System equivalence for AR-systems over rings

Citation for published version (APA):

Habets, L. C. G. J. M. (1997). *System equivalence for AR-systems over rings*. (RANA : reports on applied and numerical analysis; Vol. 9718). Technische Universiteit Eindhoven.

Document status and date:

Published: 01/01/1997

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

RANA 97-18
November 1997

System equivalence for AR-systems over rings

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Reports on Applied and Numerical Analysis
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ISSN: 0926-4507

Abstract

In this paper the notion of autoregressive systems over an integral domain $\mathcal{R}$ is introduced, as a generalization of AR-systems over the rings $\mathbb{R}[s]$ and $\mathbb{R}[s, s^{-1}]$. Unlike the behavioral approach, the signal space is considered as a module $\mathcal{M}$ over the ring $\mathcal{R}$. In this setup the problem of system equivalence is studied: when do two different AR-representations characterize the same behavior? This problem is solved using a ring extension of $\mathcal{R}$, that explicitly depends on the signal space $\mathcal{M}$. In this way the usual divisibility conditions on the system-defining matrices can be recovered. The results apply to the class of delay-differential systems with (in)commensurable delays. In this particular application, the ring extension of $\mathcal{R}$ can be characterized explicitly.

Key Words: Autoregressive (AR)-systems, systems over rings, delay-differential systems, behavioral approach, system equivalence.

1 Introduction

In the behavioral approach to dynamical systems, introduced by J.C. Willems (see e.g. [14], [15]), a system is described by a triple $(T, W, \mathcal{B})$. Here T is the time-axis, W the space in which the signals take their values, and $\mathcal{B}$ —the behavior—is a subspace of the signal space W^T . $\mathcal{B}$ can be seen as the set of all time-trajectories, satisfying the laws governing the system.

Often a dynamical system is described by a set of ordinary differential equations (for continuous-time systems) or difference equations (for discrete-time systems), together with some non-dynamic linear relations among the variables. In this situation, the behavior is the set of all time-trajectories satisfying the system-defining equations. Collecting the variables in a vector w , and the equations in a polynomial matrix $P(s) \in \mathbb{R}[s]^{p \times q}$, the behavior is described by the set $\{w \mid P(\frac{d}{dt})w = 0\}$ for continuous-time systems, or $\{w \mid P(\sigma)w = 0\}$ in the discrete-time case. Here σ denotes the shift operator $\sigma(x(t)) = x(t-1)$. In both situations we see that the behavior is the kernel of an operator, described by a polynomial matrix in the differential operator $\frac{d}{dt}$ or the shift operator σ . A system described by such a kernel representation is called an *autoregressive* (AR)-system.

In this paper we study AR-systems over a more general type of ring. Instead of behaviors described by matrices over the polynomial rings $\mathbb{R}[s]$ or $\mathbb{R}[s, s^{-1}]$, we assume that the system-defining equations are described by matrices over an arbitrary *integral domain* $\mathcal{R}$. In this way we are able to study a much larger class of systems. For example, delay-differential systems with incommensurable time-delays fit into this framework.

This generalization from the rings $\mathbb{R}[s]$ and $\mathbb{R}[s, s^{-1}]$ to arbitrary integral domains resembles in a way the extension of the theory of state space systems over fields to the ring case (see e.g. [2], [13]). However, there is an important difference. In the theory of state-space systems over rings, the system defining equations are studied in a rather formal way, without fixing explicitly the context in which these equations should be interpreted. For example, the same quadruple of matrices, representing a discrete-time system over a polynomial ring in state-space form, can be used to describe a delay system with point delays. In this approach formal manipulations of the system-defining equations are emphasized, and these transformations are applicable independent of the way the equations are interpreted.

For the problem studied in this paper—system equivalence—the context in which the defining equations should be interpreted is important. This context is fixed by defining a *module* $\mathcal{M}$, describing the class of all time-trajectories under consideration. So $\mathcal{M}$ takes the place of the signal space W^T . $\mathcal{M}$ is a module over the ring $\mathcal{R}$ under consideration. Each ring element corresponds to an operator acting on the elements of the module $\mathcal{M}$. In this framework a large class of dynamic equations can be described.

In comparison with the behavioral approach, the setup presented in this paper is slightly different. Instead of explicitly describing a time-axis T , and a space W in which the signals take their values, we take a module $\mathcal{M}$ over the ring $\mathcal{R}$ as the signal space. In this way it is possible to endow the signal space with a richer structure. Furthermore, the action of the operator corresponding to a ring element of $\mathcal{R}$, can be seen as an action on a time-trajectory in $\mathcal{M}$ as a whole.

In this paper we study the problem of *system equivalence*: when are the behaviors described by two different AR-representations the same? For AR-systems over the polynomial ring $\mathbb{R}[s]$, representing continuous-time systems described by sets of linear ordinary differential equations, it is known that system equivalence can be translated into division relations among the polynomial matrices describing the system. In this paper it becomes apparent that in general the solution to the problem of system equivalence for AR-systems over rings

explicitly depends on the module $\mathcal{M}$ of all time-trajectories under consideration. Using the module $\mathcal{M}$, the ring $\mathcal{R}$ is extended to a ring $\mathcal{R}_{\mathcal{M}}$, and system equivalence is characterized by division properties of the system defining matrices over this extended ring. In this way, the well known result for AR-systems over $\mathbb{R}[s]$ is generalized to arbitrary integral domains $\mathcal{R}$.

This paper is organized as follows. First we give a formal definition of an AR-system over a ring $\mathcal{R}$. In this definition the module $\mathcal{M}$ of all time-trajectories under consideration plays an important role. It is explained how this set $\mathcal{M}$ can be seen as a module over the ring $\mathcal{R}$. In Section 3 we describe how the ring $\mathcal{R}$ can be extended to a ring $\mathcal{R}_{\mathcal{M}}$, explicitly depending on the module $\mathcal{M}$. Furthermore we show that $\mathcal{M}$ can be considered as a module over the ring extension $\mathcal{R}_{\mathcal{M}}$. Using the tools developed in Section 3, it is possible to tackle the problem of system equivalence. This solution is presented in Section 4, and consists of two parts: first we consider the square case, then the general case. In Section 5 we present some applications of the results obtained in this paper. We show how the problem of system equivalence for delay-differential systems with (in)commensurable point delays can be solved within our framework, and derive an alternative characterization of the ring $\mathcal{R}_{\mathcal{M}}$ for this particular case.

The results of this paper can be seen as a generalization of the case of differential-difference systems with commensurable delays described in [5]. In fact, [5] contains already some of the main ideas used here, but in a rather hidden way. Moreover, the proofs are quite different. Whereas in [5] most results are based on the fact that for differential-difference systems with commensurable delays the ring $\mathcal{R}_{\mathcal{M}}$ is a Bezout ring, we here use a more direct approach. In this way we solve the problem of system equivalence for a larger class of systems. Nevertheless, the approach in [5] remains interesting in its own right, because the Bezout ring property helps to solve several other system theoretic problems, for example the question of controllability of differential-difference systems with commensurable delays in a behavioral setting. Recently this question was also solved in [10], using completely different techniques.

2 AR-systems over rings

Let $\mathcal{R}$ be an *integral domain*, i.e. $\mathcal{R}$ is a commutative ring with identity and without zero divisors. Let $\mathcal{M}$ be a module over $\mathcal{R}$. We think of $\mathcal{M}$ as the space of all time-trajectories under consideration. Each element of $\mathcal{R}$ has an action on the time-trajectories in the module $\mathcal{M}$. To distinguish between an element $r \in \mathcal{R}$ and its action on the module $\mathcal{M}$, we associate to every element $r \in \mathcal{R}$ an operator $\bar{r}$:

$$\bar{r} : \mathcal{M} \longrightarrow \mathcal{M} : \quad \bar{r}(m) = r \cdot m. \quad (1)$$

The set of all operators that can be obtained in this way is again a ring, which we denote by $\overline{\mathcal{R}}$:

$$\overline{\mathcal{R}} := \{\bar{r} : \mathcal{M} \longrightarrow \mathcal{M} \mid r \in \mathcal{R}\}. \quad (2)$$

Of course we want that different ring elements correspond to different operators. Throughout the paper we therefore make the following assumption.

Assumption 2.1 If $r \in \mathcal{R}$ is such that $\forall m \in \mathcal{M} : \bar{r}(m) = 0$, then $r = 0$.

Formally, Assumption 2.1 means that the surjective ring homomorphism T between $\mathcal{R}$ and $\overline{\mathcal{R}}$, described by $T(r) = \bar{r}$, is also injective. So $\mathcal{R}$ and $\overline{\mathcal{R}}$ are assumed to be *isomorphic*. Note that Assumption 2.1 puts a condition on the module $\mathcal{M}$. This module should contain enough elements in order to distinguish between operators corresponding to different elements of $\mathcal{R}$.

