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Subset selection from
generalized logistic populations

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Eindhoven, June 1997
The Netherlands
Subset Selection from Generalized Logistic Populations

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Abstract

We give an introduction to the logistic and generalized logistic distributions. These generalized logistic distributions Type-I, Type-II and Type-III are indexed by a real valued parameter. They have been derived as mixtures with the standard logistic distribution and for discrete values of the parameter they describe the distribution of the minimum, maximum, and median, respectively, of a i.i.d. sample from a logistic distribution. We obtain exact results for the probability of correct subset selection from Type-I, Type-II and Type-III generalized logistic populations which only differ in their location parameter.

In the course of establishing these exact results, we derive an explicit expression for the cdf of the median based on a sample from the logistic distribution of size $2b-1$; it is a sum of $b$ terms involving binomial coefficients. By the CLT we can use this explicit cdf to approximate the unknown cdf of the normal distribution. We show some numerical results which show that for $b=5$ the approximation error is of the order 0.001 in the middle and 0.0001 in the tails. This implies that for practical purposes the Type-III family with $b=5$ is indistinguishable from the normal family.

AMS classification: Primary 62F07; secondary 62E15.
Key Words and Phrases: Subset Selection, Type-I, Type-II and Type-III generalized logistic distribution, probability of correct selection.

1 Introduction

Subset selection has been introduced by Gupta (1965). Many contributions to this field of statistical interest have been given. Given are $k \geq 2$ random variables $X_1, \ldots, X_k$, which may be sample means or sample quantiles, associated with $k$ populations $\pi_1, \ldots, \pi_k$, respectively. We assume that the distributions of these random variables differ only in their location parameter. We are interested in choosing the best population, that is the population with the largest value of the location parameter. If there are more than one contenders for the highest rank, we suppose that one of these is appropriately tagged. Subset selection has as its goal to indicate a subset of the collection of $k$ populations in order to include the best
population with a certain confidence and with the requirement that the size of the subset is as small as possible. Gupta’s subset selection rule is defined by:

Select \( \pi_i \) if and only if \( x_i \geq \max_{1 \leq j \leq k} x_j - d \),

where \( x_i \) is the observed value of \( X_i \) \((i = 1, ..., k)\). The selection constant \( d \) \((\geq 0)\) has to be chosen such that the probability is at least \( P^* \) \((k^{-1} < P^* < 1)\) that the subset contains the best population. A correct selection \( CS \) is defined as a selection of any subset which includes the best population.

The selection constant and the probability of \( CS \) depend on the form of the underlying distribution. Lorentzen and McDonald (1981) considered the selection problem for logistic populations using sample medians. Some exact distributional results for subset selection from logistic populations are given in Van der Laan (1989, 1992) for Bechhofer’s indifference zone approach (1954) and Gupta’s subset selection approach, respectively.

The probability of \( CS \) is equal to

\[
P(CS) = P \left( X_{(k)} \geq \max_{1 \leq j \leq k} X_j - d \right),
\]

where \( X_{(k)} \) is the unknown random variable associated with \( \theta_{[k]} \) and where the ranked location parameters \( \theta_1, ..., \theta_k \) are denoted by \( \theta_{[1]} \leq ... \leq \theta_{[k]} \). From Gupta (1965) we have

\[
\inf P(CS) = \int_{-\infty}^{\infty} F^{k-1}(x + d) dF(x),
\]

which is attained for the least favourable configuration (LFC): \( \theta_{[1]} = \theta_{[k]} \), and where \( F(\cdot) \) is the cumulative distribution function (cdf) of the populations under the LFC. To be sure that \( P(CS) \geq P^* \) for all configurations of \( \theta_1, ..., \theta_k \) the smallest value of the selection constant \( d \) has to be chosen for which

\[
\int_{-\infty}^{\infty} F^{k-1}(x + d) dF(x) = P^*.
\] 

It is of interest to note that the left-hand side of this equation also equals the minimal probability of correct selection if we only select \( \max_i X_i \) and where the minimum is over all configurations for which \( \theta_{[k]} - \theta_{[k-1]} \geq d \). The size \( S \) of the selected subset with Gupta’s selection rule is a random variable with possible outcomes \( \{1, 2, ..., k\} \). The expected value of this size of the subset can be used as an efficiency criterion. From Gupta (1965) we have

\[
M \equiv \max_{\Omega} E(S) = k \int_{-\infty}^{\infty} F^{k-1}(x + d) dF(x) = kP^*,
\]

where \( \Omega \) is the parameter space consisting of all configurations of \( \theta \)'s. Instead of taking the maximum over the whole parameter space one could also compute the maximum \( M(\delta) \) over \( \Omega(\delta) \), where \( \Omega(\delta) \) consists of all possible configurations for which \( \theta_{[k-1]} - \theta_{[k]} \geq \delta \) apart.

The left-hand side of the equation (1) and \( M(\delta) \) have been analytically determined for the logistic distribution by Van der Laan (1989, 1992). Here it is also highlighted with computations that \( M \) and \( M(\delta) \) are hardly different for practical values of \( \delta \); so only if the distance between the best population and the remaining is large, then the expected subset size decreases substantially by using this. This indicates that in many practical applications it
is hard to improve the conservative $P^*$, which is based on the least favourable configuration, with more sophisticated approaches and hence that Gupta’s approach is also sensible in practice.

A generalization is to consider Type-I, Type-II and Type-III generalized logistic populations. The logistic distribution is a special case of the class of Type-I, Type-II and Type-III generalized logistic distributions. In section 2 some introductory remarks concerning the class of generalized logistic distributions are given. In section 3, we provide an explicit expression for the cumulative distribution function of the Type-III generalized logistic distribution and a numerical study to establish its approximation rate to the normal family. In section 4 we shall consider subset selection, and thus the principal equation (1), for Type-I, Type-II and Type-III generalized logistic distributions. We conclude with some remarks and open problems.

2 Generalizations of the standard logistic family

The probability density function of a logistic variable $X$ with expectation $\mu$ and variance $\sigma^2$ is given by:

$$f(x; \mu, \sigma) = \frac{\pi}{\sigma \sqrt{3}} \frac{e^{-\pi(x-\mu)/(\sigma \sqrt{3})}}{\left[1 + e^{-\pi(x-\mu)/(\sigma \sqrt{3})}\right]^2} \quad \text{with} \quad -\infty < x < \infty. \tag{2}$$

From a mathematical point of view it is sometimes simpler to work with the random variable

$$Z = \frac{\pi(X - \mu)}{\sigma \sqrt{3}} \quad \text{where} \quad EZ = 0 \quad \text{and} \quad \text{VAR}(Z) = \frac{\pi^2}{3}.$$ 

The density and cdf of $Z$ are

$$g(z) = \frac{e^{-z}}{(1 + e^{-z})^2} \quad \text{and} \quad G(z) = \left(1 + e^{-z}\right)^{-1}, \quad \text{respectively, with} \quad -\infty < z < \infty. \tag{3}$$

The logistic distribution is symmetric and its shape is similar to that of the normal distribution, but the density is more peaked in the center than the normal density. The cumulative distribution functions of the standardized normal and logistic variates differ by a maximum of 0.0228. However, the logistic tails are more heavy than the normal tails. The logistic distribution has a number of Biological, Industrial and Engineering applications. Applications in the field of Bio-assay, Quantal Response Data, Probit Analysis, and Dosage Response Analysis are well-known. The logistic distribution appears also as limiting distribution of standardized statistics. An important aspect of the logistic distribution is its relative simplicity, relative in comparison with for instance the normal distribution. This makes it sometimes profitable to develop exact theories under the logistic distribution without causing a too great discrepancy with the corresponding normal theory.

