} (I^+ + I^-) dt', \quad (68)$$

$$\frac{\partial I^\pm}{\partial t} \pm \frac{k_\infty^2}{\varepsilon_r^A \omega} \frac{\partial I^\pm}{\partial z} = \left( a(N^t - n_t) - \mu_0 e(\mu_n^A + \mu_p^A) N^t \frac{\omega}{\varepsilon_r^A} \right) I^\pm + \beta BN^{12}, \quad (69)$$
with the boundary conditions $I^+(0, t) = R(1)I^-(0, t)$ and $I^-(L, t) = R(2)I^+(L, t)$. The intensities must also satisfy the periodicity condition given by $I^+(z, t) = I^+(z + 2Lk/\omega)$ and $I^-(z, t) = I^-(z + 2Lk/\omega)$. The value of $k$ is obtained from the dispersion relation for the waveguide, that is

$$k^2 = \frac{c^2 \omega^2}{c^2} - \frac{n^2}{\omega^2}.$$  

The last term on the right-hand side of (69) models the contribution of the spontaneous emission to the lasing mode.

Now let $\bar{N}(t)$ be the electron concentration, $\bar{S}(t)$ be the intensity and $\bar{J}$ be the current density averaged over the active region. For the fully lumped model (or single-mode rate equations; cf. equations (66)-(67)), we have

$$\frac{d\bar{N}}{dt} = \frac{\bar{J}}{ed} - \frac{\bar{N}}{A_p + A_n} - B\bar{N}^2 - (C_n + C_p)\bar{N}^3 - a(\bar{N} - n_t)\frac{\omega}{2Lk}\int_{t'=t}^{t-2Lk/\omega} \bar{S} dt', \quad \text{(70)}$$

$$\frac{d\bar{S}}{dt} = \left( a\bar{N} - n_t \right) - \frac{\epsilon_0 \alpha}{\mu} - \mu_0 e(\mu_n^A + \mu_p^A)\frac{c^2}{\epsilon_0^A} \bar{S} + 2B\bar{N}^2. \quad \text{(71)}$$

The periodicity condition for $\bar{S}$ is given by $\bar{S}(z, t) = \bar{S}(z, t + 2Lk/\omega)$. Here the constant $\alpha$ represents the averaged loss through the mirrors of the photon density and $\mu$ is the group refractive index. We note that the appearance of the quantities $\mu_n^A$ and $\mu_p^A$ in the relevant terms gives a physical interpretation to free carrier absorption coefficients adopted by other authors (see [18]).

**Appendix B. The longitudinal spatial-hole burning model**

This appendix briefly summarises the dimensionless formulation of the partially-lumped single-mode model for competition between longitudinal spatial-hole burning (local variations of electron and hole concentrations in the longitudinal direction) and carrier diffusion. The nature of the derivation is mentioned in Subsection 5.1. The model consists of three ordinary differential equations and four first-order wave equations. The ordinary differential equations for the coefficients of functions 1, $\sin(2kz)$ and $\cos(2kz)$ in the truncated Fourier expansion for the electron concentration are given by

$$\frac{\partial N_0}{\partial \tau} = -\frac{N_0}{A_p + A_n} - B\left(N_0^2 + \frac{1}{2}N_0^3 + \frac{1}{2}N_0^2 \right) - (C_n + C_p)\left(N_0^3 + \frac{3}{2}N_0N_2 + \frac{3}{2}N_0N_2^2 \right) + J_p$$

$$-e_A^0\mathcal{H}\omega^3 \int_{T=0}^{2k/\omega} (N_0 - n^*) \left( (A_0^+)^2 + (B_0^+)^2 + (A_0^-)^2 + (B_0^-)^2 \right) + N_3 (A_0^+B_0^- - B_0^+A_0^-) = N_4 (A_0^+A_0^- + B_0^+B_0^-) dt, \quad \text{(72)}$$

$$\frac{\partial N_3}{\partial \tau} = -\frac{N_3}{A_p + A_n} - 2BN_0N_3 - (C_n + C_p)\left(\frac{3}{4}N_3^3 + 3N_0^2N_3 + \frac{3}{4}N_3^2N_4 \right) - 4k^2N_3 \frac{D_nV_p + D_pV_n}{V_n + V_p}$$

$$-e_A^0\mathcal{H}\omega^3 \int_{T=0}^{2k/\omega} N_3 \left( (A_0^+)^2 + (B_0^+)^2 + (A_0^-)^2 + (B_0^-)^2 \right) + 2(N_0 - n^*) (A_0^+B_0^- - B_0^+A_0^-) dt, \quad \text{(73)}$$

$$\frac{\partial N_4}{\partial \tau} = -\frac{N_4}{A_p + A_n} - 2BN_0N_4 - (C_n + C_p)\left(\frac{3}{4}N_4^3 + 3N_0^2N_4 + \frac{3}{4}N_3^2N_4 \right) - 4k^2N_4 \frac{D_nV_p + D_pV_n}{V_n + V_p}$$

$$-e_A^0\mathcal{H}\omega^3 \int_{T=0}^{2k/\omega} N_4 \left( (A_0^+)^2 + (B_0^+)^2 + (A_0^-)^2 + (B_0^-)^2 \right) + 2(N_0 - n^*) (A_0^+A_0^- + B_0^+B_0^-) dt, \quad \text{(74)}$$
respectively. The first three terms on the right-hand side of (72)-(74) represent recombination. The fourth term on the right-hand side of (72) models the supply current. The fourth term on the right-hand side of (73) and (74) represents carrier diffusion. The remaining terms model the stimulated emission and absorption. The first-order wave equations associated with $A^+_2$ and $B^+_2$ now become

$$
e^A_2 \dot{A}^+_2 + \dot{k} A^+_2 = A^+_2 (2A_1 (N_0 - n^*) - 2A_2 N_0) + A^+_2 (A_1 - A_2) N_4 + B^+_2 (A_1 - A_2) N_3, \tag{75}$$

$$
e^A_2 \dot{B}^+_2 + \dot{k} B^+_2 = B^+_2 (2A_1 (N_0 - n^*) - 2A_2 N_0) - A^+_2 (A_1 - A_2) N_3 + B^+_2 (A_1 - A_2) N_4, \tag{76}$$

where

$$\Lambda_1 = \frac{\epsilon^A CH}{4}, \quad \Lambda_2 = \frac{C(V_n + V_p)\omega}{4},$$

and the equations for $A^-_2$ and $B^-_2$ can again be obtained by symmetry in the direction of propagation. The boundary conditions (59)-(60) and periodicity conditions (61)-(62) are required to complete the model.
Figure 1: Typical layered structure of a double-heterostructure semiconductor laser. The lateral direction is denoted by $x$, the transverse direction by $y$ and the longitudinal (or axial) direction by $z$. The battery potential is applied in the transverse direction, the mirrors are located at $z = 0$ and $z = L$ and $X_1 = w/2$. 
Figure 2: A schematic of the absorption process and stimulated gain process. In the former process a dipole is created and the electric field is decreased and in the latter process a dipole is removed and the electric field is increased; see (29).

Figure 3: A schematic of the light intensity of the wave travelling in the positive $z$ direction, $I^+(z,t)$ and of the light intensity of the wave travelling in the negative $z$ direction, $I^-(z,t)$. 