Since $\mathcal{R}$ and $\overline{\mathcal{R}}$ are isomorphic, $\mathcal{M}$ may also be considered as a module over $\overline{\mathcal{R}}$. This means that the pair $(\mathcal{R}, \mathcal{M})$ satisfies the following conditions:

- (i) $\mathcal{M}$ is a commutative group with respect to addition,
- (ii) for all $r, r_1, r_2 \in \mathcal{R}$ and for all $m, m_1, m_2 \in \mathcal{M}$ we have

$$\overline{r}(m_1 + m_2) = \overline{r}(m_1) + \overline{r}(m_2), \quad (3)$$

$$\overline{(r_1 + r_2)}(m) = \overline{r_1}(m) + \overline{r_2}(m), \quad (4)$$

$$\overline{(r_1 r_2)}(m) = \overline{r_1}(\overline{r_2}(m)), \quad (5)$$

$$\overline{1}(m) = m, \quad (6)$$

where 1 denotes the identity element of $\mathcal{R}$.

Next we extend our framework to the multivariable case. Let P be a $p \times q$ matrix over the integral domain $\mathcal{R}$. Let $\mathcal{M}$ be a module over $\mathcal{R}$. By replacing each entry p_{ij} of P by the corresponding operator $\overline{p}_{ij} : \mathcal{M} \rightarrow \mathcal{M}$, the matrix P is turned into an operator $\overline{P} : \mathcal{M}^q \rightarrow \mathcal{M}^p$, where $\mathcal{M}^q$ and $\mathcal{M}^p$ are spaces of *multivariable* time-trajectories.

Definition 2.2 Let $\mathcal{R}$ be an integral domain, and $\mathcal{M}$ a module over $\mathcal{R}$. Let $P \in \mathcal{R}^{p \times q}$. Then the pair $(P, \mathcal{M}^q)$ describes the *AR-system in $\mathcal{M}^q$, belonging to P* . The *behavior* of this system is given by

$$\mathcal{B}(P, \mathcal{M}^q) = \{m \in \mathcal{M}^q \mid \overline{P}(m) = 0\}. \quad (7)$$

In Definition 2.2 the matrix P determines the algebraic structure of the system: it contains the equations describing the laws governing the system. The module $\mathcal{M}$ determines how the equations described by P should be interpreted. The behavior of a system $(P, \mathcal{M}^q)$ is the kernel of the operator $\overline{P}$ in $\mathcal{M}^q$. If the module $\mathcal{M}$ has been fixed, or if there is no confusion on the the interpretation at hand, the behavior $\mathcal{B}(P, \mathcal{M}^q)$ is also denoted by $\ker(\overline{P})$.

Example 2.3 Let $\mathcal{R} = \mathbb{R}[s]$ and $\mathcal{M}_1 = \mathcal{C}^\infty(\mathbb{R})$, the space of all arbitrarily often continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. To each polynomial $r \in \mathcal{R}$ corresponds an operator $\overline{r} : \mathcal{M}_1 \rightarrow \mathcal{M}_1$, obtained after substitution of the differentiation operator $\frac{d}{dt}$ for s . For $P \in \mathcal{R}^{p \times q}$ the behavior $\mathcal{B}(P, \mathcal{M}_1^q)$ is the solution set of a p -tuple of linear algebraic- and ordinary differential equations in q variables.

Next consider the same ring, but now in combination with the module $\mathcal{M}_2 = \mathbb{R}^{\mathbb{Z}}$ of all functions from $\mathbb{Z}$ to $\mathbb{R}$. After substitution of the unit shift operator σ defined by

$$\sigma : \mathcal{M}_2 \rightarrow \mathcal{M}_2 : \quad (\sigma x)(t) = x(t - 1),$$

a polynomial $r \in \mathcal{R}$ is turned into a difference operator $\overline{r}$. The behavior $\mathcal{B}(P, \mathcal{M}_2^q)$, determined by a matrix $P \in \mathcal{R}^{p \times q}$, represents a discrete-time system with q variables, governed by a set of p difference equations.

If P and Q are matrices over the ring $\mathcal{R}$ with q columns, and a module $\mathcal{M}$ over $\mathcal{R}$ has been fixed, the behaviors $\mathcal{B}(P, \mathcal{M}^q)$ and $\mathcal{B}(Q, \mathcal{M}^q)$ are uniquely determined. Although the matrices P and Q may be different, it is still possible that their behaviors are the same.

Definition 2.4 Let $P \in \mathcal{R}^{p_1 \times q}$, and $Q \in \mathcal{R}^{p_2 \times q}$. Let $\mathcal{M}$ be a module over $\mathcal{R}$. Then the AR-systems $(P, \mathcal{M}^q)$ and $(Q, \mathcal{M}^q)$ are called *equivalent* if

$$\mathcal{B}(P, \mathcal{M}^q) = \mathcal{B}(Q, \mathcal{M}^q). \quad (8)$$

Example 2.5 Consider the polynomial ring $\mathcal{R} = \mathbb{R}[s, z]$, and let $\mathcal{M} = \mathcal{C}^\infty(\mathbb{R})$. A polynomial $p(s, z) \in \mathbb{R}[s, z]$ is turned into an operator on $\mathcal{M} = \mathcal{C}^\infty(\mathbb{R})$ by substituting the differential operator $\frac{d}{dt}$ for s , and the unit shift operator σ ($\sigma : \mathcal{M} \rightarrow \mathcal{M} : (\sigma x)(t) = x(t - 1)$) for z . Since the operators $\frac{d}{dt}$ and σ commute, and Assumption 2.1 is satisfied, the polynomial ring $\mathcal{R}$ and the ring of operators $\overline{\mathcal{R}}$ are isomorphic. It is obvious that $\mathcal{M}$ is a module over $\overline{\mathcal{R}}$.

Now we consider two matrices over $\mathbb{R}[s, z]$:

$$P = \begin{pmatrix} s & s^2 + sz & 0 \\ 0 & z & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} z - 1 & (z - 1)s + z^2 & 1 \\ sz & zs^2 + (z^2 + z)s & s \end{pmatrix}.$$

The behavior $\mathcal{B}(Q, \mathcal{M}^3)$ is characterized by all triples $(x_1, x_2, x_3) \in \mathcal{M}^3$, satisfying the two differential-difference equations:

$$x_1(t - 1) - x_1(t) + \dot{x}_2(t - 1) - \dot{x}_2(t) + x_2(t - 2) + x_3(t) = 0, \quad (9)$$

$$\dot{x}_1(t - 1) + \ddot{x}_2(t - 1) + \dot{x}_2(t - 2) + \dot{x}_2(t - 1) + \dot{x}_3(t) = 0. \quad (10)$$

Differentiation of (9), and subtracting this equation from (10) yields

$$\dot{x}_1(t) + \ddot{x}_2(t) + \dot{x}_2(t - 1) = 0.$$

Therefore every trajectory $(x_1, x_2, x_3) \in \mathcal{B}(Q, \mathcal{M}^3)$ also satisfies the first differential-difference equation describing the behavior $\mathcal{B}(P, \mathcal{M}^3)$.

Since $(x_1, x_2, x_3) \in \mathcal{M}^3$, equation (10) can be integrated; in this way we obtain

$$x_1(t - 1) + \dot{x}_2(t - 1) + x_2(t - 2) + x_2(t - 1) + x_3(t) = K, \quad (11)$$

with K a constant in $\mathbb{R}$. Application of the shift operator on (11), and multiplication by -1 yields

$$-x_1(t - 2) - \dot{x}_2(t - 2) - x_2(t - 3) - x_2(t - 2) - x_3(t - 1) = -K. \quad (12)$$

Also equation (9) may be shifted in time:

$$x_1(t - 2) - x_1(t - 1) + \dot{x}_2(t - 2) - \dot{x}_2(t - 1) + x_2(t - 3) + x_3(t - 1) = 0. \quad (13)$$

Addition of (11), (12), and (13) yields

$$x_2(t - 1) + x_3(t) = 0.$$

Therefore $(x_1, x_2, x_3) \in \mathcal{B}(Q, \mathcal{M}^3)$ also satisfies the second equation describing the behavior $\mathcal{B}(P, \mathcal{M}^3)$. We conclude that

$$\mathcal{B}(Q, \mathcal{M}^3) \subset \mathcal{B}(P, \mathcal{M}^3).$$

Using similar techniques, one may also prove that the inclusion in the opposite direction holds. So P and Q describe the same behavior; the AR-systems $(P, \mathcal{M}^3)$ and $(Q, \mathcal{M}^3)$ are equivalent.