Three interesting generalizations of the standard logistic family have been obtained by Balakrishnan (1992) and Balakrishnan and Leung (1988). The Type-I generalized logistic random variable $X$ has density function and cdf given by:

$$h(x; \mu, \sigma) = \frac{b}{\sigma} \frac{e^{-(x-\mu)/\sigma}}{\left[1 + e^{-(x-\mu)/\sigma}\right]^{b+1}} \quad \text{and} \quad H(x; \mu, \sigma) = \left\{1 + e^{-(x-\mu)/\sigma}\right\}^{-b},$$

respectively, with $-\infty < x < \infty$ and $b > 0$. The density (4) was obtained by compounding an extreme-value distribution of the double exponential type with a gamma distribution. They showed that given

$$f(x|\alpha) = \alpha e^{-x} e^{-\alpha e^{-x}} \quad (-\infty < x < \infty, \alpha > 0),$$

3
an extreme-value density of the double exponential type, and
\[ g(a) = e^{-\alpha} \alpha^{-b-1} / \Gamma(b) \quad (\alpha, b > 0), \]
a gamma density, the compounding distribution (the Type-I generalized logistic distribution) has density
\[ f_I(x) = \int_0^\infty f(x|\alpha)g(\alpha)d\alpha = \frac{be^{-x}}{(1 + e^{-x})^{b+1}}, \quad -\infty < x < \infty, \ b > 0 \]
(5)
and cdf \( F_I(x) = (1 + e^{-x})^{-b} \). The special case \( b = 1 \) corresponds to the logistic density. For (positive) integer values of \( b \) the density (5) is also the density function of the largest order statistic in a random sample of size \( b \) from the logistic distribution (3); this follows from the fact that \( F_I(x) = G(x)^b \). Relations for mean, variance, skewness and kurtosis, and some numerical results for \( b = 1(0.5)5(1)8 \) are given in Balakrishnan and Leung (1988). This family consists of positively skewed distributions with its coefficient of kurtosis greater than that of the logistic.

The Type-II generalized logistic distribution has density and cdf given by:
\[ f_{II}(x) = \frac{be^{-bx}}{(1 + e^{-x})^{b+1}}, \quad F_{II}(x) = 1 - \frac{e^{-bx}}{(1 + e^{-x})^b}, \quad -\infty < x < \infty \text{ and } b > 0. \]
(6)
The special case \( b = 1 \) corresponds again with the ordinary logistic distribution. For (positive) integer values of \( b \) the density (6) becomes the density function of the smallest order statistic in a random sample of size \( b \) from the logistic distribution with density (3). If \( X \) is a random variable distributed as Type-I, then \( Z = -X \) has a Type-II distribution. Hence this family consists of negatively skewed distributions with its coefficient of kurtosis greater than that of the logistic.

Finally, the Type-III generalized logistic distribution has density
\[ f_{III}(x) = \frac{1}{B(b, b)} \frac{e^{-bx}}{(1 + e^{-x})^{2b}}, \quad -\infty < x < \infty \text{ and } b > 0 \]
where \( B(b, b) = (\Gamma(b))^2 / \Gamma(2b) \) is the complete Beta function. The special case \( b = 1 \) corresponds again with the logistic density (3). For (positive) integer values of \( b \) the Type-III generalized logistic density is the density of the sample median in a random sample of size \( 2b - 1 \) from the logistic distribution with density (3). The distribution is symmetric about zero and very close to the normal distribution.

The Type-III distribution has proved to be useful as an approximation to other symmetric distributions, for instance the t-distribution. The distribution can be used in robustness studies of classical procedures based on normality and as alternative to the normal distribution in power studies. The form of the Type-III generalized logistic distribution and some exact results have already been presented and used for subset selection for logistic populations using sample medians by Lorenzen and McDonald (1981). They used a numerical quadrature routine to determine the desired integrals.

### 3 The cumulative distribution function of the Type-III generalized logistic family

The following theorem presents an explicit expression with only positive terms of the cumulative distribution of the Type-III generalized logistic distribution.
**Theorem 3.1** The cumulative distribution of the Type-III distribution is given by:

\[
F_{III,b}(x) = (1 - G(x))^b \sum_{i=0}^{b-1} \binom{i+b-1}{b-1} (G(x))^i,
\]

(7) where \(G(x) = (1 + \exp(x))^{-1}\) is the standard logistic cumulative distribution function.

**Proof.**

\[
B(b, b) F_{III,b}(x) = \int_{-\infty}^{\infty} \frac{\exp(-bs)}{(1 + \exp(-s))^{2b}} ds
\]

\[
= \int_{\exp(-x)}^{\infty} \frac{z^{b-1}}{(1 + z)^{2b}} dz
\]

\[
= \frac{1}{(2b-1)} \exp(-(b-1)x) + \frac{b-1}{2b-1} \int_{\exp(-x)}^{\infty} \frac{z^{b-2}}{(1 + z)^{2b-1}} dz,
\]

using partial integration. Define

\[
I(i, x) \equiv \int_{\exp(-x)}^{\infty} \frac{z^{b-1-i}}{(1 + z)^{2b-i}} dz, \quad i = 0, 1, \ldots, b - 1.
\]

Above we expressed \(I(0, x)\) in \(I(1, x)\). Similarly, we express \(I(1, x)\) in \(I(2, x)\), etc. In this way we obtain:

\[
B(b, b) F_{III,b}(x) = \frac{1}{(2b-1)} \exp(-(b-1)x) + \frac{b-1}{(2b-1)(2b-2)} I(1, x)
\]

\[
= \frac{1}{(2b-1)} \exp(-(b-1)x) + \frac{b-1}{(2b-1)(2b-2)} I(1, x)
\]

\[
+ \frac{(b-1)(b-2)}{(2b-1)(2b-2)(2b-3)} \exp(-(b-3)x) + \ldots + \frac{(b-1)(b-2) \ldots 2}{(2b-1) \ldots (b+1)} \exp(-(b-1)x) + \frac{(b-1) \ldots 1}{(2b-1) \ldots (b+1)} I(b-1, x).
\]

