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Distribution planning for a divergent depotless two-echelon network under service constraints

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Abstract

In this paper we discuss a distribution planning procedure for a system consisting of one central depot supplying a number of end stockpoints. The central depot is not allowed to hold stock and allocates all incoming goods immediately to these end stockpoints. An ordering and allocation policy is presented which is based on a decomposition method. The emphasis lies on the realization of pre-determined target service levels at the end stockpoints. In this paper we present two adjustment methods which improve the service performance considerably in certain cases. Another important contribution of this paper is the generalization of the concept of imbalance. An analytical approximation of the probability of imbalance is presented. An extensive simulation study validates the analytical results.

Keywords: Imbalance; Inventory; Multi-echelon; Service; Simulation

1. Introduction

In this paper we consider a divergent two-echelon inventory model that operates according to a periodic review policy without batch size restrictions. This model applies to a distribution network consisting of a central depot which supplies a number of end stockpoints. No intermediate stocks are held at the central depot; i.e. stocks are only held at the most downstream stockpoints of the network, where customer demand is realized. Every order that arrives at the depot is immediately allocated to the end stockpoints. The depot serves as a pure distribution centre. The model also applies to situations where, according to a hierarchical product structure, two successive decisions are made in time concerning planned production for an aggregate production volume (family or group of products) and for individual products. Production planning for individual products obeys the aggregate production volume constraint defined at the preceding level. We restrict ourselves to the application of the model to a distribution environment. For hierarchical production planning application we refer to De Kok (1990).

The emphasis in this paper is on the determination of the echelon order-up-to-level at the central depot in order to ensure that the demand satisfied from stock on hand at the end stock-
points equals a pre-determined target level. These target levels may differ per end stockpoint. This approach differs from most approaches reported in the literature. Usually one defines a cost structure and the aim is to find cost-optimal policies (for example, Clark and Scarf, 1960; Eppen and Schrage, 1981; Zipkin, 1984; Federgruen and Zipkin, 1984; Rosling, 1989; Langenhoff and Zijm, 1990; Van Houtum and Zijm, 1991; Svoronos and Zipkin, 1991). In other cases one assumes that the stockout probability (i.e. the probability of negative stock immediately before the arrival of a replenishment order) is equal for all end stockpoints (Eppen and Schrage, 1981).

It should be noted that in practice neither of these approaches is applicable. The cost-optimal policies cannot be used since in most cases penalty costs for shortages are unknown (Tijms and Groenevelt, 1984). The equal stockout probability assumption is not valid, since in most practical cases one tends to differentiate service levels. More important, one uses the service criterion mentioned above, i.e. the fraction of demand satisfied from stock on hand (Tijms and Groenevelt, 1984; Silver and Peterson, 1985; De Kok, 1990, Lagodimos, 1992).

The contributions of this paper are the following. Firstly, the concept of imbalance is generalized. Instead of assuming equal stockout probabilities, we define imbalance as the occurrence of negative allocation quantities after application of a straightforward allocation policy. Through this generalization, we can in principle determine echelon policies that satisfy target service levels under any service criterion. Furthermore, we derive analytical approximations for the probability of a negative allocation to a particular stockpoint as an indication for the probability of imbalance. We note that the approach in this paper is based on the approach used in De Kok (1990) for tackling a related problem.

The paper is organized as follows. In Section 2 we introduce some important assumptions and definitions. In Section 3 we analyze the echelon policy for the two-echelon network we consider here (this analysis relies heavily on De Kok, 1990). Two improvement methods are presented in Section 4 to correct the computational deficiencies found by applying the original algorithm by De Kok (1990). In Section 5 attention is given to the important phenomenon of inventory imbalance. Finally, in Section 6 we present some conclusions and recommendations for further research. Throughout the paper simulation results are presented to validate the analytical results.

2. Assumptions and definitions

Throughout this paper we make use of a number of general assumptions. Similar assumptions have been made in most previous work in the area.
1. External demand is imposed at end stockpoints.
2. Demand at end stockpoints are independent stochastic variables, uncorrelated in time, with known mean ($\mu$) and standard deviation ($\sigma$).
3. All demand not satisfied from stock on hand is backlogged.
4. Lead times are constant.
5. There are no fixed order quantities nor capacity constraints.

We now introduce some important definitions. The net inventory of an end stockpoint is defined as the physical stock at this stockpoint minus backorders. The echelon inventory position of an end stockpoint is defined as the net inventory of this stockpoint plus all stock that has been allocated to this stockpoint but has not yet arrived. The echelon inventory position of the central depot is defined as the sum of the inventory positions of all end stockpoints plus the physical stock at the depot plus outstanding orders that have not yet arrived at the depot.