In this paper we study the problem of system equivalence: given a module $\mathcal{M}$ over the ring $\mathcal{R}$, we want to characterize all matrices over $\mathcal{R}$ describing the same behavior. Instead of using ad hoc methods like in Example 2.5, this question is solved using division properties of the system defining matrices P and Q over a ring extension of $\mathcal{R}$, determined by the module $\mathcal{M}$. So the interpretation of the system defining equations plays a role in the problem of system equivalence. This is already suggested in Example 2.5. There the behaviors $\mathcal{B}(P, \mathcal{M}^3)$ and $\mathcal{B}(Q, \mathcal{M}^3)$ are the same, although there does not exist a matrix U over $\mathbb{R}[s, z]$ such that $Q = UP$. In Chapter 5 we will see that P and Q do exhibit division properties in a ring extension of $\mathbb{R}[s, z]$, depending on the module $\mathcal{M}$.

3 An important ring extension

Given a module $\mathcal{M}$ over the integral domain $\mathcal{R}$, the ring $\overline{\mathcal{R}}$ is not a complete description of all linear operators from $\mathcal{M}$ to $\mathcal{M}$. In the problem of system equivalence it turns out that the ring $\overline{\mathcal{R}}$ is sometimes too small. To overcome this difficulty we introduce a ring extension of $\mathcal{R}$, explicitly depending on the module $\mathcal{M}$. Each element of this ring extension corresponds to a linear operator from $\mathcal{M}$ to $\mathcal{M}$.

Definition 3.1 Let $\mathcal{R}$ be an integral domain, and let $\mathcal{M}$ be a module over $\mathcal{R}$. Let $r \in \mathcal{Q}(\mathcal{R})$, where $\mathcal{Q}(\mathcal{R})$ denotes the quotient field of $\mathcal{R}$. Then a pair $p, q \in \mathcal{R}$ is called an *admissible fractional representation* of r if

- (i) $r = \frac{p}{q}$,
- (ii) $\overline{q} : \mathcal{M} \rightarrow \mathcal{M}$ is surjective,
- (iii) $\ker(\overline{q}) \subset \ker(\overline{p})$.

The set $\mathcal{R}_{\mathcal{M}}$ is defined as the ring extension of $\mathcal{R}$, consisting of all elements in $\mathcal{Q}(\mathcal{R})$, for which there exists an admissible fractional representation.

Note that not every element in $\mathcal{Q}(\mathcal{R})$ has an admissible fractional representation, i.e. $\mathcal{R}_{\mathcal{M}} \subsetneq \mathcal{Q}(\mathcal{R})$.

Lemma 3.2 $\mathcal{R}_{\mathcal{M}}$ is a subring of $\mathcal{Q}(\mathcal{R})$. $\mathcal{R}_{\mathcal{M}}$ is an integral domain and $\mathcal{R} \subset \mathcal{R}_{\mathcal{M}}$.

Proof: Let $r_1, r_2 \in \mathcal{R}_{\mathcal{M}}$, with admissible fractional representations $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$, respectively. To show that $\mathcal{R}_{\mathcal{M}}$ is a subring of $\mathcal{Q}(\mathcal{R})$ we have to verify that both $r_1 - r_2$ and $r_1 \cdot r_2$ are elements of $\mathcal{R}_{\mathcal{M}}$.

Clearly we have $r_1 - r_2 = \frac{p_1 q_2 - p_2 q_1}{q_1 q_2}$. Moreover, $\overline{q_1 q_2}$ is surjective since both $\overline{q_1}$ and $\overline{q_2}$ are surjective. Let $x \in \ker(\overline{q_1 q_2})$, i.e. $\overline{q_1 q_2}(x) = 0$. Then $\overline{q_2}(x) \in \ker(\overline{q_1}) \subset \ker(\overline{p_1})$, hence $\overline{p_1 q_2}(x) = \overline{p_1}(\overline{q_2}(x)) = 0$. Analogously, since q_1 and q_2 , and therefore also $\overline{q_1}$ and $\overline{q_2}$ commute, we have $\overline{q_1}(x) \in \ker(\overline{q_2}) \subset \ker(\overline{p_2})$, and therefore $\overline{p_2 q_1}(x) = 0$. So $x \in \ker(\overline{p_1 q_2 - p_2 q_1})$, and thus $p_1 q_2 - p_2 q_1, q_1 q_2$ is an admissible fractional representation of $r_1 - r_2$, i.e. $r_1 - r_2 \in \mathcal{R}_{\mathcal{M}}$.

Next, consider $r_1 \cdot r_2 = \frac{p_1 p_2}{q_1 q_2}$. Again, it is obvious that $\overline{q_1 q_2}$ is surjective. Let $x \in \ker(\overline{q_1 q_2})$. Then $\overline{q_2}(x) \in \ker(\overline{q_1}) \subset \ker(\overline{p_1})$, so $\overline{p_1 q_2}(x) = 0$. Since $\overline{p_1}$ and $\overline{q_2}$ commute, it follows that $\overline{p_1}(x) \in \ker(\overline{q_2}) \subset \ker(\overline{p_2})$. We conclude that $x \in \ker(\overline{p_2 p_1}) = \ker(\overline{p_1 p_2})$, and thus $\frac{p_1 p_2}{q_1 q_2}$ is an admissible fractional representation of $r_1 r_2$. Hence $r_1 r_2 \in \mathcal{R}_{\mathcal{M}}$.

Since every element $p \in \mathcal{R}$ has an admissible fractional representation $\frac{p}{1}$, it is obvious that $\mathcal{R} \subset \mathcal{R}_{\mathcal{M}}$. Moreover, since $\mathcal{R}$ is an integral domain, we have $1 \in \mathcal{R}_{\mathcal{M}}$, and $\mathcal{R}_{\mathcal{M}}$ is also an integral domain. ■

Definition 3.3 Let $r \in \mathcal{R}_{\mathcal{M}}$, with admissible fractional representation $\frac{p}{q}$. Then the operator $\overline{r} : \mathcal{M} \rightarrow \mathcal{M}$ is defined in the following way. Let $m \in \mathcal{M}$. Choose $w \in \mathcal{M}$ such that $\overline{q}(w) = m$. Then $\overline{r}(m) := \overline{p}(w)$.

It is not immediately clear that the action of $\overline{r}$ (with $r \in \mathcal{R}_{\mathcal{M}}$) on elements of $\mathcal{M}$ is well defined. If $\frac{p}{q}$ is an admissible fractional representation of r , we know that $\overline{q}$ is surjective. So there exists a $w \in \mathcal{M}$ such that $\overline{q}(w) = m$, but w is not necessarily unique. Therefore we have to show that the value of $\overline{r}(m)$ is independent of the solution w of the equation $\overline{q}(w) = m$.

Moreover, since $r \in \mathcal{R}_{\mathcal{M}}$ may not have a unique admissible fractional representation, we also have to prove that the operator $\bar{r}$ is independent of the chosen fractional representation $\frac{p}{q}$.

Let $m \in \mathcal{M}$. Let $w_1, w_2 \in \mathcal{M}$ be such that $\bar{q}(w_1) = \bar{q}(w_2) = m$. Since $\bar{q}$ is a linear operator, we have $\bar{q}(w_1 - w_2) = 0$. Therefore $w_1 - w_2 \in \ker(\bar{q}) \subset \ker(\bar{p})$. Hence $\bar{p}(w_1 - w_2) = 0$, and thus $\bar{p}(w_1) = \bar{p}(w_2)$. This implies that $\bar{r}(m)$ is independent of the solution w of $\bar{q}(w) = m$.

Next, let $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be two admissible fractional representations of $r \in \mathcal{R}_{\mathcal{M}}$. Then $p_1 q_2 = p_2 q_1$. Let $m \in \mathcal{M}$, and let $w_1, w_2 \in \mathcal{M}$ be such that $\bar{q}_1(w_1) = \bar{q}_2(w_2) = m$. To show that $\bar{r}$ is independent of a particular fractional representation, we have to verify that $\bar{p}_1(w_1) = \bar{p}_2(w_2)$. Since $\bar{q}_1$ is surjective, there exists a $y \in \mathcal{M}$ such that $\bar{q}_1(y) = w_2$. Because q_1 and q_2 commute, we have

$$\bar{q}_1(\bar{q}_2(y) - w_1) = \bar{q}_2(\bar{q}_1(y)) - \bar{q}_1(w_1) = \bar{q}_2(w_2) - \bar{q}_1(w_1) = m - m = 0.$$

Hence $\bar{q}_2(y) - w_1 \in \ker(\bar{q}_1) \subset \ker(\bar{p}_1)$, and therefore also $\bar{p}_1(\bar{q}_2(y) - w_1) = 0$. Recalling that $p_1 q_2 = p_2 q_1$, we find

$$\bar{p}_1(w_1) = \bar{p}_1 \bar{q}_2(y) = \bar{p}_2 \bar{q}_1(y) = \bar{p}_2(w_2).$$

Let $\overline{\mathcal{R}_{\mathcal{M}}}$ denote the set of all operators from $\mathcal{M}$ to $\mathcal{M}$ that are obtained by applying Definition 3.3 to the elements of $\mathcal{R}_{\mathcal{M}}$. Then Assumption 2.1 guarantees that different ring elements in $\mathcal{R}_{\mathcal{M}}$ correspond to different operators in $\overline{\mathcal{R}_{\mathcal{M}}}$:

Lemma 3.4 *Let $r \in \mathcal{R}_{\mathcal{M}}$, and assume that*

$$\forall m \in \mathcal{M} : \bar{r}(m) = 0.$$

Then $r = 0$.