However,

\[
I(b-1, x) = \frac{1}{b} \frac{1}{\exp(-x) + 1}.
\]

So we have

\[
B(b, b) F_{III,b}(x) = \frac{1}{b} \sum_{i=0}^{b-1} \frac{\exp(ix)}{(1 + \exp(-x))^{b+i}} \frac{b(b-1) \ldots (i+1)}{(2b-1)(2b-2) \ldots (i+b)}
\]

\[
= 2 \sum_{i=0}^{b-1} \frac{\exp(-ix)}{(1 + \exp(-x))^{b+i}} \frac{bl(i+b-1)!}{i!(2b)!}.
\]

Notice now that

\[
\frac{2b!}{B(b, b) i!(2b)!} = \frac{(2b-1)!}{(2b)!} \frac{b!}{(b-1)!} \frac{(i+b-1)!}{i!(b-1)!} = \binom{i+b-1}{b-1}.
\]
Hence,

\[ F_{III,b}(x) = \sum_{i=0}^{b-1} \binom{i+b-1}{b-1} \frac{\exp(-ix)}{(\exp(-x) + 1)^{b+i}}, \]

which proves the theorem. □

The proof given above is a constructive proof. It is easy to verify directly that \( F_{III,b} \) as defined in (7) has density \( f_{III,b} \). Firstly, by differentiation of (7) it follows that:

\[
\frac{dF_{III,b}(x)}{dx} = \sum_{i=0}^{b-1} \frac{(b-1+i)! (b+i) \exp(-(i+1)x) - i \exp(-ix)(\exp(-x)+1)}{(b-1)!i! (\exp(-x) + 1)^{b+1+i}}.
\]

Notice now that the sum of the first part of term \( i \) and the second part of the next term \( i+1 \) equals zero. Hence

\[
\frac{dF_{III,b}(x)}{dx} = \frac{(2b-1)! \exp(-bx)}{((b-1)!)^2 (\exp(-x) + 1)^{2b}},
\]

which equals, as a matter of fact, \( f_{III,b}(x) \).

3.1 Approximation of the normal distribution

Suppose that \( X \sim F_{III,b} \). Let \( \sigma_b \) be the standard deviation of \( X \). Then \( X/\sigma_b \) has mean zero and variance 1 with cdf \( F_{III,b}(\sigma_b \cdot) \). By the CLT we know that \( F_{III,b}(\sigma_b \cdot) \) converges to the cdf of the \( N(0,1) \) for \( b \to \infty \). Therefore one could use \( F_{III,b}(\sigma_b \cdot) \) as an approximation of the \( N(0,1) \) cdf. More general, the \( F_{III,b} \) family can be used to approximate the \( N(\mu, \sigma^2) \) for any \( \mu, \sigma^2 \). It is of interest to know how good this approximation is for finite \( b \). Since \( f_{III,1} \), i.e. the standard logistic distribution, is already close to the \( N(0,1) \) density one expects a very good approximation. The results are given by table 1. It shows that \( F_{III,2} \) differs from the cdf of the \( N(0,1) \) at the given quantiles by not more than 0.012 in the middle of the distribution and the differences become essentially smaller towards the tail. In fact, \( F_{III,2} \), which is just a sum of two terms, is visually hardly distinguishable from the standard normal cdf. Moreover, note that \( F_{III,5} \) is for practical purposes a perfect approximation of the cdf of \( N(0,1) \) whose density is visually and empirically not distinguishable from the standard normal cdf. The differences are always of the order 0.001 or smaller. Because \( F_{III,5} \) is computed by a simple sum of 5 terms this provides us with a representation of the standard normal cdf which can be quickly computed. In particular, this means that our exact formula for \( P^*(d) \) with (say) \( b = 5, d \) the selection constant, can also be used for subset selection from normal populations which only differ in their location parameter \( \mu_i, i = 1, \ldots, k \), based on equal size sample means. Similarly, this holds for the exact formula for the expectation of the size of the subset (see section 1).

As remarked in section 2 it some cases it is profitable to develop exact theories under the logistic distribution without causing a too great discrepancy with the corresponding normal theory. By developing exact theories for the family

\[ F_{III,5} \left( \sigma_S \frac{x - \mu}{\sigma} \right), \quad \mu \in \mathbb{R}, \sigma > 0 \]

this discrepancy is even negligible. Of course, one might even be satisfied with \( F_{III,2} \) as an approximation to the normal family.
In this section explicit expressions for the minimal probability of correct selection will be derived for Type-I, Type-II and Type-III generalized logistic populations.

Table 1: Below we report the values of \( F_{III}(x \sigma \text{b}) \), where \( \sigma^2 = \text{VAR}_{FIII,b}(X) \), at the values \( x \in \{0.38, 0.70, 1.645, 1.96, 2.33\} \). These values correspond with the 0.6480, 0.7580, 0.95, 0.9750, 0.9901 quantiles of the \( \mathcal{N}(0, 1) \). The last two rows give the differences with the normal quantiles for \( b = 5 \) and \( b = 20 \).

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<th>0.7580</th>
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</table>

| Diff, b=5 | -0.003505 | -0.004725 | 0.000461 | 0.000734 | 0.001181 |
| Diff, b=20 | -0.000875 | -0.001188 | 0.000121 | 0.000194 | 0.000311 |

The extended family

\[
\frac{1}{\sigma} f_{III,b} \left( \frac{x - \mu}{\sigma} \right), \; \mu \in \mathbb{R}, \sigma > 0, \; b \in \mathbb{N}
\]

can be used as a statistical model which includes the normal family. When \( b \) ranges from 0 till 5 the tail ranges from the logistic tail to the normal tail. This simple model allows one to estimate and test the tail behavior of the underlying random variable.

4 Subset selection for generalized logistic populations

In this section explicit expressions for the minimal probability of correct selection will be derived for Type-I, Type-II and Type-III generalized logistic populations.
4.1 Subset selection from Type-I logistic populations.

For Gupta's subset selection rule given in section 1 the minimal probability of correct selection for Type-I generalized logistic populations described by (5) is given by

\[ I(b) = \int_{-\infty}^{\infty} F_{I}^{k-1}(x + d) f_{I}(x) \, dx, \]

where \( d \) is the selection constant. We get

\[ I(b) = \int_{-\infty}^{\infty} \frac{be^{-x}}{(1 + e^{-x})^{b+1}(1 + e^{-x-d})^{b(k-1)}} \, dx. \]

We have the following theorem.