In this paper the central depot is not allowed to hold stock. Every order that arrives at the depot is immediately allocated to end stockpoints according to some allocation policy. There is, however, one case where the central depot is allowed to hold stock. That is when the central depot delivers directly to an external customer and therefore assumption (1) no longer holds. The stock at the depot is then exclusively reserved for this particular customer and may not be used for replenishment of end stockpoints.
This situation can be modelled as an extra end stockpoint (representing the external customer) with zero lead time.

The service criterion considered in this paper is the fraction of demand satisfied directly from stock on hand. This definition of service is considered to be the most widely used in practice (Tijms and Groenevelt, 1984; Silver and Peterson, 1985; Lagodimos, 1993).

The lead times are assumed to be constant. However, it can be shown that stochastic lead times for the end stockpoints can be implemented easily. A stochastic lead time for the central depot on the other hand complicates the analysis considerably when order crossing is not allowed.

### 3. System dynamics

In this section we analyse the echelon policy for the two-echelon model as reported by De Kok (1990). We apply the policy for a distribution environment, while De Kok (1990) uses a hierarchical production planning structure.

#### 3.1. The model

The network we consider is shown in Fig. 1. It consists of a central depot (CD) supplying N individual stockpoints where the external demand is realized. The central depot operates a periodic review $(R, S)$ ordering policy. Every shipment that arrives at the CD is immediately allocated and distributed to the end stockpoints, which may have different lead times. Management will try to realize specified service levels for every stockpoint. We refer to these desired service levels as target levels. These target levels may also differ for the various stockpoints. To describe the network operation we will use the following notation:

- $L$: lead time for CD,
- $L_k$: lead time for stockpoint $k$,
- $D_{k,s}$: demand at stockpoint $k$ during time period $[s-1, s)$,
- $\mu_k$: mean period demand at stockpoint $k$,
- $\sigma_k$: standard deviation of period demand at stockpoint $k$,
- $\hat{\beta}_k$: target level value for stockpoint $k$,
- $\beta_k$: service level measure at stockpoint $k$,
- $Z_s$: echelon inventory position of CD just before an order is issued by the CD at time $s$,
- $Z_{k,s}$: echelon inventory position of stockpoint $k$ just before the allocation decision at CD is taken at time $s$,

![Fig. 1. Two-echelon model.](image)

\[
D_0 := \sum_{k=1}^{N} \sum_{s=t+1}^{t+L} D_{k,s}
\]

(aggregate demand during $[t, t+L)$),

\[
D^{(1)}_k := \sum_{s=t+L+1}^{t+L+L_k} D_{k,s}
\]

(demand at stockpoint $k$ during $[t+L, t+L+L_k]$),

\[
D^{(2)}_k := \sum_{s=t+L+1}^{t+L+L_k+R} D_{k,s}
\]

(demand at stockpoint $k$ during $[t+L, t+L+L_k+R]$),

\[
\nu_k := E[D^{(2)}_k], \quad \nu_0 := \sum_{k=1}^{N} \nu_k.
\]

The parameters $R$, $L$, and $L_k$ are assumed to be nonnegative integers, but the lead times $L$ and $L_k$ do not have to be multiples of $R$. We can now examine the system operation over time. Since the CD uses a periodic $(R, S)$ policy, at the beginning of every review period of length $R$ its echelon inventory position is increased to an order-up-to-level $S$. So the quantity ordered by the CD at the beginning of a review period equals the
aggregate realized demand at all stockpoints during the previous review period. Suppose that at time $t$ the CD orders a quantity $Q_t$. Then $Q_t = S - Z_t$. (1)

At the second decision level, after the arrival of order $Q_t$ at time $t + L$ at the CD, we have to allocate the quantity $Q_t$ to $N$ different stockpoints. Let $q_{k,t+L}$ be the quantity allocated to stockpoint $k$. Since the depot holds no inventory:

$$\sum_{k=1}^{N} q_{k,t+L} = Q_t,$$ (2)

which implies that all arriving material at the CD is immediately allocated to the end stockpoints. Clearly, we need an allocation rule for determining these quantities $q_{k,t+L}$. For notational ease, we will omit in the remainder of this section the time subscript of these allocation quantities.

In order to achieve the pre-determined service performance, we need to specify the following parameters:

1. the order-up-to-level $S$ for the CD, and
2. the allocation rule to determine the quantities $q_k$ for the $N$ separate stockpoints.

