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Oscillating boundary layers
in polymer extrusion

by

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OSCILLATING BOUNDARY LAYERS IN POLYMER EXTRUSION

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1. INTRODUCTION

Many polymer products are manufactured by extrusion. The devices involved may considerably vary in the technical details. In Fig.1 a very simple one is drawn, which still shows the essential parts: a barrel, in which the melt is kept under pressure via a plunger, and a capillary, through which the melt is extruded. For reasons of efficiency the extrusion rate should be as high as possible. However, this rate is limited by the occurrence of melt flow instabilities. At relatively low flow rates the extruded polymer strands can develop a surface with a more or less regular pattern of ridges resulting in an observable mattness. This phenomenon is usually referred to as the 'sharkskin' effect. The period of the sharkskin oscillations is typically in the order of 0.01 second. At higher flow rates the extrudate comes out of the capillary in bursts. This is commonly referred to as the 'spurt' phenomenon. During spurt the complete flow behaves in a periodic fashion with a period which is typically in the order of 10 seconds. Further increase of the flow rate leads to a disappearance of the oscillations. However, the extrudate is then most of the time distorted, and thus useless. This regime is known as 'gross melt fracture'.

The dynamics in the spurt regime can be appropriately described in terms of relaxation oscillations as proposed in [Molenaar and Koopmans] and used by [Durand et al.]. In this report we investigate the sharkskin effect. The observations indicate that the typical pattern of ridges is the result of an oscillating boundary layer. In the literature there is much discussion about the reason of this. One approach is to propose that it is essentially a slip-stick phenomenon: at low flow rates the boundary condition at the wall is that of vanishing velocity, but if some critical shear rate is exceeded at the wall this condition could be violated and 'slip' occurs. Because of the slip the shear rate would decrease and the flow comes back in the 'stick' regime, after which this scenario would repeat itself.

In the present report we shall show that another explanation is possible, which does not invoke the violation of the 'classical' stick condition. It is assumed that small oscillations appear near the capillary wall, whereas the core of the capillary flow is stationary. The instability of the boundary layer is shown to be a possible consequence of the non-monotonicity of the stress-strain relation of these materials.

The analyses are given for two visco-elastic models, the Kaye-Bernstein-Kearsly-Zapas (KBKZ) model and the Johnson-Segalman-Oldroyd (JSO) model. Though the details of the derivations are quite different, the results from both models are very similar.

The structure of this report is as follows. In §2 the relevant conservation laws are presented and the resulting equations are brought into dimensionless form in §3. Both the KBKZ and the JSO model allow for constant solutions. In §4 it is shown that for low values of the pressure these constant solutions are unique, but that for pressure values above a critical one infinitely many constant solutions exist. In §5 it is shown
that some of them have an oscillating boundary layer. In §6 it is discussed that such solutions are a highly appropriate explanation of the occurrence of sharkskin.

![Diagram of extrusion device](image)

Figure 1: Schematical drawing of the extrusion device.
2. MODELING THE EXTRUSION PROCESS

We consider the extrusion device shown in Fig. 1, which consists of a barrel with a plunger, coupled to a capillary. Important characteristics of this system are the pressure difference $P$ over the capillary, the volume flux $Q_{\text{out}}$ leaving the capillary, and the volume flux $Q_{\text{in}}$ pushed into the barrel by the plunger. The pressure in the barrel is assumed to be uniform.

The system can be operated under two conditions. First, the plunger velocity $v_p$ can be kept constant, so that

\begin{equation}
Q_{\text{in}} = v_p A
\end{equation}

is constant, where $A$ is the area of the plunger. Second, the pressure $P$ can be kept constant by adjusting the force exerted on the plunger.

The flow in the capillary has extensively been studied in [Aarts and van de Ven, Malkus et al. (1990), Malkus et al. (1991)]. It turns out that the flow in the capillary can be described as an axisymmetric, incompressible, shear flow. The velocity profile $v(t, r)$ in the capillary is a function of time $t$ and the radial coordinate $r$ only, and not of the position along the capillary. The shear rate $\omega(t, r)$ is then given by

\begin{equation}
\omega(t, r) = -\frac{\partial v(t, r)}{\partial r}.
\end{equation}

Because of axisymmetry we have for all $t$ the condition

\begin{equation}
\omega(t, 0) = 0.
\end{equation}

At the wall we impose the no-slip condition

\begin{equation}
v(t, R) = 0,
\end{equation}

where $R$ is the radius of the capillary. Using these conditions we can write the volume flux $Q_{\text{out}}$ through the capillary as

\begin{equation}
Q_{\text{out}}(t) = 2\pi \int_0^R v(t, r) r dr = \pi \int_0^R \omega(t, r) r^2 dr.
\end{equation}