**Theorem 4.1** We have for integer \( b (> 0) \)

\[ I(b) = 1 + \left( \frac{b(k - 1) + b - 2}{b - 1} \right) \times \]
\[ \left[ \sum_{i=1}^{b-1} (-1)^i \left( \frac{b(k - 1) + b - 2}{b - i - 1} \right)^{-1} a^{-i} + (-1)^{b} \{ b(k - 1) + b - 1 \} a^{-b} C_{b(k-1)+b-1}(a) \right], \]

where \( a = e^d, C_m(c) = \left( \frac{c}{c - 1} \right)^{m+1} \cdot \ln c - \sum_{i=1}^{m} \frac{1}{i} (1 - \frac{1}{c})^i \}, \sum_{i=1}^{m} \frac{1}{i} (1 - \frac{1}{c})^i = 0 \) for integer \( m \leq 0 \) and \( c > 0 \).

For \( b = g + \frac{1}{2} \), with \( g \) is an integer \((\geq 0)\), and \( k \) is odd we have

\[ I(b) = 1 + \frac{1}{bka^{1/2}} \sum_{j=0}^{b-1} \left( \frac{2bk(2bk - 2) \cdots (2bk - 2j)}{(2bk - 1)(2bk - 3) \cdots (2bk - 1 - 2j)} \left( \frac{a}{a - 1} \right)^{j+1} \right), \]

with \( n!! = n(n-2)\cdots 1 \) for \( n \) is odd and \( n(n-2)\cdots 2 \) for \( n \) is even, and with \((b-1)\cdots \frac{1}{2} = 1\) for \( b-1 < \frac{1}{2} \), thus the case \( b = \frac{1}{2} \).

For \( b = g + \frac{1}{2} \) and \( k \) is even, we have

\[ I(b) = 1 + \frac{1}{bka^1/2} \sum_{j=0}^{b-1} \left( \frac{2bk(2bk - 2) \cdots (2bk - 2j)}{(2bk - 1)(2bk - 3) \cdots (2bk - 1 - 2j)} \left( \frac{a}{a - 1} \right)^{j+1} \right), \]

with \((2bk - 1)\cdots 3 = 1 \) for \( 2bk - 1 < 3 \), thus the case \( k = 2 \) and \( b = \frac{1}{2} \).

**Proof:** See Appendix.
4.2 Subset selection from Type-II logistic populations

Let \( X_1, \ldots, X_k \) be \( k \) random variables associated with \( k \) Type-II logistic distributions \( \pi_1, \ldots, \pi_k \) which only differ in their location parameter. Define \( Z_1 \equiv -X_1, \ldots, Z_k \equiv -X_k \) and recall that the \( Z_i \)'s have a Type-I logistic distribution. Notice now that Gupta's subset selection rule on the \( X_i \)'s is in terms of the \( Z_i \)'s given by:

Select \( \pi_i \) if and only if \( Z_i \leq \min_{1\leq j\leq k} Z_j + d \).

Correct selection means now that one selects the \( Z_i \) with minimal location parameter. The minimal probability of correct selection (i.e. we select the \( \pi_i \) with minimal location parameter) is now given by:

\[
II(b) = \int_{-\infty}^{\infty} (1 - F_I)^{k-1}(x + d) dF_I(x).
\]

Denote the integral (7) with \( I(b, k) \) to stress the dependence of this minimal probability for correct selection on \( k \):

\[
I(b, k) \equiv \int_{-\infty}^{\infty} F_I^{k-1}(x + d) f_I(x) \, dx.
\]  

Because

\[
(1 - F_I)^{k-1} = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j F_I^j
\]

we can now express \( II(b) \) in terms of \( I(b, k) \) which is explicitly given in Theorem 4.1. This proves the following corollary of Theorem 4.1.

**Corollary 4.1** Let

\[
II(b) = \int_{-\infty}^{\infty} F_{II}^{k-1}(x + d) dF_{II}(x)
\]

be the minimal probability of correct selection from \( k \) Type-II logistic populations. Let \( I(b, k) \), as defined by (10), be the minimal probability of correct selection from \( k \) Type-I logistic populations. We have

\[
II(b) = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j I(b, j + 1).
\]

By Theorem 4.1 this provides us with an explicit solution for \( II(b) \), \( b \) integer and \( b = g + 0.5 \), \( g \) integer.

4.3 Subset selection for Type-III generalized logistic populations

For Gupta's subset selection rule given in section 1 the minimal probability of correct selection for Type-III generalized logistic populations is given by

\[
III(b) = \int_{-\infty}^{\infty} F_{III}^{k-1}(x + d) f_{III}(x) \, dx,
\]
where \( d > 0 \) is the selection constant. The theorem below provides us with an explicit formula for \( III(b) \) in terms of binomial and multinomial coefficients. For notational convenience, we will denote the multinomial coefficients with \( C^{k-1}_{i_0, \ldots, i_{b-1}} \):

\[
C^{k-1}_{i_0, \ldots, i_{b-1}} = \binom{k-1}{i_0 \ i_1 \ \ldots \ i_{b-1}}, \text{ with } 0 \leq i_0, \ldots, i_{b-1} \leq k-1 \text{ and } \sum_{j=0}^{b-1} i_j = k - 1
\]

and the binomial coefficient with \( C^n_k \):

\[
C^n_k \equiv \binom{n}{k}.
\]

We have the following theorem, which is proved in the appendix:

**Theorem 4.2** Let \( \alpha = \exp(-d) \). Let \( C^n_k, C^{k-1}_{i_0, \ldots, i_{b-1}} \) be the binomial and multinomial coefficients as defined in (13) and (12), respectively.

Define

\[
C_2 = C_2(k-1, i_0, \ldots, i_{b-1}) \equiv C^{b-1}_{i_0, \ldots, i_{b-1}} (C^b_{b-1})^{i_0} (C^b_{b-1})^{i_1} \ldots (C^2_{b-1})^{i_{b-1}}.
\]

Moreover, define

\[
s = s(i_0, \ldots, i_{b-1}) \equiv \sum_{j=0}^{b-1} j i_j.
\]

We have that:

\[
III(b) = \frac{\exp(-bd)}{B(b, b)} \sum_{j=0}^{b-1} C_2 \sum_{i=0}^{s + b - 1} \binom{s + b - 1}{i} (-1)^i A(b(k-2)+i+1, 2b, \alpha),
\]

where \( b(k-2)+i+1 \geq 1 \text{ and } 2b \geq 2 \). Recall that \( C_2 = C_2(k-1, i_0, \ldots, i_{b-1}) \) and \( s = s(i_0, \ldots, i_{b-1}) \). Here \( A(b(k-2)+i+1, 2b, \alpha) \) is given by the following formulas for \( A(p, q, \alpha) \) for positive integers \( p, q \) with \( p \geq 1, q \geq 2 \): We have

\[
A(1, q, \alpha) = \frac{\alpha^{-q+1}}{q-1} - \frac{1}{q-1} A(2, q-1, \alpha), \text{ if } q \geq 3
\]

\[
A(1, 2, \alpha) = \frac{1 - \alpha + \alpha \log(\alpha)}{(\alpha - 1)^2 \alpha}.
\]