De Kok (1990) introduced the concept of allocation fractions $p_k$ ($k = 1, \ldots, N$). The allocation fraction $p_k$ for end stockpoint $k$ represents the expected projected net inventory in end stockpoint $k$ (as a result of the allocation at the CD at time $t + L$) as a fraction of the expected aggregate projected net inventory in all end stockpoints (as a result of that same allocation at the CD).

$$p_k = \frac{Z_{k,t+L} + q_k - \nu_k}{\sum_{i=1}^{N} (Z_{i,t+L} + q_i - \nu_i)} = \frac{Z_{k,t+L} + q_k - \nu_k}{S - D_0 - \nu_0},$$ (3)

where $Z_{k,t+L} + q_k$ represents the echelon inventory position of stockpoint $k$ directly after the allocation decision at time $t + L$. The numerator represents the expected projected net inventory for stockpoint $k$, as a result of the allocation at time $t + L$. The denominator represents the expected aggregate projected net inventory in all stockpoints.

From Eq. (3) we have the following allocation rule:

$$q_k = p_k \{ S - D_0 - \nu_0 \} + \nu_k - Z_{k,t+L},$$ (4)

where:

$$0 \leq p_k \leq 1 \text{ and } \sum_{k=1}^{N} p_k = 1.$$

The inventory position of stockpoint $k$ directly after application of rule (4) now equals:

$$Z_{k,t+L} + q_k = \nu_k + p_k \{ S - D_0 - \nu_0 \}.$$ (5)

It is obvious from Eq. (5) that a fraction $p_k$ of the aggregate expected projected net inventory is allocated to stockpoint $k$. Application of allocation rule (4) should result in a quantity $q_k$ for stockpoint $k$ that is sufficient to realize a service level equal to $\beta_k$ in stockpoint $k$. Here we introduce an assumption whose importance is discussed in Section 3.2.

**Generalised balance assumption.** The allocation of the aggregate quantity $Q_t$ at the CD is such that all allocation quantities $q_k$ in (4) are positive ($k = 1, \ldots, N$).

It can easily be seen that the expected shortage in stockpoint $k$ in the time interval between two successive order arrivals equals

$$E \left[ \left( D_{k}^{(2)} - \left( Z_{k,t+L} + q_k \right) \right)^+ \right] - E \left[ \left( D_{k}^{(1)} - \left( Z_{k,t+L} + q_k \right) \right)^+ \right].$$ (6)

Eq. 6 represents the expected shortage in stockpoint $k$ at time $t + L + L_k + R$ (just before an order arrival) minus the expected shortage in stockpoint $k$ at time $t + L + L_k$ (directly after an order arrival). Using the definition of service level (i.e. fraction of demand satisfied from stock on hand, see e.g. De Kok, 1990), we get the following service level equation for stockpoint $k$:

$$\beta_k = 1 - \left[ E \left[ \left( D_{k}^{(2)} - \left( Z_{k,t+L} + q_k \right) \right)^+ \right] - E \left[ \left( D_{k}^{(1)} - \left( Z_{k,t+L} + q_k \right) \right)^+ \right] \right] \cdot \left( R \cdot \mu_k \right)^{-1},$$ (7)
where \( R \cdot \mu_k \) represents the expected demand in stockpoint \( k \) during a review period.

Substitution of (5) in (7) gives the following general expression for \( \beta_k \) (as a function of \( S \) and \( p_k \)):

\[
\beta_k = 1 - \left\{ \mathbb{E}\left[ \left( (D_k^2 + p_k D_0) - (p_k S - p_k v_0 + v_k) \right)^+ \right] - \mathbb{E}\left[ \left( (D_k^{(1)} + p_k D_0) - (p_k S - p_k v_0 + v_k) \right)^+ \right] \right\} \cdot \{ R \cdot \mu_k \}^{-1}. \tag{8}
\]

3.2. The generalised balance assumption

The generalised balance assumption that is stated in the model description differs significantly from the balance assumption commonly used by other authors (Eppen and Schrage, 1981; Van Donselaar and Wijngaard, 1986; Jönsson and Silver, 1987; Lagodimos, 1992). They define the allocation assumption as the situation in which the allocation quantities are sufficient to ensure equal stockout probabilities for all stockpoints. In this paper we use a different definition of service level (fraction of demand delivered from stock on hand) and we allow for different target levels (service levels to be realized) for the end stockpoints. The generalised balance assumption states that these target levels can be realized with positive allocation quantities. Therefore the new generalised balance assumption can be seen as a generalization of the traditional one. In principle it is possible to find an allocation rule that yields any target value for any service criterion.