The modeling of the extrusion process is based on conservation laws. In the following subsections we shall apply the laws of conservation of mass and momentum respectively.
2a. Conservation of mass

Since the flow in the capillary is assumed to be incompressible, conservation of mass is there trivially satisfied. In the barrel the compressibility of the polymer melt has to be taken into account. Denoting the height of the barrel by $h(t)$ and the (uniform) density of the melt by $\rho(t)$, the total mass in the barrel is given by $Ah\rho$, while the mass flux leaving the barrel into the capillary is given by $\rho Q_{\text{out}}$. Conservation of mass in the barrel is then expressed by the equation

\begin{equation}
\frac{d}{dt} (Ah\rho) = -\rho Q_{\text{out}} .
\end{equation}

The melt compressibility $\chi$ of the polymer is defined by

\begin{equation}
\frac{1}{\rho} \dot{\rho} = \chi \dot{P} ,
\end{equation}

where the notation $\dot{\rho} \equiv d\rho/dt$ is used. Using $\dot{h} = -v_p$, (2.1), and (2.7), we may rewrite (2.6) in the form

\begin{equation}
\dot{P} = \frac{1}{C} \Delta Q ,
\end{equation}

with the definitions $\Delta Q = Q_{\text{in}} - Q_{\text{out}}$ and $C = A\chi h$. The parameter $C$ is time dependent through $h(t)$. However, the change of $h$, given by $v_p$, is much slower than the typical period of the sharkskin phenomenon. Because of that $C$ is assumed to be constant throughout the present analysis.

2b. Conservation of momentum

The velocity profile in the capillary is determined by the following balance of linear momentum:

\begin{equation}
\rho \left( \frac{\partial v}{\partial t} + (v \cdot \nabla) v \right) = \rho b + \nabla \cdot T ,
\end{equation}

where $v$ is the velocity, $b$ the total body force, and $T$ the total, symmetric stress tensor. In the present application we have $b = 0$. The form of $T$ follows from the specific constitutive model under consideration. We assume that $T$ can be written as

\begin{equation}
T = -pI + 2\eta D + S .
\end{equation}

Here, $I$ is the unit tensor, $p$ the pressure in the capillary, $\eta$ the coefficient of the Newtonian viscosity, and $D$ the rate-of-deformation tensor.
The polymer melt is assumed to consist of a small-molecule solvent, which behaves like a viscous fluid, and a dissolved polymer, which behaves elastically. The polymer contribution is represented by $S$. The explicit form of $S$ depends on the model. In the following two subsections we shall present the momentum equations based on the KBKZ and the JSO models respectively.

In view of the symmetry of the capillary we shall use cylindrical coordinates, with radial coordinate $r$ and axial coordinate $z$. As origin we choose the point where the capillary merges into the barrel.

### 2b1. The KBKZ model

The KBKZ model is a nonlinear viscoelastic model including memory effects. In [Tanner, Bird et al.] full expositions of this model are given. Here we use the version which is appropriate for the present system and described in [Aarts and van de Yen, Grob]. This version includes only one relaxation rate; inclusion of more relaxation rates is assumed to leave the qualitative aspects of the present results unchanged. In this approach the pressure $p$ in the capillary has the form

\begin{equation}
\tag{2.12}
 p(r, z, t) = P(t) \frac{L - z}{L} + \mu \lambda \int_{-\infty}^{t} \frac{1}{c + \gamma^2(r, t, \tau)} e^{-\lambda(t-\tau)} d\tau + P_0(t).
\end{equation}

Here, $P_0(t)$ is a further irrelevant term, $L$ is the length of the capillary, $\mu$ is the elastic shear modulus, $\lambda$ the relaxation rate, and $c$ a dimensionless constant. The factor $\gamma$ is the magnitude of the shear strain at time $t$ if the strain is applied in the past at time $\tau$:

\begin{equation}
\tag{2.13}
 \gamma(r, t, \tau) = \int_{\tau}^{t} \omega(s, r) ds.
\end{equation}

From (2.12) it is seen that $p$ is the sum of an ‘internal’ term including memory effects through an integral over the past, and an ‘external’ term, which linearly varies along the capillary.

In [Aarts and van de Yen] it is shown that the momentum equation can be written as

\begin{equation}
\tag{2.14}
 \eta \omega + \mu \gamma(r, t, 0) e^{-\lambda t} + \mu \lambda \int_{0}^{t} \frac{\gamma(r, t, \tau)}{c + \gamma^2(r, t, \tau)} e^{-\lambda(t-\tau)} d\tau =
\end{equation}

\[
= \frac{1}{2} \frac{P(t)}{L} - \frac{\rho}{r} \int_{0}^{r} \xi \frac{\partial v(t, \xi)}{\partial t} d\xi
\]
for $0 \leq r \leq R$ and $t \geq 0$. The extrusion is assumed to have started at $t = 0$.