For \( q \geq 2 \):

\[
A(2, q, \alpha) = \frac{\alpha - q C(q)(\alpha)}{\alpha^{q+1}},
\]

where

\[
C(q)(\alpha) = \left(\frac{\alpha}{\alpha - 1}\right)^{q+1} \left\{ \log(\alpha) - \sum_{i=1}^{q} \frac{1}{i} (1 - \frac{1}{\alpha})^i \right\}.
\]

If \( p \geq 3 \) and \( q \geq 2 \), then we have

\[
A(p, q, \alpha) = \frac{\alpha^{-q}}{p-1} + \sum_{i=1}^{p-3} \frac{(-1)^i q(q+1) \ldots (q+i-1)}{(p-1)(p-2) \ldots (p-1-i)} \alpha^{-(q+i)}
\]

\[
+ (-1)^{p-2} \frac{q(q+1) \ldots (q+p-3)}{(p-1)(p-2) \ldots 2} A(2, q+p-2, \alpha).
\]

Here the summation equals zero if \( p = 3 \).
5 Concluding remarks and a few open problems

The Type-I and Type-II generalized logistic distribution can be used as a probabilistic model in a number of practical situations. Varying the parameter $b$ a large number of interesting alternatives of the logistic can be obtained. If $X_1, \ldots, X_k$ have negatively or positively skewed generalized logistic distributions which only differ in their location parameter, then our results provide the probability of correct selection (see (1)) for the subset selection approach as well as for the indifference zone approach. If $X_1, \ldots, X_k$ are the minimum (or maximum) of a sample of size $b$ from logistic populations which only differ in their location parameter, then Theorem 4.1 can be used to compute the exact probability of correct selection $P^*$. Using the minimum or maximum in selecting the best population might be sensible in cases where their is a lot of censoring or if $b$ is small. In particular, we believe that using the minimum in subset selection is useful in survival time applications described below. Suppose that there are $k$ treatments available for a particular type of patient. Assume that it is reasonable to assume that the survival times have logistic distributions which only differ in their location parameter.

In order to compare the treatments one constructs $k$ equal size treatment groups and one starts observing the survival time of the patients. When one has observed the first failure in each group one can construct a subset which contains with probability (at least) $P^*$ the best treatment, using our exact results for correct selection. In situations where the lifetime of a person is at stake it is very important to have such methods available which do not need all the data so that certain treatments can already be excluded. In AIDS-applications one is often confronted with a new treatment (where this treatment might be split up in several treatments by dose) and one wants to determine if this treatment improves the mean survival time relative to the old treatment. Again, in such applications it is important that one does not need to finish the experiments before one can draw conclusions; this is especially true since these experiments require full cooperation of the patients. In light of these potential applications of subset selection based on order statistics we believe that it might be worthwhile to develop exact results for popular distributions in survival time applications like the Weibull and exponential distribution.

Another application for the Type-I and Type-II logistic distributions is the following. Consider the situation where $X_1, \ldots, X_k$ are sample means based on small samples such that they are not normally distributed, yet. In this case, the parametric or nonparametric (depending on how much is known about the distributions we have been sampling from) bootstrap could be used to fit a Type-I or Type-II logistic distribution and the probability of correct selection could be computed under the fitted parameter $b$. The performance of such a method remains to be investigated, but one can expect better results than the one obtained by (wrongly) assuming normality.

Exact results for $P^*$ for $F_{III,b}$ the cdf of the Type-III generalized logistic distribution with parameter $b$, is of interest because of the following reasons. If we are interested in selecting the best of $k$ logistic populations which only differ in their location parameter and we have a sample of size $b$ from each of these populations, then the median seems to be a perfect candidate for $X_i$, $i = 1, \ldots, k$ (see Lorenzen and McDonald, 1981). The median has in this case a smaller variance than, for example, the mean, in contrary with the normal distribution where the mean is known to be an efficient estimator of its median (which equals its mean). Since the median has the Type-III distribution an exact result for $P^*$ is very useful. Secondly, obtaining an exact result for $P^*$ for the Type-III leads to a similar result for the normal distribution by the fact that the approximation is almost perfect.

Above we mentioned that the median is a better candidate for $X_i$, $i = 1, \ldots, k$, than
the mean if we are concerned with subset selection from logistic populations and we have a sample of size \( b \) available from each population. Let \( K \) be an increasing function on \((-\infty, \infty)\) and \( K(0) = 0 \). Let \( F \) be a logistic distribution and define \( \mu \) by

\[
\int K(x - \mu) dF(x) = 0.
\]

Then \( \mu \) is a location parameter of \( F \). If \( K(x) = x \), then \( \mu \) is the mean of \( X \) and if \( K(x) = \text{sign}(x) \), then \( \mu \) is the median. By varying \( K \) we obtain all possible location parameters. Let \( \mu_n \) be an efficient estimator of \( \mu \) like the MLE. The best choice of location parameter for the purpose of subset selection is the one which can be estimated most precisely. We wonder if the median is from this point of view the best location parameter to choose. In order to answer this question we could compute the Cramér-Rao lower bound for \( \mu \), corresponding with \( K \), the model being the logistic family. Then one could try to minimize this lower bound at the standard logistic distribution over an interesting class of \( K \)'s.

Finally, we discuss subset selection with right censored data. Let \( T_1, \ldots, T_k \) be \( k \) survival time random variables associated with \( k \) different treatments. Assume that we have \( n \) i.i.d. copies of each \( T_i \), \( i = 1, \ldots, k \); say we have \( k \) treatment groups of \( n \) patients. We assume that each individual is subject to right censoring by a censoring variable with distribution \( G \) (the same distribution for each treatment group): for individual \( j \) in group \( i \) we observe

\[
(T_{ij} = \hat{T}_{ij} \wedge C_{ij}, I(T_{ij} = \hat{T}_{ij})),
\]

where \( C_{ij} \sim G, i = 1, \ldots, k, j = 1, \ldots, n \). We are concerned with selecting the population with maximal \( \mu_i^* = E(T_i) \).

In this case we could estimate a location parameter \( \mu_i \), like the median or some other quantile, with the Kaplan-Meier estimator for each group \( i \), \( i = 1, \ldots, k \). It is well known that \( \sqrt{n}(\mu_{i,KM} - \mu_i) \) is asymptotically normal with mean zero and a variance \( \sigma_i^2 \) which is a known function of \( F_i \) and \( G \): Gill (1989). Suppose treatment group 1 has a smaller mean than treatment group 2. Then the treatment group 2 will be subject to more censoring. Therefore the variance of \( \mu_{1,KM} \) is smaller than the variance of \( \mu_{2,KM} \). Assume that \( \mu_{i,KM} \) is approximately normally distributed (say \( n \) is large). Then the minimal probability of correct selection for Gupta's selection rule applied to \( \mu_{i,KM} \), corresponding with the least favourable configuration \( \mu_1 = \ldots = \mu_k \), is given by:

\[
P^* = \int \prod_{i=1}^{k-1} F((x + d)/\sigma_i) dF(x/\sigma_k),
\]

\( F \) being the standard normal cumulative distribution. Hence \( P^* \) involves unknown variances \( \sigma_i^2, i = 1, \ldots, k \), which need to be estimated from the data. In order to obtain a conservative estimate of \( P^* \) one could replace \( \sigma_i \) by the the right-end point of a 95% confidence interval for \( \sigma_i, i = 1, \ldots, k \).