Proof: Let $r \in \mathcal{R}_{\mathcal{M}}$, and let $\frac{p}{q}$ be an admissible fractional representation of r . Assume that $\bar{r}(m) = 0$ for all $m \in \mathcal{M}$. Let $x \in \mathcal{M}$. Choose $m := \bar{q}(x)$. Then $\bar{p}(x) = \bar{r}(m) = 0$. So, for all $x \in \mathcal{M} : \bar{p}(x) = 0$, and Assumption 2.1 implies that $p = 0$, hence $r = 0$. ■

In the next theorem it is shown that the module structure of $\mathcal{M}$ over $\mathcal{R}$ carries over to the ring extension $\mathcal{R}_{\mathcal{M}}$. This observation plays a crucial role in our solution to the problem of system equivalence.

Theorem 3.5 *Let $\mathcal{R}$ be an integral domain and $\mathcal{M}$ be a module over $\mathcal{R}$. Then $\mathcal{M}$ is also a module over $\mathcal{R}_{\mathcal{M}}$.*

Proof: Since $\mathcal{M}$ is a module over $\mathcal{R}$, $\mathcal{M}$ is—by definition—a commutative group with respect to addition. So we only have to prove that for all $r, r_1, r_2 \in \mathcal{R}_{\mathcal{M}}$, and for all $m, m_1, m_2 \in \mathcal{M}$, formulae (3) — (6) hold.

Let $r \in \mathcal{R}_{\mathcal{M}}$ with admissible fractional representation $r = \frac{p}{q}$. Let $m_1, m_2 \in \mathcal{M}$. Choose $w_1, w_2 \in \mathcal{M}$ such that $\bar{q}(w_1) = m_1$ and $\bar{q}(w_2) = m_2$. Then $\bar{r}(m_1) + \bar{r}(m_2) = \bar{p}(w_1) + \bar{p}(w_2)$. On the other hand, $\bar{q}(w_1 + w_2) = \bar{q}(w_1) + \bar{q}(w_2) = m_1 + m_2$, and thus

$$\bar{r}(m_1 + m_2) = \bar{p}(w_1 + w_2) = \bar{p}(w_1) + \bar{p}(w_2) = \bar{r}(m_1) + \bar{r}(m_2).$$

To prove (4), let $r_1, r_2 \in \mathcal{R}_{\mathcal{M}}$, with admissible fractional representations $r_1 = \frac{p_1}{q_1}$, and $r_2 = \frac{p_2}{q_2}$, respectively. Let $m \in \mathcal{M}$. Choose $w_1, w_2 \in \mathcal{M}$ such that $\bar{q}_1(w_1) = m$ and $\bar{q}_2(w_2) = m$. Then

$$\bar{r}_1(m) + \bar{r}_2(m) = \bar{p}_1(w_1) + \bar{p}_2(w_2).$$

Using the same arguments as in the proof of Lemma 3.2, one verifies that $\frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$ is an admissible fractional representation of $r_1 + r_2$. Since $\overline{q_2}$ is surjective, there exists a $v \in \mathcal{M}$ such that $\overline{q_2}(v) = w_1$. Now $\overline{q_1}$ and $\overline{q_2}$ commute, and thus $\overline{q_2}(w_2 - \overline{q_1}(v)) = \overline{q_2}(w_2) - \overline{q_1}(\overline{q_2}(v)) = m - m = 0$, i.e. $(w_2 - \overline{q_1}(v)) \in \ker(\overline{q_2}) \subset \ker(\overline{p_2})$. So $\overline{p_2}(w_2 - \overline{q_1}(v)) = 0$, and therefore $\overline{p_2 q_1}(v) = \overline{p_2}(w_2)$. This formula is used in the computation of $(r_1 + r_2)(m)$. First note that $\overline{q_1 q_2}(v) = \overline{q_1}(w_1) = m$. Then it follows that

$$\begin{aligned} \overline{(r_1 + r_2)}(m) &= \overline{(p_1 q_2 + p_2 q_1)}(v) = \overline{p_1}(\overline{q_2}(v)) + \overline{p_2 q_1}(v) = \\ &= \overline{p_1}(w_1) + \overline{p_2}(w_2) = \overline{r_1}(m) + \overline{r_2}(m). \end{aligned} \quad (14)$$

Next we prove (5). Let again $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ be admissible fractional representations of $r_1, r_2 \in \mathcal{R}_{\mathcal{M}}$, respectively. Let $m \in \mathcal{M}$, and choose $w_2 \in \mathcal{M}$ such that $\overline{q_2}(w_2) = m$. Then $\overline{r_2}(m) = \overline{p_2}(w_2)$. Choose $w_1 \in \mathcal{M}$ such that $\overline{q_1}(w_1) = \overline{p_2}(w_2)$. Then by definition

$$\overline{r_1}(\overline{r_2}(m)) = \overline{p_1}(w_1).$$

According the proof of Lemma 3.2, $r_1 r_2 = \frac{p_1 p_2}{q_1 q_2}$ is an admissible fractional representation of $r_1 r_2$. Since $\overline{q_1}$ is surjective, there exists an $x \in \mathcal{M}$ such that $\overline{q_1}(x) = w_2$. Recalling that $\overline{q_1}$ and $\overline{q_2}$ commute, we find $\overline{q_1 q_2}(x) = \overline{q_2}(\overline{q_1}(x)) = \overline{q_2}(w_2) = m$. Hence $\overline{r_1 r_2}(m) = \overline{p_1 p_2}(x)$. In order to compute $\overline{p_1 p_2}(x)$, note that $\overline{q_1}(\overline{p_2}(x) - w_1) = \overline{p_2}(\overline{q_1}(x)) - \overline{q_1}(w_1) = \overline{p_2}(w_2) - \overline{p_2}(w_2) = 0$, where we used the definitions of x, w_1, w_2 , and the fact that $\overline{q_1}$ and $\overline{p_2}$ commute. We conclude that $(\overline{p_2}(x) - w_1) \in \ker(\overline{q_1}) \subset \ker(\overline{p_1})$, so $\overline{p_1 p_2}(x) - \overline{p_1}(w_1) = \overline{p_1}(\overline{p_2}(x) - w_1) = 0$, and thus

$$\overline{r_1 r_2}(m) = \overline{p_1 p_2}(x) = \overline{p_1}(w_1) = \overline{r_1}(\overline{r_2}(m)). \quad (15)$$

Since the correctness of (6) is trivial, this completes the proof. $\blacksquare$

Theorem 3.5 implies that the rings $\mathcal{R}_{\mathcal{M}}$ and $\overline{\mathcal{R}_{\mathcal{M}}}$ are isomorphic. From (14) and (15) it follows that the mapping $T : \mathcal{R}_{\mathcal{M}} \rightarrow \overline{\mathcal{R}_{\mathcal{M}}} : T(r) = \overline{r}$ is a surjective ring homomorphism. According to Lemma 3.4, T is also injective. Therefore, $\mathcal{M}$ can also be considered as a module over $\overline{\mathcal{R}_{\mathcal{M}}}$.

Let $P \in \mathcal{R}^{p \times q}$, and consider the system $(P, \mathcal{M}^q)$ over $\mathcal{R}$. Since $\mathcal{R} \subset \mathcal{R}_{\mathcal{M}}$ and $\mathcal{M}$ is also a module over $\mathcal{R}_{\mathcal{M}}$, $(P, \mathcal{M}^q)$ can also be considered as a system over the ring $\mathcal{R}_{\mathcal{M}}$. This change in point of view does not change the behavior of the system, because both the set of laws governing the system, and the space of all time-trajectories under consideration remain the same. In fact, the class of AR-systems over $\mathcal{R}$ is embedded in the class of AR-systems over $\mathcal{R}_{\mathcal{M}}$. In this way, our freedom to manipulate the system defining equations determined by the matrix P , without actually changing the behavior, is enlarged, provided that the ring $\mathcal{R}_{\mathcal{M}}$ is indeed larger than $\mathcal{R}$. This is often the case, for example for differential-difference systems with commensurable delays.