We are interested in the behaviour of the system after the transient phenomena have died out, so in the limit $t \to \infty$. The second term in the left-hand side of (2.14) can then be neglected. After the shift $t - \tau \to \tau$ in the third term of (2.14) we obtain the equation

$$\eta \omega(t, r) + \mu \lambda \int_0^\infty \frac{\gamma(r, t, t - \tau)}{c + \gamma^2(r, t, t - \tau)} e^{-\lambda \tau} d\tau = \frac{1}{2} r \frac{P(t)}{L} - \frac{\rho}{r} \int_0^r \xi \frac{\partial v(t, \xi)}{\partial t} d\xi.$$

2b2. The JSO model

The JSO model, see [Bird et al., Tanner, Johnson and Segalman], provides an expression for the tensor $S$ in (2.10) in terms of differential equations. The application of the full JSO model to the present system is given in [Malkus et al.(1990), Malkus et al.(1991), Grob]. A small difference with the geometry used until now is that the capillary is replaced by two parallel plates of distance $R$, say. Instead of the radial coordinate $r$ used above, we use in the JSO model the Cartesian coordinate $x$ with $-\frac{R}{2} \leq x \leq \frac{R}{2}$. It leads to the following set of partial differential equations:

$$\left\{ \begin{array}{l}
\rho \frac{\partial v}{\partial t} - \frac{\partial \sigma}{\partial x} = \eta \frac{\partial^2 v}{\partial x^2} + \frac{P}{L}, \\
\frac{\partial \sigma}{\partial t} - (Z + \mu) \frac{\partial v}{\partial x} = -\lambda \sigma, \\
\frac{\partial Z}{\partial t} + b \sigma \frac{\partial v}{\partial x} = -\lambda Z.
\end{array} \right.$$ 

Here, $v(t, x)$ is the velocity, and $\sigma(t, x)$ and $Z(t, x)$ are linear combinations of the components of the $2 \times 2$ tensor $S$ in (2.10): $\sigma = S^{xx} = S^{zz}$, $Z = -\frac{1}{2} b (S^{zz} - S^{xx})$. The slip parameter $b$ varies in the range $0 \leq b \leq 1$. So, $Z$ is proportional to the principal normal stress difference.
3. DIMENSIONLESS EQUATIONS

The equations in §2 can be analysed in detail only if they are put into dimensionless form. The dimensionless quantities will be denoted using the superscript *. Natural choices for the units of length and time are \( R \) and \( \lambda^{-1} \) respectively. So,

\[
(3.1) \quad r^* = \frac{r}{R} , \quad z^* = \frac{z}{R} , \quad t^* = \lambda t .
\]

For the velocity and shear-rate profiles we then find

\[
(3.2) \quad \begin{cases} 
 v^*(t^*, r^*) = \frac{1}{R\lambda} v(t, r) \\
 \omega^*(t^*, r^*) = \frac{1}{\lambda} \omega(t, r) .
\end{cases}
\]

The fluxes in (2.1) and (2.5) are accordingly scaled:

\[
(3.3) \quad \begin{cases} 
 Q^*_{in}(t') = \frac{1}{R^3\lambda} Q_{in} \\
 Q^*_{out}(t') = \frac{1}{R^3\lambda} Q_{out}(t) = \pi \int_0^1 \omega^*(t^*, r^*) (r^*)^2 \, dr^* .
\end{cases}
\]

The pressure \( P \) is scaled using the shear modulus \( \mu \):

\[
(3.4) \quad P^*(t^*) = \frac{1}{\mu} P(t) .
\]

The mass conservation equation (2.8) then transforms into

\[
(3.5) \quad \frac{dP^*}{dT^*} = \nu (Q^*_{in}(t^*) - Q^*_{out}(t^*)) ,
\]

where the dimensionless quantity \( \nu \) is defined by

\[
(3.6) \quad \nu = \frac{R^3}{A\chi h\mu} .
\]

The accumulated shear strain \( \gamma \) in (2.13) is already dimensionless. The KBKZ momentum equation (2.15) reads in dimensionless form:

\[
(3.7) \quad \eta \lambda \omega^*(t^*, r^*) + \mu \int_0^{t^*} \frac{\gamma^*(r^*, t^*, t^* - \tau^*)}{c + (\gamma^*(r^*, t^*, t^* - \tau^*)^2} e^{-\tau^*} \, d\tau^* = 
\]
To simplify the notation we introduce the function

\[ h_c(x) = \frac{x}{c + x^2} \]  

with derivative

\[ h'_c(x) = \frac{d}{dx} h_c(x) = \frac{c - x^2}{(c + x^2)^2} . \]