It is very well possible that, due to high variation of the demand, application of the allocation rule results in some negative allocation quantities, i.e. imbalance occurs. In practice this would imply that goods that were allocated earlier on in the planning process have to be pulled back and allocated to other stockpoints. This is often impossible and therefore we assume that imbalance does not occur. The probability of imbalance and its effect on the service performance is discussed in Section 5.

4. Solution methods

Given \( p_k \) and \( S \), we can calculate the service level for stockpoint \( k \), using Eq. (8). However, we need to solve the reverse problem. Given the target levels \( \hat{\beta}_k \) for each stockpoint \( k \), calculate the required values of \( S \) and all fractions \( p_k \). To calculate this we need to solve a system with \( N + 1 \) non-linear equations and \( N + 1 \) unknowns \((p_1, \ldots, p_N, \text{ and } S)\):

\[
\sum_{k=1}^{N} p_k = 1,
\]

where \( \beta(S, p_k) = \hat{\beta}_k \) \((k = 1, \ldots, N)\),

In principle, this algebraic system of equations can be solved exactly using numerical methods. For example, De Kok (1990) used a bisection scheme for the variable \( S \) and a nested bisection scheme for all \( p_k \). In practice, however, inventory systems as the one described in this paper control thousands of items. Solving the algebraic system for such an environment within an acceptable amount of time, using an exact method, is virtually impossible. Therefore, we use an approximate decomposition method that is based on empirical findings (De Kok, 1990). As we discuss later, this method is very fast and gives good results.

4.1. Decomposition method to evaluate \( p_k \) and \( S \)

Under the assumption of normally distributed demand, equal lead times \( L \), and equal stockout probabilities \( \alpha \) for all stockpoints, it can be shown that the allocation fractions \( p_k \) become

\[
p_k = \frac{t_\alpha \sqrt{L + 1} \sigma_k}{\sum_{i=1}^{N} t_i \sqrt{L + 1} \sigma_i} = \frac{\sigma_k}{\sum_{i=1}^{N} \sigma_i} \tag{9},
\]

where \( t_\alpha \) denotes a safety factor depending on \( \alpha \).
It is interesting to note that these $p_k$ are implied by Eppen and Schrage (1981), who used a different allocation rule than the one we used here. Clearly, in this special case the allocation fraction $p_k$ can be interpreted as the fraction of the aggregate safety stock of $N$ single-echelon models allocated to stockpoint $k$.

Numerical experiments reveal that this result approximately holds for the more general conditions in our model (i.e., non identical stockpoint parameters and any demand distribution). It also appears that the allocation fractions are insensitive to the CD lead time $L$. Now we define the allocation fractions as follows:

$$p_k = \frac{ss_k^{(1)}}{\sum_{i=1}^N ss_i^{(1)}} \quad (1 \leq k \leq N),$$

where $ss_k^{(1)}$ denotes the safety stock for a single-echelon $(R, S)$ inventory system with lead time $L_k$, demand parameters $\mu_k$ and $\sigma_k$ and target level $\beta_k$.

A fast and accurate inversion algorithm (see Appendix A) enables us to compute the order-up-to-level $S_k^{(1)}$ for such a single-echelon model. The safety stock $ss_k^{(1)}$ is then computed as follows:

$$ss_k^{(1)} = S_k^{(1)} - (L_k + R) \cdot \mu_k. \quad (9)$$

Once we have computed the allocation fractions (applying the inversion algorithm to $N$ different single-echelon models), we are able to calculate the echelon order-up-to-level $S$ for the CD. This order-up-to-level can be obtained by applying the inversion algorithm to the service level equation (Eq. (8)). As a result we obtain an order-up-to-level $S_k$ for the CD which denotes a value for $S$ that is sufficient to meet the target service level for stockpoint $k$. For every stockpoint we find an associated order-up-to-level. The final order-up-to-level $S$ for the CD is then simply computed by taking the mean of all these separate order-up-to-levels

$$S = \frac{1}{N} \sum_{k=1}^N S_k. \quad (10)$$

In total we apply the inversion algorithm $2N$ times.

In general this method is justifiable because the differences between the values of $S_k$ of the different stockpoints appear to be very small. However, when we are dealing with different target levels for the different stockpoints (ranging from e.g. 0.70 to 0.95), the values of $S_k$ differ more than desirable. Averaging over these values implies that for certain stockpoints $k$ the final value of $S$ is too large (if $S_k < S$) and consequently the realized service performance too high, or the final value of $S$ is too small (if $S_k > S$) and consequently the resulting service performance too low. This results in bad performance of the inventory policy.