Example 3.6 Let $\mathcal{R} = \mathbb{R}[s, z]$ and $\mathcal{M} = \mathcal{C}^\infty(\mathbb{R})$, and consider (as in Example 2.5) $\overline{p} \in \overline{\mathcal{R}}$ as a delay-differential operator from $\mathcal{M}$ to $\mathcal{M}$, obtained after substitution of $\frac{d}{dt}$ for s and the unit delay operator σ for z . Let $r = \frac{1-z}{s} \in \mathbb{R}(s, z)$, and define $p := 1 - z$ and $q := s$. On $\mathcal{C}^\infty(\mathbb{R})$ the operator $\overline{q}$ (pure differentiation) is clearly surjective. Moreover

$\ker(\overline{q}) = \{\text{constant functions}\} \subset \{\text{periodic functions in } \mathcal{C}^\infty(\mathbb{R}) \text{ with period } 1\} = \ker(\overline{p})$, so $\frac{p}{q}$ is an admissible fractional representation of r , hence $r \in \mathcal{R}_{\mathcal{M}}$. It is obvious that $r \notin \mathcal{R}$, and therefore the ring $\mathcal{R}_{\mathcal{M}}$ is strictly larger than $\mathcal{R}$. The operator $\overline{r} \in \overline{\mathcal{R}_{\mathcal{M}}}$, corresponding to r , acts on elements $f \in \mathcal{C}^\infty(\mathbb{R})$, and is given by

$$\overline{r} : \mathcal{C}^\infty(\mathbb{R}) \rightarrow \mathcal{C}^\infty(\mathbb{R}) : \quad (\overline{r}(f))(t) = \int_{t-1}^t f(\tau) d\tau.$$

Given an integral domain $\mathcal{R}$ and a module $\mathcal{M}$ over $\mathcal{R}$, the ring $\mathcal{R}$ can be extended to the integral domain $\mathcal{R}_{\mathcal{M}}$, and $\mathcal{M}$ is a module over $\mathcal{R}_{\mathcal{M}}$. At this point we are in the same situation as before, so one might repeat the same extension procedure in order to obtain a ring extension of $\mathcal{R}_{\mathcal{M}}$. The next result indicates that such a repetition is useless since the ring extension of $\mathcal{R}_{\mathcal{M}}$ does not contain any new elements.

Proposition 3.7 *Let $\mathcal{R}$ be an integral domain, and let $\mathcal{M}$ be a module over $\mathcal{R}$. Let $\mathcal{R}_{\mathcal{M}}$ be the ring extension of $\mathcal{R}$ as described in Definition 3.1. Let $\mathcal{S}_{\mathcal{M}}$ denote the ring obtained after applying the same ring extension procedure to $\mathcal{R}_{\mathcal{M}}$, i.e. $\mathcal{S}_{\mathcal{M}} = (\mathcal{R}_{\mathcal{M}})_{\mathcal{M}}$. Then*

$$\mathcal{S}_{\mathcal{M}} = \mathcal{R}_{\mathcal{M}}. \quad (16)$$

Proof: The inclusion $\mathcal{R}_{\mathcal{M}} \subset \mathcal{S}_{\mathcal{M}}$ is trivial, so we only have to show that all elements of $\mathcal{S}_{\mathcal{M}}$ are contained in $\mathcal{R}_{\mathcal{M}}$. First note that $\mathcal{S}_{\mathcal{M}} \subset \mathcal{Q}(\mathcal{R})$: since all elements of $\mathcal{R}_{\mathcal{M}}$ belong to the quotient field $\mathcal{Q}(\mathcal{R})$, a fraction of elements of $\mathcal{R}_{\mathcal{M}}$ also belongs to $\mathcal{Q}(\mathcal{R})$. Let $u \in \mathcal{S}_{\mathcal{M}}$ with admissible fractional representation $u = \frac{r_1}{r_2}$ with respect to $\mathcal{R}_{\mathcal{M}}$, i.e. r_1 and r_2 satisfy conditions (i), (ii), and (iii) of Definition 3.1, with $\mathcal{R}$ replaced by $\mathcal{R}_{\mathcal{M}}$. Both r_1 and r_2 have an admissible fractional representation over $\mathcal{R}$, say $r_1 = \frac{p_1}{q_1}$ and $r_2 = \frac{p_2}{q_2}$, respectively. Regarding u as an element of $\mathcal{Q}(\mathcal{R})$ we have

$$u = \frac{r_1}{r_2} = \frac{\frac{p_1}{q_1}}{\frac{p_2}{q_2}} = \frac{p_1 q_2}{p_2 q_1}.$$

We have to show that $\frac{(p_1 q_2)}{(p_2 q_1)}$ is an admissible fractional representation of u over $\mathcal{R}$.

By assumption, the operator $\bar{r}_2$ is surjective. For the admissible fractional representation $r_2 = \frac{p_2}{q_2}$, this implies that $\bar{p}_2$ is surjective. Since $\frac{p_1}{q_1}$ is an admissible fractional representation of r_1 , also $\bar{q}_1$ is surjective, and therefore $\bar{p}_2 \bar{q}_1$ is surjective.

Let $w \in \ker(\bar{p}_2 \bar{q}_1)$, and define $y := \bar{q}_2(\bar{q}_1(w))$. Using Definition 3.3 it follows that $\bar{r}_2(y) = \bar{p}_2(\bar{q}_1(w)) = 0$. Since $\ker(\bar{r}_2) \subset \ker(\bar{r}_1)$, we have $\bar{r}_1(y) = 0$. On the other hand, $\bar{r}_1(y)$ may be computed directly, using Definition 3.3. Recalling $\bar{q}_1(\bar{q}_2(w)) = \bar{q}_2(\bar{q}_1(w)) = y$, we find

$$0 = \bar{r}_1(y) = \bar{p}_1(\bar{q}_2(w)) = \bar{p}_1 \bar{q}_2(w).$$

Hence $w \in \ker(\bar{p}_1 \bar{q}_2)$, and we conclude that $u = \frac{p_1 q_2}{p_2 q_1} \in \mathcal{R}_{\mathcal{M}}$. ■

Proposition 3.7 indicates that the transition from the integral domain $\mathcal{R}$ to $\mathcal{R}_{\mathcal{M}}$ is not always a ring extension. For the problem of system equivalence it is interesting to know, under which conditions $\mathcal{R} = \mathcal{R}_{\mathcal{M}}$. In this paper, this question cannot be answered completely; we only mention the following partial result.

Proposition 3.8 *Let $\mathcal{R}$ be an integral domain and let $\mathcal{M}$ be a module over $\mathcal{R}$. Assume that the following two conditions are satisfied:*

- (i) $\mathcal{R}$ is a Bezout domain.
- (ii) If $p \in \mathcal{R}$ is such that $\ker(\bar{p}) = \{0\}$, then p is a unit in $\mathcal{R}$.

Then $\mathcal{R} = \mathcal{R}_{\mathcal{M}}$.

Proof: Since the other inclusion is trivial, we only have to show that $\mathcal{R}_{\mathcal{M}} \subset \mathcal{R}$. Let $r \in \mathcal{R}_{\mathcal{M}}$, with admissible fractional representation $r = \frac{p}{q}$. Because $\mathcal{R}$ is a Bezout domain, the ideal $\langle p, q \rangle$ generated by p and q is a principal ideal. So there exist a $d \in \mathcal{R}$ such that $\langle p, q \rangle = \langle d \rangle$, and $a, b \in \mathcal{R}$ such that $p = ad$ and $q = bd$. Since $\bar{q}$ is surjective, both $\bar{b}$ and $\bar{d}$ are surjective.

Next we prove that $\ker(\bar{b}) \subset \ker(\bar{a})$. Let $w \in \mathcal{M}$ be such that $\bar{b}(w) = 0$. Since $\bar{d}$ is surjective, there exists a $y \in \mathcal{M}$ such that $w = \bar{d}(y)$. Then $\bar{b}\bar{d}(y) = \bar{b}(w) = 0$, hence $y \in \ker(\bar{q}) \subset \ker(\bar{p})$, and thus we have $0 = \bar{p}(y) = \bar{a}\bar{d}(y) = \bar{a}(w)$. So $w \in \ker(\bar{a})$, and we conclude that also $\frac{a}{b}$ is an admissible fractional representation of r .