Dividing (3.7) by \( \mu \) and using (3.8) we find

\[ \varepsilon \omega^* + \int_0^{t^*} h_c(\gamma^*(r^*, t^*, t^* - \tau^*)) e^{-r^*} d\tau^* = \frac{1}{2} \beta r^* P^* - \frac{\alpha}{r^*} \int_0^{t^*} \xi^* \frac{\partial v^*(t^*, \xi^*)}{\partial t^*} d\xi^* \]

with the definitions

\[ \left\{ \begin{array}{l}
\varepsilon = \frac{\eta \lambda}{\mu} \\
\beta = \frac{R}{L} \\
\alpha = \frac{R^2 \lambda^2 \rho}{\mu} .
\end{array} \right. \]

The parameter \( c \) is of the order 1. The parameter \( \varepsilon \) is the ratio of the Newtonian viscosity \( \eta \) and the shear viscosity \( \mu/\lambda \). Note that the quotient \( \alpha/\varepsilon \) is the Reynolds number \( R^2 \lambda \rho/\eta \). For the materials under consideration we have that \( \alpha \ll 1 \). The last term in (3.10) will therefore be neglected in the following.

With the scalings (3.1)-(3.4) and the definitions

\[ \left\{ \begin{array}{l}
\sigma^* = \frac{\sigma}{\mu} \\
Z^* = \frac{Z}{\mu}
\end{array} \right. \]

the dimensionless JSO momentum equations (2.16) read as
\[
\begin{align*}
\alpha \frac{\partial u^*}{\partial t^*} - \frac{\partial \sigma^*}{\partial x^*} &= -\varepsilon \frac{\partial \omega^*}{\partial x^*} + \beta P^* \\
\frac{\partial \sigma^*}{\partial t^*} + (Z^* + 1) \omega^* &= -\sigma^* \\
\frac{\partial Z^*}{\partial t^*} - b \sigma^* \omega^* &= -Z.
\end{align*}
\]

The factor \( b \) could be scaled out from (3.13), but this would lead to an inconsistency with the scaling as used for the KBKZ equation (3.10), and is therefore omitted. For the same reason as mentioned above we shall omit the term with \( \alpha \) in (3.13a).
4. CONSTANT SOLUTIONS OF THE MOMENTUM EQUATION

If the operating conditions are kept constant, the extrusion process is observed to converge to a constant or periodic solution if time proceeds. Here we analyse whether the momentum equations of the KBKZ and the JSO model admit constant solutions, thus with the \( \omega \) and \( v \) profiles being independent of time. In the following we shall use the dimensionless equations. Because confusion is hardly possible, the superscript * will be omitted.

If \( \omega \) does not depend on \( t \) we have from (2.13)

\[ (4.1) \quad \gamma(r, t, \tau) = \omega(r)(t - \tau) . \]

Substitution of this into (3.10) yields the KBKZ momentum equation:

\[ (4.2) \quad H_{\text{KBKZ}}(\omega(r)) = \frac{1}{2} \beta r P , \quad 0 \leq r \leq 1 , \]

with

\[ (4.3) \quad H_{\text{KBKZ}}(\omega) = \varepsilon \omega + \int_{0}^{\infty} h_c(\omega \tau) e^{-\tau} d\tau . \]

Here, we have taken the limit \( t \to \infty \). As far as the JSO equations (3.13) are concerned we notice that, after omission of the the term with \( \alpha \), (3.13a) can be integrated with respect to \( x \). This results in

\[ (4.4) \quad -\sigma = -\varepsilon \omega + \beta x P . \]

Setting the time derivatives in (3.13b,c) at zero we find that the constant \( \sigma \) and \( Z \) profiles are given by

\[ (4.5) \begin{cases} 
Z(r) \equiv Z(\omega(x)) = \frac{-b \omega^2}{b \omega^2 + 1} \\
\sigma(r) \equiv \sigma(\omega(x)) = \frac{-\omega}{b \omega^2 + 1} .
\end{cases} \]

Substituting (4.5b) into (4.4) we obtain an expression which is very similar to (4.2):

\[ (4.6) \quad H_{\text{JSO}}(\omega) = \beta x P , \quad -\frac{1}{2} \leq x \leq \frac{1}{2} \]

with
(4.7) \[ H_{JSO}(\omega) = \varepsilon \omega + \frac{1}{b} h_{b-1}(\omega). \]

Both \( H_{KKBKZ} \) and \( H_{JSO} \) have the same typical form for \( \varepsilon \ll 1 \): they are non-monotonous functions of \( \omega \) as shown in Fig.2, where the function