In these situations we can improve the results by adjusting the allocation fractions. Increasing the value of allocation fraction $p_k$ results in a decrease in the value of the matching $S_k$, in order to maintain the same service performance. Likewise a decrease of $p_k$ results in an increase of the matching $S_k$. So by adjusting the allocation fractions we are able to bring the separate order-up-to-levels closer together. Clearly, the adjusted values of the allocation fractions still have to sum up to one. The reason why we adjust the allocations and then calculate the order-up-to-levels (and not vice versa) is the very fast inversion algorithm that enables us to calculate these order-up-to-levels. Every single adjustment of the allocation fractions involves $N$ applications of this inversion algorithm. Adjusting $S_k$ and next calculating $p_k$ is very time consuming, as indicated above. We now describe two methods of adjusting the allocation fractions.

4.1.1. The group method

Divide end stockpoints ($k = 1, \ldots, N$) into two groups $A$ and $B$ in the following way:

- $k \in A$ if $S_k < S$,
- $k \in B$ if $S_k \geq S$.

The allocation fraction for each stockpoint in group $A$ has to be decreased (in order to increase the matching order-up-to-level) and the allocation fraction for each stockpoint in group $B$ has to be increased (in order to decrease the matching order-up-to-level). The adjusted values $\tilde{p}_k$ of the allocation fractions are determined in
the following way:

- if \( k \in A \), then
  
  \[
  \hat{p}_k = \frac{(1 - \delta)p_k}{1 + \delta - 2\delta \sum_{i \in A} p_i}.
  \]

- if \( k \in B \), then
  
  \[
  \hat{p}_k = \frac{(1 + \delta)p_k}{1 + \delta - 2\delta \sum_{i \in A} p_i}.
  \]

Observe that the values of \( \hat{p}_k \) sum up to one. The parameter \( \delta \) determines to what extent the allocation fractions are increased or decreased. The value of \( \delta \) is determined by a local search method aimed at minimizing the following expression:

\[
\frac{S_{\text{max}} - S_{\text{min}}}{\text{ASS}},
\]

where

\[
S_{\text{max}} := \max\{S_k \mid 1 \leq k \leq N\},
\]

\[
S_{\text{min}} := \min\{S_k \mid 1 \leq k \leq N\},
\]

\[
\text{ASS} := S - \sum_{k=1}^{N} (L + L_k + R) \cdot \mu_k.
\]

Notice that \( \text{ASS} \) represents the aggregate safety stock in all end stockpoints when using an order-up-to-level \( S \) for the CD. The exact solution would, of course, be reached when expression (11) equals zero, implying that all order-up-to-levels \( S_k \) are identical \( (S_{\text{max}} = S_{\text{min}}) \).

This procedure can be applied repeatedly until no further reduction of Eq. (11) is obtained. There is, however, no guarantee that we will obtain the exact solution.

4.1.2. The extreme case method

This method selects the order-up-to-level that differs most from \( S \) (the extreme case). Let this be \( S_k \), the order-up-to-level resulting from the service level equation for stockpoint \( k \) (i.e. \( S_k = S_{\text{min}} \) or \( S_k = S_{\text{max}} \)). The matching allocation fraction \( p_k \) is adjusted in the following way:

- if \( S_k < S \), then \( \hat{p}_k = p_k + \delta \cdot (p_i/p_{\text{rest}}) \);
- if \( S_k \geq S \), then \( \hat{p}_k = p_k - \delta \cdot (p_i/p_{\text{rest}}) \),

where \( p_{\text{rest}} := \sum_{i \neq k} p_i \).

It is evident that the adjusted \( \hat{p}_k \) values sum up to one. The value of the parameter \( \delta \) is determined in the same way as in the group method. As with the latter method, adjustments are made repeatedly until no further reduction of Eq. (11) is obtained. There is again no guarantee for optimality.

4.2. Numerical results

De Kok (1990) used the decomposition method with \( S \) given by (10) with very good results. This was due to the fact that the target levels considered varied in a limited range (0.90 and 0.95). If we broaden the target levels range, the analytical results deteriorate. We considered some typical examples: \( R = 1 \), \( N = 6 \), lead time to all stockpoints is 3, expected period demand in all stockpoints is 100, the target levels vary from 0.70 to 0.95. The common lead time \( L \) is 5 and 9, the standard deviation (equal in all stockpoints) is 50 and 200, resulting in a coefficient of variation of 0.5 and 2.0 respectively.