**Appendix: Type-I.**

The proof of Theorem 4.1.

First, we consider the case \( b \) is an integer (\( > 0 \)). By doing the transformation \( z = \exp(-x) \) we obtain

\[
I(b) = ba^{b(k-1)} \int_0^{\infty} (z + 1)^{-b-1}(z + a)^{-b(k-1)} dz .
\]
with \( a = e^d \). Repeated partial integration yields

\[
I(b) = 1 - b(k - 1)a^{b(k-1)} \int_0^\infty (z + 1)^{-b} (a + z)^{-b(k-1)-1} dz
\]

\[
= 1 - \frac{b(k - 1)}{(b - 1)a} + \frac{b(k - 1)(b(k - 1) + 1)}{(b - 1)(b - 2)a^2} + \ldots + (-1)^{b-2} \frac{b(k - 1)(b(k - 1) + 1)\ldots b(k - 1) + b - 3}{(b - 1)(b - 2)\ldots 2 \cdot a^{b-2}} + \\
+ (-1)^{b-1} \frac{b(k - 1)(b(k - 1) + 1)\ldots b(k - 1) + b - 2}{(b - 1)(b - 2)\ldots 2} a^{b(k-1)} \int_0^\infty (z + 1)^{-2} (a + z)^{-b(k-1)-b+1} dz.
\]

Using Theorem 1 in Van der Laan (1992) provides us with

\[
I(b) = 1 + \sum_{i=1}^{b-1} (-1)^i \frac{b(k - 1)\ldots (b(k - 1) + i - 1)}{(b - 1)\ldots (b - i)a^i} + (-1)^b \frac{b(k - 1)\ldots (b(k - 1) + b - 1)}{(b - 1)\ldots 1 \cdot a^b} e_{b(k-1)+b-1} (a)
\]

\[
= 1 + \sum_{i=1}^{b-1} (-1)^i \left( \frac{b(k - 1) + b - 2}{b - 1} \right) \left( \frac{b(k - 1) + b - 2}{b - i - 1} \right)^{-1} a^{-i+i} + \\
+ (-1)^b \left( \frac{b(k - 1) + b - 1}{b} \right) a^{-b} e_{b(k-1)+b-1} (a)
\]

\[
= 1 + \left( \frac{b(k - 1) + b - 2}{b - 1} \right) \left[ \sum_{i=1}^{b-1} (-1)^i \left( \frac{b(k - 1) + b - 2}{b - i - 1} \right)^{-1} a^{-i+i} + \\
+ (-1)^b \{b(k - 1) + b - 1\} a^{-b} e_{b(k-1)+b-1} (a) \right].
\]

Secondly, we consider the case \( b = g + \frac{1}{2} \), with integer \( g \geq 0 \). We have

\[
I(b) = 1 + \sum_{i=1}^g (-1)^i \frac{b(k - 1)\ldots (b(k - 1) + i - 1)}{(b - 1)\ldots (b - i)a^i} + (-1)^{g+1} \frac{b(k - 1)\ldots (b(k - 1) + g)}{(b - 1)\ldots (b - g)} a^{b(k-1)} J(p),
\]

where

\[
J(p) = \int_0^\infty (z + 1)^{-\frac{1}{2}} (a + z)^{-p} dz \quad \text{with} \quad p = kb + \frac{1}{2} \left( \frac{3}{2} \right).
\]

We have two possibilities, namely i) \( k \) is odd thus \( p \) is an integer \((\geq 2)\), and ii) \( k \) is even thus \( p = h + \frac{1}{2} \) with \( h \) an integer \((\geq 1)\).

For case i) we obtain, using the transformation \((z + 1)^{\frac{1}{2}} = x \sqrt{a - 1}\),

\[
J(p) = 2(a - 1)^{-p+\frac{1}{2}} \int_{(a-1)^{-\frac{1}{2}}}^\infty (x^2 + 1)^{-p} dx,
\]

and
\[ K(p) = \int_{a^{-\frac{1}{2}}}^{\infty} (x^2 + 1)^{-p} dx = \int_{a^{-\frac{1}{2}}}^{\infty} (x^2 + 1)^{-p+1} dx - \int_{a^{-\frac{1}{2}}}^{\infty} x^2 (x^2 + 1)^{-p} dx \]

\[ = -\frac{1}{2(p-1)} \frac{(a-1)^{p-\frac{3}{2}}}{a^{p-1}} + \frac{2p-3}{2(p-2)} K(p-1) \quad \text{(for } p \geq 2) . \]

Working out this recurrence relation yields

\[ K(p) = \frac{(2p-3)!!}{2^{p-1}(p-1)!} K(1) - \frac{1}{2(p-1)} \frac{(a-1)^{p-\frac{3}{2}}}{a^{p-1}} \sum_{i=0}^{p-2} \frac{(2p-1)\cdots(2p-1-2i)}{(2p-2)\cdots(2p-2-2i)} \left( \frac{a}{a-1} \right)^i \]

\[ = \frac{(2p-3)!!}{2^{p-1}(p-1)!} \left( \frac{\pi}{2} - \text{bgtan} \frac{1}{\sqrt{a-1}} \right) - \frac{1}{(p-\frac{1}{2})a^p} \sum_{i=0}^{p-2} \frac{(2p-1)\cdots(2p-1-2i)}{(2p-2)\cdots(2p-2-2i)} \left( \frac{a}{a-1} \right)^{p-2-i} , \]

and

\[ J(p) = \frac{(2p-3)!!}{2^{p-2}(p-1)!(a-1)^{p-\frac{1}{2}}} \left( \frac{\pi}{2} - \text{bgtan} \frac{1}{\sqrt{a-1}} \right) - \frac{1}{(p-\frac{1}{2})a^{p+1}} \sum_{i=0}^{p-2} \frac{(2p-1)\cdots(2p-1-2i)}{(2p-2)\cdots(2p-2-2i)} \left( \frac{a}{a-1} \right)^{i+1} . \]