(4.8) \[ H(\omega) = \varepsilon \omega \frac{\omega}{1+\omega^2} \]

is drawn, i.e. \( H_{JSO} \) for \( b = 1 \). The curve has a relative maximum and a relative minimum. The coordinates indicated in Fig.2 have the following approximate values in terms of \( \varepsilon \):

\[
\begin{align*}
\omega_{\text{max}} &= 1 + 2\varepsilon, \\
H_{\text{max}} &= \frac{1}{2} + \varepsilon \\
\omega_{\text{min}} &= \varepsilon^{-1/2} - \frac{3}{2} \varepsilon^{1/2}, \\
H_{\text{min}} &= 2\varepsilon^{1/2} - \varepsilon^{3/2} \\
\omega_1 &= 2\varepsilon^{1/2}.
\end{align*}
\]

From Fig.2 it is immediately seen that (4.2) and (4.6) have unique solution only as long as the right-hand sides attain values smaller than \( H_{\text{min}} \). For values of the right-hand sides in the range \((H_{\text{min}}, H_{\text{max}})\) the equations have three solutions.
5. OSCILLATING BOUNDARY LAYERS

The sharkskin effect is typically a boundary layer effect: the surface of the extrudate shows the sharkskin pattern, whereas the inner core of the extrudate does not show any sign of oscillating behaviour. Here we analyse whether the observed phenomenon can be described in terms of the models presented above. We emphasise that we shall not introduce slip at the wall. Our purpose is to show that the non-monotonicity of the stress-strain relation implies that the flow in the capillary may develop an oscillating boundary layer, which might be responsible for the observed sharkskin effect.

In §4 we have seen that both the KBKZ and the JSO model give rise to very similar momentum equations. For higher values of $Q_{\text{in}}$, and thus of $P$, these equations have infinitely many constant solutions due to the non-monotonicity of the functions $H_{\text{KBKZ}}$ and $H_{\text{JSO}}$ as shown in Fig. 2. Here we shall investigate whether the momentum equations have solutions which are oscillating near the wall and constant elsewhere in the capillary. Because the sharkskin oscillations have small amplitudes, we shall investigate small amplitude perturbations of constant solutions. In view of the fact that observed oscillations are localised near the wall, we expect to find solutions that are constant everywhere except for a very thin boundary layer. In the first instance we shall not put this localisation as an 'Ansatz' in the analysis.

The extrusion device is assumed to operate in the constant flux mode, so with $Q_{\text{in}}$ being constant. Let us denote an arbitrary constant solution of the momentum equations corresponding to the (constant) pressure $P_0$ by $\omega_0(r)$. In a constant state we have $Q_{\text{out}} = Q_{\text{in}}$. A solution oscillating around this constant solution can be written as

\[
\begin{align*}
\omega(t, r) &= \omega_0(r) + \omega_1(r) e^{i \theta t} \\
Q_{\text{out}}(t) &= Q_{\text{in}} + Q_1 e^{i (\theta t + \varphi_1)} \\
P(t) &= P_0 + P_1 e^{i (\theta t + \varphi_2)} ,
\end{align*}
\]

where $\theta \geq 0$ is the angular frequency of the oscillation, and $\varphi_1, \varphi_2$ are possible phase differences. The amplitudes $\omega_1, Q_1$, and $P_1$ of the perturbations are assumed to be small with respect to $\omega_0, Q_{\text{in}},$ and $P_0$ respectively. Because of this we may linearise the momentum equations around the constant state.

The phase differences $\varphi_1$ and $\varphi_2$ are easily found. Substituting (5.1a) into (3.3b) we obtain

\[
(5.2) \quad Q_{\text{out}}(t) = \pi \int_0^1 \omega_0(r) r^2 \, dr + \pi e^{i \theta t} \int_0^1 \omega_1(r) r^2 \, dr.
\]

Comparing this with (5.1b) we conclude from the fact that $Q_1$ is real that
\[
\begin{aligned}
Q_{in} &= \pi \int_{0}^{1} \omega_{0}(r) r^2 \, dr \\
Q_1 &= \pi \int_{0}^{1} \omega_{1}(r) r^2 \, dr \\
\varphi_1 &= 0.
\end{aligned}
\]

(5.3)

The phase difference \(\varphi_2\) follows from the mass conservation equation (3.5). Substitution of (5.1a, b) into (3.5) yields:

(5.4) \quad \iota \theta P_1 e^{i\varphi_2} = -\nu Q_1.

Because \(\varphi, P_1, \nu, \) and \(Q_1\) are real and positive, we conclude that

(5.5) \quad \varphi_2 = \frac{\pi}{2}.

In the next two subsections we investigate whether the momentum equations of the KBKZ and the JSO model admit solutions which locally oscillate.