Using the results from the decomposition method \((S \, \text{and} \, \{p_k\})\), the service levels are analytically calculated by fitting the stochastic variables in (8) to a mixture of Erlang distributions (if the coefficient of variation is less than one) or a hyperexponential distribution (if the coefficient of variation is equal or greater than one). A detailed description of these calculations is given in Verrijdt (1992). Tables 1 and 2 show the origi-

<table>
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<th>Target</th>
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<th>( cv = 2.0 )</th>
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Table 2

<table>
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<tr>
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nal analytical results, the extreme case method results and the group method results.

When determining the order-up-to-level $S$, given $p_k$ and $\beta_k$, we apply the inversion algorithm $2N$ times. When determining the service performance, given $S$ and $p_k$, we apply a two-moment fit. Due to these approximations small numerical deviations can occur. The CPU time needed to calculate the results for one configuration is less than one second on a 486 (33 MHz) computer.

Observe that the service performance deterioration is stronger for low coefficients of variation (cv = 0.5) in combination with a large CD lead time ($L = 9$). Especially the stockpoint with the highest target level (0.95) is affected most. A realized service performance of 90.6% against a target level of 95% implies that the number of backorders is almost doubled! So realizing the target level is more important for stockpoints with high target levels (> 90%) than for stockpoints with low target levels (< 80%).

Both improvement methods (extreme case and group method) adjust the allocation fractions such that the realized service levels are more in accordance with the target levels. For the parameter settings considered both methods perform equally well. Additional results are presented in Appendix B.

5. Exploring the modelling assumption

**Imbalance** is defined as the situation in which application of the allocation procedure at the CD results in one or more negative allocation quantities. In other words, the generalised balance assumption is violated. In the preceding analysis we assumed that imbalance does not occur. In reality, however, imbalance does occur and may disrupt our planning process.

When simulating the planning procedure, we tackle the imbalance problem by adjusting the allocation quantities such that no negative quantities remain. In case of imbalance at the CD at allocation time $t$ we adjust the allocation quantities $q_{k,t}$ ($k = 1, \ldots, N$) as follows:

- if $q_{k,t} < 0$, then $\tilde{q}_{k,t} := 0$;
- if $q_{k,t} \geq 0$, then $\tilde{q}_{k,t} := q_{k,t} + (q_{k,t}/q_{pos}) \cdot q_{neg}$, with $q_{pos} := \sum_{k: q_{k,t} > 0} q_{k,t}$ and $q_{neg} := \sum_{k: q_{k,t} < 0} q_{k,t}$.

Notice that Eq. (2) still holds for the adjusted quantities $\tilde{q}_{k,t}$.

In order to quantify the impact of imbalance on our planning procedure, and therefore on the realized service levels, we need some analytical measure of imbalance. An obvious such measure is the probability of imbalance at the CD:

$$P(\exists k : q_{k,t} < 0).$$

Because it is extremely difficult to derive an expression for this measure (Eppen and Schrage, 1981; Lagodimos, 1992), we use a surrogate measure: the probability that the allocation quantity $q_{k,t}$ for a certain stockpoint $k$ is negative:

$$P(q_{k,t} < 0) \quad (1 \leq k \leq N).$$

We make the important restriction that the generalised balance assumption was not violated at the previous allocation period. In other words, at time $t - R$ all allocation quantities $q_{k,t-R}$ ($k = 1, \ldots, N$) are positive. A similar modelling assumption is used by Eppen and Schrage (1981) and Lagodimos (1992).

From the analysis of the 2-echelon model in Section 3.1 we know

$$V_{k,t} = Z_{k,t} + q_{k,t} = p_k \cdot \{S - D(t-L_t^*) - v_0 \} + v_k,$$

(12)
with:

- \( V_{k,t} \): echelon inventory position of stock point \( k \) directly after the allocation at time \( t \),
- \( p_k \): allocation fraction for stockpoint \( k \),
- \( S \): order-up-to-level for the CD,
- \( D_{t-L,t} \): aggregate demand at all stockpoints during \([t - L, t)\),
- \( D_k^{(t, t+R)} \): demand at stockpoint \( k \) during \([t, t + R)\).