Thus

\[ J(b) = 1 + \sum_{i=1}^{b} (-1)^i \frac{b(k-1)\cdots b(k-1) + i - 1}{(b-1)\cdots(b-i)a^i} + \]

\[ + (-1)^{b+\frac{1}{2}} \frac{b(k-1)\cdots(b(k-1) + b - \frac{1}{2})}{(b-1)\cdots\frac{1}{2}} a^{b(k-1)} \left[ \frac{2bk}{2^{bk-\frac{3}{2}}(bk-\frac{1}{2})!(a-1)bk} \left( \frac{\pi}{2} - \text{bgtan} \frac{1}{\sqrt{a-1}} \right) - \right. \]

\[ - \frac{1}{bk\alpha^{bk+\frac{1}{2}}} \sum_{j=0}^{bk-\frac{3}{2}} \frac{2bk(2bk-2)\cdots(2bk-2j)}{(2bk-1)(2bk-3)\cdots(2bk-1-2j)} \left( \frac{a}{a-1} \right)^{j+1} \right] . \]

For case ii) we have

\[ K(p) = -\frac{1}{2(p-1)} \frac{(a-1)^{p-\frac{3}{2}}}{a^{p-1}} - \frac{2p-3}{(2p-2)(2p-4)} \frac{(a-1)^{p-\frac{3}{2}}}{a^{p-2}} - \]

\[ \cdots - \frac{(2p-3)(2p-5)\cdots4}{(2p-2)(2p-4)\cdots3} \frac{a-1}{a^\frac{3}{2}} + \frac{(2p-3)(2p-5)\cdots2}{(2p-2)(2p-4)\cdots3} K\left( \frac{3}{2} \right) \quad \text{(for } p \geq \frac{5}{2} ) , \]

with

\[ K\left( \frac{3}{2} \right) = \int_{\frac{1}{\sqrt{a-1}}}^{\infty} (x^2 + 1)^{-\frac{3}{2}} dx \]

\[ = 4 \int_{0}^{\infty} \frac{y}{(y^2 + 1)^2} dy \]
using the transformation \((x^2 + 1)^{\frac{1}{2}} = x + y\).
Some basic calculations yield
\[
K(\frac{3}{2}) = 1 - \frac{1}{\sqrt{a}}.
\]
Thus
\[
I(b) = 1 + \sum_{i=1}^{b-\frac{1}{2}} (-1)^i \frac{b(k-1) \cdots (b(k-1)+i-1)}{(b-1) \cdots (b-i)a^i}
+
\]
\[
+(-1)^{b+\frac{1}{2}} \frac{b(k-1) \cdots (b(k-1)+b-\frac{1}{2})}{(b-1) \cdots \frac{1}{2}} a^{b(k-1)}2(a-1)^{-bk} \left[ - \frac{1}{2p-2} \frac{(a-1)^{p-\frac{3}{2}}}{a^{p-2}} \right.
- \frac{2p-3}{(2p-2)(2p-4)} \frac{(a-1)^{p-\frac{3}{2}}}{a^{p-2}} - \cdots
- \frac{(2p-3) \cdots 4}{(2p-2) \cdots 3} \frac{a-1}{a^{\frac{3}{2}}} + \frac{(2p-3) \cdots 2}{(2p-2) \cdots 3} \left( 1 - \frac{1}{\sqrt{a}} \right) \right]
= 1 + \sum_{i=1}^{b-\frac{1}{2}} (-1)^i \frac{b(k-1) \cdots (b(k-1)+i-1)}{(b-1) \cdots (b-i)a^i}
+
\]
\[
+(-1)^{b+\frac{1}{2}} \frac{b(k-1) \cdots (b(k-1)+b-\frac{1}{2})}{(b-1) \cdots \frac{1}{2}} \left[ - \frac{1}{2bk-1} \frac{1}{a^{b-\frac{1}{2}}(a-1)} - \frac{2bk-2}{(2bk-1)(2bk-3)} \frac{1}{a^{b-\frac{1}{2}}(a-1)^2} - \cdots
- \frac{(2bk-2) \cdots 4}{(2bk-1) \cdots 3} \frac{1}{a^{b-bk}(a-1)^{bk-1}} + \frac{(2bk-2) \cdots 2}{(2bk-1) \cdots 3} \frac{1}{a^{b-bk}(a-1)^{bk}} \left( 1 - \frac{1}{\sqrt{a}} \right) \right]
= 1 + \sum_{i=1}^{b-\frac{1}{2}} (-1)^i \frac{b(k-1) \cdots (b(k-1)+i-1)}{(b-1) \cdots (b-i)a^i}
+(-1)^{b-\frac{1}{2}} \frac{b(k-1) \cdots (b(k-1)+b-\frac{1}{2})}{b(b-1) \cdots \frac{1}{2}}
\]
\[
\left[ \sum_{j=0}^{bk-2} \frac{2bk(2bk-2) \cdots (2bk-2j)}{(2bk-1)(2bk-3) \cdots (2bk-1-2j)} \frac{a^{j-b+\frac{1}{2}}}{(a-1)^{j+1}} - \frac{2bk(2bk-2) \cdots 2}{(2bk-1) \cdots 3} \frac{1}{a^{b-bk}(a-1)^{bk}} \left( 1 - \frac{1}{\sqrt{a}} \right) \right],
\]
with \((2bk-1) \cdots 3 = 1\) for \(2bk-1 < 3\) (thus the case \(k = 2\) and \(b = \frac{1}{2}\)).

Appendix: Type-III.

Proof of theorem 4.2. Substitution of the representation (7) for \(F_{III}\) into (11) provides us with:
\[
III(b) = \int_{-\infty}^{\infty} \left\{ \sum_{i=0}^{b-1} C_{b-1}^{i+b-1} \frac{\exp(-i(x+d))}{(\exp(-x-d)+1)^{b+i}} \right\}^{k-1} \frac{1}{B(b,b)} \frac{\exp(-bx)}{(1+\exp(-x))^{2b}} dx. \quad (14)
\]
We define
\[
a_i(x) \equiv C_{b-1}^{i+b-1} \frac{\exp(-i(x+d))}{(\exp(-x-d)+1)^{b+i}}, \quad i = 0, \ldots, b-1.
\]
Then we can represent the term between accolades in (14) as $\sum_{i=0}^{b-1} a_i(x)$. Then

$$\left(\sum_{i=0}^{b-1} a_i(x)\right)^{k-1} = \sum_{i_0 \ldots i_{b-1} \geq 0, \sum_{j=0}^{b-1} i_j = k-1} C_{i_0 \ldots i_{b-1}}^{k-1} a_0(x)^{i_0} a_1(x)^{i_1} \ldots a_{b-1}(x)^{i_{b-1}}.$$

We can substitute this expression into (14) to obtain:

$$III(b) = \frac{1}{B(b, b)} \sum_{i_0 \ldots i_{b-1} \geq 0, \sum_{j=0}^{b-1} i_j = k-1} C_{i_0 \ldots i_{b-1}}^{k-1} a_0(x)^{i_0} a_1(x)^{i_1} \ldots a_{b-1}(x)^{i_{b-1}} \frac{\exp(-bx)}{(1 + \exp(-x))^{2b}} dx.$$