5a. The KBKZ model

Here we study solutions of the KBKZ equation (3.10). The nonlinearity of (3.10) comes from the integral

(5.6) \quad \int_{0}^{t} h_{c}(\gamma(r,t,t-\tau)) \, e^{-\tau} \, d\tau

with

(5.7) \quad \gamma(r,t,t-\tau) = \int_{t-\tau}^{t} \omega(s,r) \, ds.

Using representation (5.1a) we can evaluate this integral explicitly. We then obtain

(5.8) \quad \gamma(r,t,t-\tau) = \omega_{0}(r) \tau + \omega_{1}(r) f(\theta, \tau) \, e^{i\varphi_1}

with

(5.9) \quad f(\theta, \tau) = -i \frac{1 - e^{-i\varphi_1}}{\theta}.

We note that \(|f(\theta, \tau)| \sim 1/\theta\) if \(\theta \to \infty\), and \(|f(\theta, \tau)| \sim \tau\) if \(\theta \to 0\). The second term at the right-hand side of (5.8) is thus small with respect to the first term. This allows for the linearisation

\[
(5.10) \quad h_c (\gamma(r, t, t - \tau)) = h_c(\omega_0 \tau) + \omega_1 f(\theta, \tau) e^{i\theta t} h_c' (\omega_0 \tau)
\]

with \(h_c'\) given by (3.9). Substituting (5.1a), (5.1c), and the linearisation (5.10) into (3.10) we obtain the sum of the unperturbed part, satisfying (4.2), and the perturbed part, given by

\[
(5.11) \quad \varepsilon \omega_1(r) + \omega_1(r) \int_0^\infty f(\theta, \tau) h_c'(\omega_0 (r) \tau) e^{-\tau} d\tau = \frac{1}{2} i \beta r P_1.
\]

Writing the factor \(f(\theta, \tau)\) in the form

\[
(5.12) \quad f(\theta, \tau) = \frac{1}{\theta} (\sin \theta r + i (\cos \theta r - 1)),
\]

we can split (5.11) into its real and imaginary parts. This yields the equations \((0 \leq r \leq 1)\)

\[
(5.13) \quad \omega_1(r) (\varepsilon \theta + g_1(\omega_0(r), \theta)) = 0,
\]

and

\[
(5.14) \quad \omega_1(r) g_2(\omega_0(r), \theta) = \frac{1}{2} \beta \theta r P_1,
\]

where the functions \(g_1\) and \(g_2\) are defined as

\[
(5.15) \quad g_1(\omega_0, \theta) = \int_0^\infty h_c'(\omega_0 \tau) \sin (\theta \tau) e^{-\tau} d\tau
\]

and

\[
(5.16) \quad g_2(\omega_0, \theta) = \int_0^\infty h_c'(\omega_0 \tau) (\cos \theta \tau - 1) e^{-\tau} d\tau.
\]

Equation (5.13) has as solution the trivial one \(\omega_1(r) \equiv 0\), which corresponds to the constant solution of the momentum equation. Also a nontrivial solution exists, corresponding to a periodic perturbation, provided that the equation

15
(5.17) \[ \varepsilon \vartheta + g_1(\omega_0(r), \vartheta) = 0 \]

is satisfied for some \( \vartheta > 0 \). The \( \omega_1(r) \) profile corresponding to the latter solution follows from (5.14), except for a multiplicative constant factor. We remark that equation (5.17) cannot be satisfied for \( r = 0 \), because \( \omega_0(0) = 0 \) and

\[ g_1(0, \vartheta) = \int_0^\infty \sin \vartheta r e^{-r} dr > 0 . \]

In Fig.3 \(-g_1(\omega_0, \vartheta)\) is plotted as a function of \( \vartheta \) for several values of \( \omega_0 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3}
\caption{The function \(-g_1\) defined in (5.15).}
\end{figure}

From this figure it is seen that equation (5.17) has a solution only if the slope of \(-g_1(\omega_0, \vartheta)\) at the origin is positive and steeper than the slope of the line \( \varepsilon \vartheta \). This leads to the condition

\[ (5.19) \quad \frac{\partial}{\partial \vartheta} g_1(\omega_0, \vartheta) \bigg|_{\vartheta=0} = \int_0^\infty h'(\omega_0 r) e^{-r} dr < -\varepsilon . \]

From (4.3) we find that condition (5.19) is equivalent to the condition
This implies that (5.17) has a solution only if $\omega_0(r)$ lies in the range where $H_{KBKZ}$ has negative slope. Confer also Fig.2 where a typical form of $H_{KBKZ}$ is drawn. Conditions (5.19) and (5.20) thus imply $\omega_{\text{max}} < \omega_0(r) < \omega_{\text{min}}$ as indicated in Fig.4. The resulting $\omega_0(r)$ profile has thus a discontinuity as shown in Fig.5. The corresponding velocity profile, obtained by integrating the $\omega_0$ profile, has a discontinuity in the derivative at $r = 1 - \delta$ as shown in Fig.6. The frequency $\theta$ of the oscillations follows from (5.17). This frequency depends on $\omega_0(r)$, the value of the shear rate of the unperturbed solution in the boundary layer. Because it is physically unacceptable that the frequency would depend on position, we conclude that in the oscillating layer $\omega_0$ has to be constant within the accuracy of the approximations we have made in the derivations. This implies that the boundary layer must be fairly thin.