By construction, the elements $a, b \in \mathcal{R}$ are coprime in the Bezout ring $\mathcal{R}$, so there exist $g, h \in \mathcal{R}$ such that

$$a \cdot g + b \cdot h = 1.$$

This implies that $\ker(\bar{b}) = \{0\}$, because if $x \in \mathcal{M}$ is such that $\bar{b}(x) = 0$, also $\bar{a}(x) = 0$, and thus

$$x = \bar{1}(x) = \overline{ag + bh}(x) = \bar{g}(\bar{a}(x)) + \bar{h}(\bar{b}(x)) = 0.$$

Using Assumption (ii), we conclude that b is a unit in $\mathcal{R}$, i.e. b^{-1} exists and is an element of $\mathcal{R}$. Therefore $r = \frac{a}{b} = a \cdot b^{-1} \in \mathcal{R}$. ■

Example 3.9 Consider the situation described in Example 2.3, with $\mathcal{R} = \mathbb{R}[s]$, i.e. $\mathcal{R}$ is a principal ideal domain, and $\mathcal{M}_1 = \mathcal{C}^\infty(\mathbb{R})$, and $\mathcal{M}_2 = \mathbb{R}^{\mathbb{Z}}$.

In the continuous-time interpretation, an operator $\bar{p}$ corresponding to a polynomial $p \in \mathcal{R}$ has kernel $\{0\}$ if and only if p is a nonzero constant. Since every linear ordinary differential equation has a nontrivial solution, only elements of $\mathbb{R} \setminus \{0\}$ correspond to injective operators from $\mathcal{M}_1$ to $\mathcal{M}_1$. Since $\mathbb{R} \setminus \{0\}$ is exactly the set of units of $\mathcal{R}$, condition (ii) of Proposition 3.8 is satisfied, and $\mathcal{R}_{\mathcal{M}_1} = \mathcal{R}$.

In the discrete-time interpretation the polynomial $p_1(s) = s$ corresponds to the unit shift operator from $\mathcal{M}_2$ to $\mathcal{M}_2$. Although this operator is injective, the polynomial p_1 is not a unit in $\mathcal{R}$, and condition (ii) of Proposition 3.8 is violated. All rational functions of the form $\frac{1}{s^k} \cdot p(s)$, with $p \in \mathcal{R}$ and $k \in \mathbb{N} \cup \{0\}$ belong to $\mathcal{R}_{\mathcal{M}_2}$. Moreover, all elements of $\mathcal{R}_{\mathcal{M}_2}$ can be written in this form. Hence $\mathcal{R}_{\mathcal{M}_2} = \mathbb{R}[s, s^{-1}]$.

4 System equivalence

For the study of the relationship between system equivalence and division properties (over the ring $\mathcal{R}_{\mathcal{M}}$) of the matrices characterizing an AR-system, we need an additional assumption.

Assumption 4.1 $\forall q \in \mathcal{R} \setminus \{0\} : \bar{q} : \mathcal{M} \longrightarrow \mathcal{M}$ is surjective.

Assumption 4.1 implies that condition (ii) in Definition 3.1 for admissible fractional representations is always satisfied. For continuous-time systems the assumption is often satisfied, e.g. for differential-difference operators with (in)commensurable time-delays (see Section 5). However, unlike Assumption 2.1, Assumption 4.1 is restrictive. For several discrete-time systems the condition is not satisfied. Some of these situations may be treated using different techniques, not included in this paper.

Remark 4.2 Most results in this section can be adapted to the situation in which Assumption 4.1 does not hold. However, using this assumption (valid in a lot of interesting applications) the theory becomes more elegant.

The results on system equivalence are divided into two groups. First we consider square matrices; subsequently these results are used in the solution of the general case.

Proposition 4.3 Let $P \in \mathcal{R}^{p \times p}$ and $Q \in \mathcal{R}^{q \times p}$, and assume that $\det(P) \neq 0$. Then

$$\begin{aligned} & \mathcal{B}(P, \mathcal{M}^p) \subset \mathcal{B}(Q, \mathcal{M}^p) \\ \iff & \\ & \exists U \in \mathcal{R}_{\mathcal{M}}^{q \times p} : Q = U \cdot P. \end{aligned}$$

In particular, if $\mathcal{B}(P, \mathcal{M}^p) \subset \mathcal{B}(Q, \mathcal{M}^p)$, then $U := Q \cdot P^{-1}$ is a matrix over $\mathcal{R}_{\mathcal{M}}^{q \times p}$, satisfying $Q = U \cdot P$.

Proof: “ $\Leftarrow$ ” Let $U \in \mathcal{R}_{\mathcal{M}}^{q \times p}$ be such that $Q = UP$, and let $w \in \mathcal{B}(P, \mathcal{M}^p)$, i.e. $\overline{P}(w) = 0$. Then $\overline{Q}(w) = \overline{UP}(w) = \overline{U}(\overline{P}(w)) = \overline{U}(0) = 0$. So $w \in \mathcal{B}(Q, \mathcal{M}^p)$.

“ $\Rightarrow$ ” Assume that $\mathcal{B}(P, \mathcal{M}^p) \subset \mathcal{B}(Q, \mathcal{M}^p)$, and define

$$U := Q \cdot P^{-1} = \frac{1}{\det(P)} \cdot Q \cdot \text{adj}(P). \quad (17)$$

We prove that $U \in \mathcal{R}_{\mathcal{M}}^{q \times p}$.

Since $\det(P) \neq 0$, $\overline{\det(P)}$ is surjective. For $i = 1, \dots, q$ and for $j = 1, \dots, p$, we denote by e_i and e_j the i -th and j -th unit vector, and by $(Q \cdot \text{adj}(P))_{ij}$ the (i, j) -th entry of $Q \cdot \text{adj}(P)$. We show that for all these entries the inclusion $\ker(\overline{\det(P)}) \subset \ker(\overline{(Q \cdot \text{adj}(P))_{ij}})$ holds.

Let $w_1 \in \ker(\overline{\det(P)})$, and define $w := w_1 \cdot e_j \in \mathcal{M}^p$. Then, according to Cramer’s rule

$$\overline{P \cdot \text{adj}(P)}(w) = \overline{\det(P)} \cdot \overline{I}(w) = \overline{\det(P)}(w_1) \cdot e_j = 0.$$

Hence—by assumption— $\overline{\text{adj}(P)}(w) \in \ker(\overline{P}) \subset \ker(\overline{Q})$, so in particular $\overline{Q \cdot \text{adj}(P)}(w) = 0$. This implies that

$$0 = e_i^T \overline{Q \cdot \text{adj}(P)}(w) = e_i^T \overline{Q \cdot \text{adj}(P)}(e_j w_1) = \overline{(Q \cdot \text{adj}(P))_{ij}}(w_1),$$

and thus $w_1 \in \ker(\overline{(Q \cdot \text{adj}(P))_{ij}})$. Since $w_1 \in \ker(\overline{\det(P)})$ and the entry (i, j) were chosen arbitrarily, we conclude that $U \in \mathcal{R}_{\mathcal{M}}^{q \times p}$. Using (17) it is obvious that $UP = QP^{-1}P = Q$. ■

Lemma 4.4 Let $P \in \mathcal{R}^{p \times p}$ and $Q \in \mathcal{R}^{q \times p}$. Assume that $\det(P) \neq 0$ and $\mathcal{B}(Q, \mathcal{M}^p) \subset \mathcal{B}(P, \mathcal{M}^p)$. Then $\text{rank}(Q) = p$, so in particular $q \geq p$.

Proof: Assume that $\text{rank}(Q) < p$. Then there exists a $0 \neq v \in \mathcal{R}^p$ such that $Qv = 0$. This implies that for all $w \in \mathcal{M}$, $\overline{v}(w) \in \mathcal{M}^p$ is an element of $\ker(\overline{Q})$. Since $\ker(\overline{Q}) \subset \ker(\overline{P})$, it follows that for all $w \in \mathcal{M}$, $\overline{Pv}(w) = \overline{P}(\overline{v}(w)) = 0$. So, according to Assumption 2.1, $Pv = 0$, and because $\det(P) \neq 0$, we conclude $v = 0$. This contradicts our assumption, and therefore $p = \text{rank}(Q) \leq \min(p, q) \leq q$. ■

Theorem 4.5 Let $P, Q \in \mathcal{R}^{p \times p}$, and assume that $\det(P) \neq 0$. Then

$$\begin{aligned} & \mathcal{B}(P, \mathcal{M}^p) = \mathcal{B}(Q, \mathcal{M}^p) \\ \iff & \\ & \exists U \in \mathcal{R}_{\mathcal{M}}^{p \times p}, U \text{ invertible over } \mathcal{R}_{\mathcal{M}}, \text{ s.t. } Q = U \cdot P. \end{aligned}$$

In particular, if $\mathcal{B}(P, \mathcal{M}^p) = \mathcal{B}(Q, \mathcal{M}^p)$, then $U := Q \cdot P^{-1}$ is a matrix over $\mathcal{R}_{\mathcal{M}}$, that is invertible over $\mathcal{R}_{\mathcal{M}}$, and satisfies $Q = U \cdot P$.