We now have the following expression for the echelon inventory position of stockpoint \( k \) after allocation at time \( t + R \):

\[
V_{k,t+R} = V_{k,t} - D_k^{(t+R)} + q_{k,t+R}.
\]  
(13)

Under the condition that the generalised balance assumption holds at allocation time \( t \) (no imbalance at CD!) we can derive the following expression for \( q_{k,t+R} \):

\[
q_{k,t+R} = V_{k,t+R} - V_{k,t} + D_k^{(t+R)}
= p_k (D_{t-L,t} - D_{t+R-L,t+R}) + D_k^{(t+R)}
= Y - X.
\]  
(14)

If \( R \leq L \), then

\[
Y = p_k D_{t-L,t} + (1 - p_k) D_k^{(t+R)},
X = p_k \sum_{i+k} D_i^{(t,L+R)};
\]

if \( R > L \) then

\[
Y = p_k D_{t-L,t} + D_k^{(t,L+R)}
+ (1 - p_k) D_k^{(t+R-L,R+R)},
X = p_k \sum_{i+k} D_i^{(t,L+R)}.
\]

Since we assume independent period demands, the stochastic variables \( X \) and \( Y \) are also independent. We can now calculate the probability \( \pi_k \) of a negative allocation quantity for stockpoint \( k \) as follows:

\[
\pi_k = P(q_k < 0) = P(Y - X < 0) = P(Y < X)
= \int_{0}^{x} \left( \int_{0}^{y} f_Y(y) \, dy \right) f_X(x) \, dx.
\]  
(15)

5.1. Numerical and simulation results

We now examine some numerical and simulation results. For a more extensive numerical summary we refer to Appendix C. The simulation time is 30,000 time periods. In the simulation we assume that the demand per period is distributed according to a mixture of Erlang distributions,
which fits to $\mu_k$ and $\sigma_k$ (Tijms, 1986). The parameter setting is as follows: $N = 6$, $R = 1$, $L = 9$, $L_k = 3$, $\mu_k = 100$. The target service levels ($\beta_k$) are identical for all stockpoints (0.70 and 0.95). The standard deviation has three alternatives: $\sigma_1 = 50$, $\sigma_2 = 200$ (for all $i$) or $\sigma_3 = \sigma_5 = \sigma_6 = 50$ and $\sigma_4 = 0.6 = 200$. Tables 3 and 4 show the analytical results versus the simulation results. The analytical results are obtained after application of the extreme case improvement method. The realized service levels $\beta_k$ and the probabilities of imbalance $\pi_k$ for each stockpoint $k$ are tabulated.

It is clear from these results that $\pi_k$ is strongly related to the coefficient of variation of the demand processes. A high coefficient of variation at a stockpoint ($\text{cv} = 2.0$) results in a high probability of imbalance at that stockpoint. The effect of imbalance on the service performance depends on the target levels and the coefficients of variation in the separate stockpoints. In case of low target levels the high probabilities of imbalance hardly affect the realized service performance. However, for high target levels in combination with high imbalance probabilities, the realized service levels are significantly lower than the target levels. The worst results are obtained in asymmetric configurations: different coefficients of variation in the various stockpoints (Table 4). The high imbalance probabilities for the stockpoints with high coefficients of variation affect the realized service performance in the stockpoints with low coefficients of variation enormously, resulting in very low service levels (compared to the target levels). This negative effect on the service performance can be noticed for situations with low target levels as well as high target levels.

With respect to the probability of imbalance we can conclude that agreement between the analytical and simulation results is very satisfactory. The small differences that occur (especially for high probabilities of imbalance in asymmetric configurations: Table 4) can be partly explained by the assumption of balance at the previous allocation period. In the simulation, however, it is possible that imbalance at a stockpoint occurs at consecutive allocation times. Recall that in the simulation the allocation quantities are adjusted when imbalance occurs, such that no negative quantities remain. This adjustment affects the proposed optimal allocation procedure and will therefore enlarge the probability of imbalance at the next allocation time. This may also explain the differences between analytical and simulation results.

When we consider configurations with a wide range of target levels (Table C.2, Appendix C), we can again observe deviations between the analytical and simulation results for situations with a high coefficient of variation. Furthermore, it is evident that the stockpoint with a high target level (0.95) and a low coefficient of variation (0.5) has a significantly higher probability of imbalance than other stockpoints. In situations with a high coefficient of variation (2.0) the analytically calculated probabilities of imbalance appear to be independent of the target levels. However, the simulation results indicate that stockpoints with high target service levels have a significant larger probability of imbalance.