5b. The JSO model

Here we study solutions of the JSO equation (3.13). In addition to the perturbations (5.1) we assume that we may write

\begin{equation}
Z(x, t) = Z_0(x) + Z_1(x) e^{i(\theta t + \phi_3)}
\end{equation}

\begin{equation}
\sigma(x, t) = \sigma_0(x) + \sigma_1(x) e^{i(\theta t + \phi_4)},
\end{equation}

Figure 4: Construction of a possible $\omega_0$ profile.
where \( Z_0 \) and \( \sigma_0 \) are given by (4.5), and \( |Z_1| \) and \( |\sigma_1| \) are assumed to be small with respect to \( |Z_0| \) and \( |\sigma_0| \).

In the following we shall neglect the terms which are quadratic in the perturbations. Substituting (5.21) into (3.13b) we find that

\[
(1 + i\theta)\sigma_1 e^{i\varphi_3} = -\omega_0 Z_1 e^{i\varphi_3} - (Z_0 + 1)\omega_1 ,
\]

Substitution into (3.13c) yields

\[
(1 + i\theta)Z_1 e^{i\varphi_3} = b \sigma_0 \omega_1 + \sigma_1 \omega_0 e^{i\varphi_4} .
\]

From (3.13a) in the integrated form (4.4) we obtain

\[
-\sigma_1 e^{i\varphi_4} = -\varepsilon \omega_1 + i \beta x P_1 .
\]

In (5.22), (5.23), and (5.24) we have 6 real equations for the 7 unknowns \( \omega_1, Z_1, \sigma_1, P_1, \varphi_3, \varphi_4, \) and \( \theta \). As extra equation (5.4) with (5.5) can be used:

\[
\theta P_1 = \nu Q_1 .
\]

Multiplying (5.22) with \( (1 + i\theta) \) and (5.23) with \( \omega_0 \) and subtracting the resulting equations we find a relation between \( \sigma_1, e^{i\varphi_4} \) and \( \omega_1 \).
Using (5.24) and (5.25) we obtain a relation between $w_1$ and $Q_1$:

\[(5.27) \quad (A\varepsilon + B) \omega_1 = i \frac{\beta \mu \alpha x}{\theta} Q_1\]

with the notations

\[(5.28) \quad \begin{cases} 
A = (1 + i\theta)^2 + \omega_0^2 \\
B = (1 + i\theta)(Z_0 + 1) + b\omega_0 \sigma_0 
\end{cases}\]

We note that the geometry used to derive the JSO model is slightly different from the geometry used for the KBKZ model. Adjusting expression (5.3b) accordingly, the flux perturbation $Q_1$ is given by

\[(5.29) \quad Q_1 = \int_{-1/2}^{-1/2+\delta} \omega_1 x \, dx + \int_{1/2-\delta}^{1/2} \omega_1 x \, dx ,\]

where we use the fact that $\omega_1$ is expected to be nonvanishing only in two thin boundary layers, given by $\frac{1}{2} - \delta \leq |x| \leq \frac{1}{2}$ with $\delta \ll 1$ say. $Q_1$ is measured per unit length along the horizontal slice.
Multiplying both sides of (5.27) by \( \varepsilon \) and integrating over the two boundary layers we find in linear order in \( \delta \):

\[
A \varepsilon + B = \frac{1}{2} i \frac{\delta B A V}{\vartheta} .
\]

This complex equation contains two real equations for the unknowns \( \vartheta \) and \( \delta \). The real part yields an equation for \( \vartheta \) only:

\[
(5.30) \quad \varepsilon + \text{Re} \frac{B}{A} = 0 ,
\]

which is equivalent to the condition

\[
(5.31a) \quad \varepsilon |A|^2 + \text{Re}(\bar{A}B) = 0
\]

with \( \bar{A} \) the complex conjugate of \( A \). From (5.28) we have

\[
(5.32) \quad \begin{cases}
\text{Re} A = 1 + \omega_0^2 - \vartheta^2 , \\
\text{Im} A = 2\vartheta .
\end{cases}
\]