Hence, it remains to determine

$$M(i_0, \ldots, i_{b-1}) \equiv \int_{-\infty}^{\infty} a_0(x)^{i_0} a_1(x)^{i_1} \ldots a_{b-1}(x)^{i_{b-1}} \frac{\exp(-bx)}{(1 + \exp(-x))^{2b}} dx.$$

We have that $a_0(x)^{i_0} a_1(x)^{i_1} \ldots a_{b-1}(x)^{i_{b-1}}$ is given by:

$$\left(C_{i_0 \ldots i_{b-1}}^{b-1}\right)^{i_0} \left(C_{b-i_0 \ldots i_{b-1}}^{b-1}\right)^{i_1} \ldots \left(C_{b-i_0 \ldots i_{b-1}}^{b-1}\right)^{i_{b-1}} \frac{\exp(-\sum_{j=0}^{b-1} j i_j (x+d))}{(1 + \exp(-(x+d)))^{\sum_{j=0}^{b-1} j (b+j)}} \equiv c_i(i_0, \ldots, i_{b-1}) \frac{\exp(-\sum_{j=0}^{b-1} j i_j (x+d))}{(1 + \exp(-(x+d)))^{\sum_{j=0}^{b-1} j (b+j)}}.$$

Notice that $\sum_{j=0}^{b-1} i_j (b+j) = s(i_0, \ldots, i_{b-1}) + b(k-1)$. Let $k' \equiv k-1$. So we have that $III(b)$ is given by:

$$\frac{1}{B(b, b)} \sum_{i_0 \ldots i_{b-1} \geq 0, \sum_{j=0}^{b-1} i_j = k-1} C_{i_0 \ldots i_{b-1}}^{k-1} \frac{\exp(-s(i_0, \ldots, i_{b-1})(x+d))}{(1 + \exp(-(x+d)))^{s(i_0, \ldots, i_{b-1}) + bk'/(1 + \exp(-x))^{2b}} dx.$$

So it remains to determine:

$$M_1(i_0, \ldots, i_{b-1}) \equiv \int \frac{\exp(-s(i_0, \ldots, i_{b-1})(x+d))}{(1 + \exp(-(x+d)))^{s(i_0, \ldots, i_{b-1}) + bk'/(1 + \exp(-x))^{2b}} dx.$$

Now, do the substitution $\exp(-x) = z$ (for notational convenience we leave out the indices $i_0, i_1, \ldots, i_{b-1}$ in $s(i_0, \ldots, i_{b-1})$):

$$M_1(i_0, \ldots, i_{b-1}) = \exp(bk') \int_0^{\infty} \frac{z^{s+b-1}}{(1+z)^{2b}(\exp(d)+z)^{s+b(k-1)}} dz \equiv \exp(bk')I_1.$$

Here $b \geq 1$ and $0 \leq s \leq (b-1)(k-1)$. Since $2b \geq 2$ and $s + b - 1 < s + b(k-1)$ the last integral $I_1$ is always convergent. Let $\alpha = \exp(-d)$. By substitution of $z = x/\alpha$ we obtain

$$I_1 = \alpha^{s+b(k-1)} \int_0^{\infty} \frac{z^{s+b-1}}{(1+z)^{2b}(1+\alpha z)^{s+b(k-1)}} dz = \alpha^{bk'} \int_0^{\infty} \frac{x^{s+b-1}}{(\alpha + x)^{2b}(1+x)^{s+b(k-1)}} dx.$$
Let \( n \equiv s + b - 1 \) and \( m \equiv s + b(k - 1) \). Notice that \( 0 \leq n \leq (2b - 1)(k - 1) \). By writing \( x^n = ((z + 1) - 1)^n \) it follows that

\[
I_1 = \alpha^{bk} \sum_{i=0}^{n} \binom{n}{i} (-1)^i \int_0^\infty \frac{dx}{(\alpha + x)^{2b(i + x)^{m - n + i}}}
\]

\[
= \alpha^{bk} \sum_{i=0}^{n} \binom{n}{i} (-1)^i A(m - n + i, 2b, \alpha) \quad \text{with} \quad m - n + i \geq 1 \quad \text{and} \quad b \geq 1.
\]

Notice that indeed \( m - n + i = b(k - 2) + i + 1 \geq 1 \).

Finally, we solve for \( A(p, q, \alpha) \) with \( p \geq 1 \) and \( q \geq 2 \). Theorem 1 in van der Laan (1992) provides us with the following lemma:

**Lemma 5.1** For positive integer \( q \) we have:

\[
A(2, q, \alpha) = \int_0^\infty \frac{1}{(z + 1)^2(z + \alpha)^q} \, dz = \frac{\alpha - q C_q(\alpha)}{\alpha^{q+1}},
\]

where

\[
C_q(\alpha) = \left( \frac{\alpha}{\alpha - 1} \right)^{q+1} \left\{ \log(\alpha) - \sum_{i=1}^{q} \frac{1}{i} (1 - \frac{1}{\alpha})^i \right\}.
\]

Now, let \( p \geq 3 \). Then by integration by parts we have:

\[
\int (z + 1)^{-p}(z + \alpha)^{-q} \, dz = \frac{\alpha^{-q}}{p-1} - \frac{q}{p-1} \int (z + 1)^{-p+1}(z + \alpha)^{-q-1} \, dz.
\]

By repeating this (as in the appendix for Type-I above) we obtain the following lemma:

**Lemma 5.2** For positive integers \( p, q, q \geq 2, p \geq 3 \) we have

\[
A(p, q, \alpha) = \frac{\alpha^{-q}}{p-1} + \sum_{i=1}^{p-3} (-1)^i \frac{q(q + 1) \cdots (q + i - 1)}{(p-1)(p-2) \cdots (p-i)} \alpha^{-q+i}
\]

\[
+ (-1)^{p-2} \frac{q(q + 1) \cdots (q + p - 3)}{(p-1)(p-2) \cdots 2} A(2, q + p - 2, \alpha),
\]

where \( A(2, q + p - 2, \alpha) \) is explicitly given in lemma 5.1. Here the summation equals zero if \( p = 3 \).

It remains to solve \( A(1, q, \alpha) \) for \( q \geq 2 \). If \( q \geq 3 \), then by integration by parts it follows that

\[
A(1, q, \alpha) = \frac{\alpha^{-q+1}}{q-1} - \frac{1}{q-1} A(2, q - 1, \alpha),
\]

where \( A(2, q - 1, \alpha) \) is explicitly given in lemma 5.1. The integral \( A(1, 2, \alpha) \) can be computed and is given in the following lemma.

**Lemma 5.3** We have

\[
A(1, q, \alpha) = \frac{\alpha^{-q+1}}{q-1} - \frac{1}{q-1} A(2, q - 1, \alpha) \quad \text{if} \quad q \geq 3
\]

\[
A(1, 2, \alpha) = \frac{1 - \alpha + \alpha \log(\alpha)}{(\alpha - 1)^2 \alpha}.
\]

This proves theorem 4.2.
References