### Table 4

Realized service and imbalance with different $\text{cv}$ per stockpoint

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6. Conclusions

In this paper we developed a planning procedure for a divergent two-echelon distribution network. Given the lead times, the demand parameters and the desired service performance for all end stockpoints, the decomposition algorithm developed here evaluates the required echelon order-up-to-level (defining the ordering policy) and the allocation fractions (defining the allocation policy). While the results of the algorithm are only approximations, these can be obtained very fast and yield excellent results. We can, however, identify two major problems.

First, when dealing with a wide range of target levels the analytically calculated service levels deviate from the target levels (especially the high service levels are affected!). Two improvement methods are presented which compensate these deviations by adjusting the allocation fractions. Both methods improve the analytical results considerably.

The second problem relates to the phenomenon of imbalance. High coefficients of variation at end stockpoints may disrupt the allocation policy and result in bad service performance. Our numerical experiments show that the algorithm defined in this paper yields excellent results with a negligible computation time if the probability of imbalance is small. Clearly, in case of large imbalances, the quality of the approximations deteriorate. One way of dealing with this problem is to hold some stock at the central depot (Van Donselaar, 1990). When imbalance occurs, this stock can be used to compensate for the negative allocation quantities. Another way is to smooth the highly variable market demand by satisfying large portions of demand directly from the central depot. As a result, the coefficient of variation at the end stockpoints will decrease. These suggestions will be the subject of further research.

Acknowledgements

The authors would like to thank Thanos Lagodimos for his comments and careful reading, which helped us to improve the readability of the paper.

Appendix A. Inversion algorithm

The algorithm described in this appendix enables us to determine the order-up-to-level $S$ for a single-echelon model such that a predetermined target level is realized. In this model an $(R, S)$ inventory strategy is applied: at the beginning of every review period of length $R$ the echelon inventory position is increased to a level $S$. We need the following data:

- $\beta$: target level,
- $L$: lead time,
- $\mu$: mean period demand,
- $\sigma$: standard deviation in mean period demand.

It can be easily shown that the service level can be written as a function of $S$,

$$\beta(S) = 1 - \frac{E[(D_{L+R} - S)^+] - E[(D_L - S)^+]}{E[D_R]}, \quad (A.1)$$

where $D_L$ is the demand during a lead time, $D_R$ is the demand during a review period, and $D_{L+R}$ is the demand during a lead time plus a review period.

$\beta(S)$ is a monotone increasing function in $S$ with $\beta(0) = 0$ and $\beta(\infty) = 1$ and can therefore be considered as a probability distribution function of a random variable $X_\beta$, i.e. $P(X_\beta \leq S) = \beta(S)$. Next we make a two-moment gamma fit $\beta(\cdot)$ of $\beta(\cdot)$. The first two moments of $X_\beta$ can be determined as follows:

$$E[X_\beta^k] = k \int_0^\infty y^{k-1}(1 - \beta(y)) \, dy. \quad (A.2)$$
Table B.1
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Given a target level $\hat{\beta}$ we now need to solve the following equation:

$$\beta(S) = \hat{\beta}.$$  \hfill (A.3)

In order to solve (A.3) for $S$ we need to invert the gamma function $\beta(\cdot)$:

$$S = \beta^{-1}(\hat{\beta}).$$  \hfill (A.4)

For an exact description of this gamma inversion we refer to De Kok (1989). The final value of $S$ follows from:

$$S = (1 + \text{vc}_\beta \cdot k_\beta)E[X_\beta],$$

with

$$\text{vc}_\beta = \sqrt{\frac{E[X_\beta^2] - E^2[X_\beta]}{E[X_\beta]}},$$

$$k_\beta = (1 - \text{vc}_\beta) \cdot k_0 + \text{vc}_\beta \cdot k_1,$$

$$k_0 = \Phi^{-1}(\hat{\beta}), \quad k_1 = -1 - \ln(1 - \hat{\beta}).$$

$\Phi^{-1}(\cdot)$ represents the inverted standardized normal probability distribution function, which is approximated polynomially (Abramowitz and Stegun (1965)).

**Appendix B**

Tables B.1 and B.2 show the analytically calculated service levels for a number of configurations.

A are realized service levels, B are realized service levels after applying the extreme case method, and C are realized service levels after applying the group method.

The coefficient of variation (cv) and the end stockpoint lead times are equal in all stockpoints. The expected period demand is 100. The review period ($R$) is 1.

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### Appendix C

Tables C.1 and C.2 present the numerical and simulation results referred to in Section 5.1.

### References


de Kok, A.G. (1989), “The inverse incomplete gamma function and an important set of equations from inventory theory”, CQM Note Nr. 083, Centre for Quantitative Methods, Philips Electronics, Eindhoven.