Proof: “ $\Leftarrow$ ” This direction is obvious: since both U and U^{-1} are matrices over $\mathcal{R}_{\mathcal{M}}$, one may use the same argument as in the proof of Proposition 4.3.

“ $\Rightarrow$ ” Since $\mathcal{B}(P, \mathcal{M}^p) \subset \mathcal{B}(Q, \mathcal{M}^p)$, Proposition 4.3 implies that $U := Q \cdot P^{-1}$ is a matrix over $\mathcal{R}_{\mathcal{M}}$, satisfying $U \cdot P = Q$. We show that U is invertible over $\mathcal{R}_{\mathcal{M}}$.

Because $\mathcal{B}(Q, \mathcal{M}^p) \subset \mathcal{B}(P, \mathcal{M}^p)$, Lemma 4.4 indicates that $\det(Q) \neq 0$. Define $V := P \cdot Q^{-1} = \frac{1}{\det(Q)} P \cdot \text{adj}(Q)$. According to Proposition 4.3, $V \in \mathcal{R}_{\mathcal{M}}^{p \times p}$, and $VQ = P$. Now $P = VQ = VUP$. Since $\det(P) \neq 0$, we have $VU = I$, and U is invertible over $\mathcal{R}_{\mathcal{M}}$ with inverse V . ■

Theorem 4.5 solves the question of system equivalence for AR-systems described by a square matrix P (with $\det(P) \neq 0$, but this condition is not very restrictive). According to Proposition 4.3, the behavior $\ker(\overline{P})$ is contained in the behavior $\ker(\overline{Q})$, if and only if P is a right divisor of Q over $\mathcal{R}_{\mathcal{M}}$. If additionally Q is a right divisor of P over $\mathcal{R}_{\mathcal{M}}$, the behaviors $\ker(\overline{P})$ and $\ker(\overline{Q})$ are the same. Conversely, Theorem 4.5 also characterizes the transformations on P that do not change its behavior: premultiplication with invertible matrices over $\mathcal{R}_{\mathcal{M}}$ is allowed.

Remark 4.6 To test system equivalence it suffices to compute the matrix $U = QP^{-1}$. Then $\ker(\overline{P}) \subset \ker(\overline{Q})$ if and only if U is a matrix over $\mathcal{R}_{\mathcal{M}}$. If additionally U is invertible over $\mathcal{R}_{\mathcal{M}}$, the behaviors $\ker(\overline{P})$ and $\ker(\overline{Q})$ are equal.

Next we consider the general (non-square) case.

Proposition 4.7 Let $P = (P_1 \mid P_2)$, with $P_1 \in \mathcal{R}^{p \times p}$ and $P_2 \in \mathcal{R}^{p \times m}$, and assume that $\det(P_1) \neq 0$. Let $Q = (Q_1 \mid Q_2)$, with $Q_1 \in \mathcal{R}^{q \times p}$ and $Q_2 \in \mathcal{R}^{q \times m}$. Then

$$\begin{aligned} \mathcal{B}(P, \mathcal{M}^{p+m}) &\subset \mathcal{B}(Q, \mathcal{M}^{p+m}) \\ \iff \\ \exists U \in \mathcal{R}_{\mathcal{M}}^{q \times p} : Q &= U \cdot P. \end{aligned}$$

In particular, if $\mathcal{B}(P, \mathcal{M}^{p+m}) \subset \mathcal{B}(Q, \mathcal{M}^{p+m})$, then $U := Q_1 P_1^{-1}$ is a $q \times p$ matrix over $\mathcal{R}_{\mathcal{M}}$, satisfying $Q = U \cdot P$.

In the proof of this result we need the following lemma:

Lemma 4.8 Let $P \in \mathcal{R}^{p \times p}$ with $\det(P) \neq 0$. Then $\overline{P} : \mathcal{M}^p \rightarrow \mathcal{M}^p$ is surjective.

Proof: Let $y \in \mathcal{M}^p$. Since $\det(P) \neq 0$, $\overline{\det(P)}$ is surjective, and thus there exists a $w \in \mathcal{M}^p$ such that $\overline{\det(P)} \cdot \overline{I}(w) = y$. Define $x := \overline{\text{adj}(P)}(w)$. Then, according to Cramer’s rule

$$\overline{P}(x) = \overline{P}(\overline{\text{adj}(P)}(w)) = \overline{P \cdot \text{adj}(P)}(w) = \overline{\det(P)} \cdot \overline{I}(w) = y. \quad \blacksquare$$

Proof of Proposition 4.7: Since “ $\Leftarrow$ ” is obvious, we only prove “ $\Rightarrow$ ”.

Assume that $\mathcal{B}(P, \mathcal{M}^{p+m}) \subset \mathcal{B}(Q, \mathcal{M}^{p+m})$. Then also $\mathcal{B}(P_1, \mathcal{M}^p) \subset \mathcal{B}(Q_1, \mathcal{M}^p)$, which can be seen as follows. Let $w_1 \in \mathcal{B}(P_1, \mathcal{M}^p)$, and extend w_1 to a vector $w \in \mathcal{M}^{p+m}$, by defining the last m entries of w equal to 0. Then $w \in \ker(\overline{P}) \subset \ker(\overline{Q})$, and thus $0 = \overline{Q}(w) = \overline{Q_1}(w_1) + \overline{Q_2}(0) = \overline{Q_1}(w_1)$, i.e. $w_1 \in \ker(\overline{Q_1})$.

Next apply Proposition 4.3 to the matrices P_1 and Q_1 . Defining $U := Q_1 P_1^{-1}$, we know that $U \in \mathcal{R}_{\mathcal{M}}^{q \times p}$ and $Q_1 = U P_1$. So it suffices to show that $Q_2 = U P_2$.

Let $w_2 \in \mathcal{M}^m$. Since $\det(P_1) \neq 0$, we know, according to Lemma 4.8, that there exists a $w_1 \in \mathcal{M}^p$ such that $\overline{P_1}(w_1) = -\overline{P_2}(w_2)$. Then

$$\overline{(P_1 \mid P_2)} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \overline{P_1}(w_1) + \overline{P_2}(w_2) = 0,$$

and thus also $\overline{(Q_1 \mid Q_2)} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$. Therefore

$$\overline{(Q_2 - UP_2)}(w_2) = \overline{Q_2}(w_2) - \overline{UP_2}(w_2) = \overline{Q_2}(w_2) + \overline{UP_1}(w_1) = \overline{Q_2}(w_2) + \overline{Q_1}(w_1) = 0.$$

Since $w_2 \in \mathcal{M}^m$ was arbitrary, Lemma 3.4 implies that $Q_2 - UP_2 = 0$. $\blacksquare$

At this point we may generalize Theorem 4.5 to the non-square case. Two matrices P and Q of full row rank characterize the same behavior if and only if P and Q are right divisors of each other over the ring $\mathcal{R}_{\mathcal{M}}$.

Theorem 4.9 *Let $P \in \mathcal{R}^{p \times m}$, with $\text{rank}(P) = p$, and $Q \in \mathcal{R}^{q \times m}$, with $\text{rank}(Q) = q$. Then*

$$\mathcal{B}(P, \mathcal{M}^m) = \mathcal{B}(Q, \mathcal{M}^m)$$

$\iff$

$$p = q \text{ and } \exists U \in \mathcal{R}_{\mathcal{M}}^{p \times p}, U \text{ invertible over } \mathcal{R}_{\mathcal{M}}, \text{ such that } Q = U \cdot P.$$

In particular, if $\mathcal{B}(P, \mathcal{M}^m) = \mathcal{B}(Q, \mathcal{M}^m)$, and if P_1 is a $p \times p$ block of P satisfying $\det(P_1) \neq 0$, and Q_1 is the corresponding $p \times p$ block of Q , then $U := Q_1 P_1^{-1}$ is an element of $\mathcal{R}_{\mathcal{M}}^{p \times p}$, that is invertible over $\mathcal{R}_{\mathcal{M}}$, and satisfies $Q = U \cdot P$.

Proof: “ $\Leftarrow$ ” is trivial, so we only prove “ $\Rightarrow$ ”.

Since $\mathcal{B}(P, \mathcal{M}^m) \subset \mathcal{B}(Q, \mathcal{M}^m)$, and $m \geq p = \text{rank}(P)$, we may apply Proposition 4.7: there exists a $U \in \mathcal{R}_{\mathcal{M}}^{q \times p}$ such that $Q = UP$.