Substituting (4.5a) for \( Z_0 \) in (5.28b) we find

\[
(5.33) \quad B = \frac{1}{b \omega_0^2 + 1} \left( 1 - b \omega_0^2 + i\vartheta \right).
\]

Using (5.32) and (5.33) we can write (5.31b) in the form

\[
(5.34) \quad f(\omega_0^2, \vartheta^2) = 0
\]

with \( f \) defined as

\[
(5.35) \quad f = \varepsilon (b \omega_0^2 + 1) \{(1 + \omega_0^2 - \vartheta^2)^2 + 4\vartheta^2 \} + (1 + \omega_0^2 - \vartheta^2)(1 - b \omega_0^2) + 2\omega_0^2 .
\]

In Fig.7 the function \( f \) is plotted as a function of \( \vartheta^2 \) for \( b = 1 \), \( \varepsilon = 0.01 \), and and various values of \( \omega_0^2 \). From this it is seen that \( f \) vanishes only if it has negative value at \( \vartheta^2 = 0 \). This leads to the condition, with \( b = 1 \),

\[
(5.36) \quad f(\omega^2, 0) = (\omega^2 + 1) \{ \varepsilon(1 + \omega^2)^2 + (1 - \omega^2) \} < 0 .
\]

From (4.7) and (4.8) we conclude that this condition is equivalent to

\[
(5.37) \quad \frac{d}{d\omega} H_{JSO}(\omega) < 0 .
\]

Comparing this with (5.20) we find that the KBKZ and JSO models lead to very similar results, although the formulae involved are quite different.
6. DISCUSSION

In §5 we have found that the momentum equations of both the KBKZ and the JSO model admit solutions with a time independent shear rate profile in the inner core of the capillary and a periodic shear rate profile in a thin boundary layer. The frequency of the oscillations is given by (5.17) for the KBKZ model and by (5.34) for the JSO model. These frequencies depend on \( \omega_0 \), the value of the shear rate of the unperturbed solution in the boundary layer. Because it is physically unacceptable that the frequency would depend on position, we conclude that in the oscillating layer \( \omega_0 \) has to be constant within the accuracy of the approximations we have made in the derivations. It is found that the value of \( \omega_0 \) in the boundary layer must lie in the interval \( \omega_{\text{max}} < \omega < \omega_{\text{min}} \) as indicated in Fig.4. In this interval the functions \( H_{\text{KBKZ}} \) and \( H_{\text{JSO}} \) in (4.3) and (4.7) have negative slope. Because \( \omega_0 \) may hardly vary in the boundary layer, this layer must be very thin.

In view of these insights we conclude that we may construct a solution of the momentum equation which can appropriately be associated to the phenomena observed in the sharkskin effect. The construction of this solution is shown in Fig.4. The pressure \( P_0 \) is chosen such that \( H_{\text{min}} < P_0 < H_{\text{max}} \), so that the solution is not unique. For \( 0 < r < 1 - \delta \) we take the \( \omega_0(r) \) values on the left branch of \( H_{\text{KBKZ}} \) (or \( H_{\text{JSO}} \)) and for \( 1 - \delta < r < 1 \) the \( \omega_0(r) \) values are chosen on the middle branch. The analysis in §5 shows that the stability of this solution is such that the boundary layer may be oscillating. Because in a practical device perturbations are always present, these oscillations are expected to indeed occur in practice giving rise to the surface mattness.
observed as the sharkskin effect.

An important question is for which $Q_{in}$ value this instability will occur for the first time. It is clear from the construction in Fig.6 that sharkskin can only occur if $P \geq H_{\text{min}}$. The $Q_{in}$ value corresponding to $P = H_{\text{min}}$ can easily be calculated. However, it is by no means clear that this $Q_{in}$ value is the critical one. The critical situation is reached for the $P$ value for which the system prefers for the first time to take the value $\omega(1)$ of the shear rate at the wall from the middle branch, as indicated in Fig.4, and not from the left branch. In the modelling used until now no criterion is present for this choice. The minimisation of the energy dissipation might be the criterion we are looking for. For the JSO model an explicit expression for the energy dissipation $E(r)$ is given in [Malkus et al. (1990)]:

$$E(r) = v(r) P + \frac{\omega^2(r)}{1 + \omega^2(r)} - \varepsilon \omega^2(r).$$

It is likely that a critical value $P_{\text{crit}}$ exists, such that the dissipation $E(1)$ at the wall is smallest for the solution with $\omega(1) < \omega_{\text{max}}$ if $P < P_{\text{crit}}$, but that for $P > P_{\text{crit}}$ $E(1)$ is smallest for a solution with $\omega_{\text{max}} < \omega(1) < \omega_{\text{min}}$. This idea deserves further investigation.
REFERENCES


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