However, also the inclusion in the opposite direction holds: $\mathcal{B}(Q, \mathcal{M}^m) \subset \mathcal{B}(P, \mathcal{M}^m)$. Furthermore, $m \geq q = \text{rank}(Q)$, and again using Proposition 4.7, we find a matrix $V \in \mathcal{R}_{\mathcal{M}}^{p \times q}$ such that $P = VQ$.

Combining both equalities, we conclude that $P = VUP$, and $Q = UVQ$. Since both P and Q have full row rank, both matrices are right-invertible over $\mathcal{Q}(\mathcal{R})$, and thus $VU = I_p$, and similarly $UV = I_q$. This implies that $p = q$, hence $U \in \mathcal{R}_{\mathcal{M}}^{p \times p}$ with inverse V satisfies the claim. $\blacksquare$

Remark 4.10 If $P \in \mathcal{R}^{p \times (p+m)}$ with $\text{rank}(P) = p$ and $Q \in \mathcal{R}^{q \times m}$ with $\text{rank}(Q) = q$, it is easily verified whether $\ker(\overline{P}) = \ker(\overline{Q})$. Let P_1 be a $p \times p$ block of P such that $\det(P_1) \neq 0$, and collect the remaining columns of P in the matrix P_2 . Partition the matrix Q correspondingly into matrices $Q_1 \in \mathcal{R}^{q \times p}$ and $Q_2 \in \mathcal{R}^{q \times (m-p)}$. Then $\ker(\overline{P}) \subset \ker(\overline{Q})$ if and only if $U := Q_1 P_1^{-1}$ is a $q \times p$ matrix over $\mathcal{R}_{\mathcal{M}}$ such that $U \cdot P_2 = Q_2$. If additionally $p = q$ and U is invertible over $\mathcal{R}_{\mathcal{M}}$, then the behaviors $\ker(\overline{P})$ and $\ker(\overline{Q})$ are equivalent.

In all results on system equivalence given in this section, the matrices P and Q describing an AR-system are assumed to be matrices over the ring $\mathcal{R}$. The transformation matrix U however, is a matrix over $\mathcal{R}_{\mathcal{M}}$. Note that in this way also the case of AR-systems over $\mathcal{R}_{\mathcal{M}}$ (i.e. with matrices P and Q over $\mathcal{R}_{\mathcal{M}}$) is included. According to Proposition 3.7, the ring $\mathcal{R}_{\mathcal{M}}$ is not extended after application of Definition 3.1: $(\mathcal{R}_{\mathcal{M}})_{\mathcal{M}} = \mathcal{R}_{\mathcal{M}}$. Therefore, the results of this section remain valid if $\mathcal{R}$ is replaced by $\mathcal{R}_{\mathcal{M}}$. This means that system equivalence for AR-systems over the ring $\mathcal{R}_{\mathcal{M}}$ has been characterized by division properties of the system-defining matrices over the same ring $\mathcal{R}_{\mathcal{M}}$. In particular, if a matrix P over $\mathcal{R}_{\mathcal{M}}$ is premultiplied by an invertible matrix over $\mathcal{R}_{\mathcal{M}}$, the behavior does not change.

5 An application: differential-difference systems

Consider a differential-difference system with k incommensurable time-delays $\tau_1, \dots, \tau_k$. Incommensurable means that the numbers $\tau_1, \dots, \tau_k \in \mathbb{R}^+$ are linearly independent over $\mathbb{Q}$. In this section we show that this class of systems fits into the framework proposed in this paper, and derive a formula for the ring $\mathcal{R}_{\mathcal{M}}$ in this particular case.

Define $\mathcal{R} := \mathbb{R}[s, z_1, \dots, z_k]$, and let $\mathcal{M} = \mathcal{E}(\mathbb{R})$ be the space of all infinitely differentiable complex functions on $\mathbb{R}$, under the topology of compact convergence in all derivatives. This means that a sequence of functions $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{E}(\mathbb{R})$ converges to $f \in \mathcal{E}(\mathbb{R})$ if f_n converges uniformly to f on every compact set $\Omega \subset \mathbb{R}$, and the same holds true for all derivatives of f_n and f , respectively. Equipped with this topology, the module $C^\infty(\mathbb{R})$ is turned into the Fréchet space $\mathcal{E}(\mathbb{R})$ (see e.g. [11, p. 107]).

For $i = 1, \dots, k$ we define σ_i as the delay operator on $\mathcal{M}$ corresponding to time-delay τ_i :

$$\sigma_i : \mathcal{M} \longrightarrow \mathcal{M} : \sigma_i(x(t)) = x(t - \tau_i).$$

Note that the differentiation operator $\frac{d}{dt}$ and the delay operators $\sigma_1, \dots, \sigma_k$ mutually commute. Each polynomial $p \in \mathcal{R}$ corresponds to an operator $\bar{p} : \mathcal{M} \longrightarrow \mathcal{M}$, obtained after substitution in p of $\frac{d}{dt}$ for s and σ_i for z_i ($i = 1, \dots, k$). Let $\bar{\mathcal{R}} = \{\bar{p} : \mathcal{M} \longrightarrow \mathcal{M} \mid p \in \mathcal{R}\}$. Since the time-delays $\tau_1, \dots, \tau_k$ are incommensurable, different ring elements $p \in \mathcal{R}$ correspond to different operators $\bar{p} \in \bar{\mathcal{R}}$, i.e. Assumption 2.1 is satisfied. So $\mathcal{R}$ and $\bar{\mathcal{R}}$ are isomorphic rings, and $\mathcal{M}$ is a module over $\bar{\mathcal{R}}$. By choosing a suitable $P \in \mathcal{R}^{n \times m}$, a set of n differential-difference equations in m variables can be described as an AR-system $(P, \mathcal{M}^m)$ over the ring $\mathcal{R}$ with behavior $\mathcal{B}(P, \mathcal{M}^m)$.

Next we have to verify whether Assumption 4.1 is satisfied: for all $q \in \mathcal{R} \setminus \{0\}$, $\bar{q} : \mathcal{M} \longrightarrow \mathcal{M}$ is surjective. In the particular application at hand, this nontrivial question was studied by Ehrenpreis in [4]. The results in [4, Section 3] guarantee that indeed Assumption 4.1 is satisfied. An alternative idea to prove the surjectivity of non-zero elements in $\bar{\mathcal{R}}$ is given in [5]. So the class of differential-difference systems fits into the framework of AR-systems over rings, and the solution to the problem of system equivalence using the ring extension $\mathcal{R}_{\mathcal{M}}$ applies. However, for this particular application, there exists an alternative characterization of the ring $\mathcal{R}_{\mathcal{M}}$, that is of great practical interest.

Let $p \in \mathcal{R}$. After substitution of the exponential functions $e^{-\tau_i s}$ for the indeterminates z_i ($i = 1, \dots, k$), the polynomial p is turned into an *exponential* (or quasi) *polynomial* $\tilde{p}$:

$$\tilde{p}(s) = p(s, e^{-\tau_1 s}, \dots, e^{-\tau_k s}).$$

Let $\tilde{\mathcal{R}} := \{\tilde{p} \mid p \in \mathcal{R}\}$ denote the ring of all exponential polynomials that can be obtained in this way. Then it is easily verified that the rings $\mathcal{R}$, $\bar{\mathcal{R}}$, and $\tilde{\mathcal{R}}$ are isomorphic.

For a polynomial $p \in \mathcal{R}$, the relationship between the differential-difference operator $\bar{p} : \mathcal{E}(\mathbb{R}) \longrightarrow \mathcal{E}(\mathbb{R})$ and the exponential polynomial $\tilde{p}$ may be illustrated using Bohl functions. For $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$, let $e_{\lambda,j}$ denote the Bohl function $e_{\lambda,j}(t) = t^{j-1} e^{\lambda t}$. Obviously $e_{\lambda,j} \in \mathcal{E}(\mathbb{R})$. Since $\frac{d}{dt}(e_{\lambda,1}) = \lambda e_{\lambda,1}$, and $\sigma_i(e_{\lambda,1}) = e^{-\lambda \tau_i} e_{\lambda,1}$ for all $i = 1, \dots, k$, we obtain for every polynomial $p \in \mathcal{R}$:

$$\bar{p}(e_{\lambda,1}) = \tilde{p}(\lambda) e_{\lambda,1}. \quad (18)$$

So, using the exponential polynomial $\tilde{p}$, the image of an exponential function under the operator $\bar{p}$ is easily computed. Next we extend formula (18) to Bohl functions $e_{\lambda,j}$ with

$j > 1$. Let $j \in \mathbb{N}$, and consider for a moment $e_{\lambda,j}$ as a function of two variables, t and λ . Then

$$\begin{aligned}\bar{p}(e_{\lambda,j}) &= \bar{p}\left(\left(\frac{d}{d\lambda}\right)^{j-1} e_{\lambda,1}\right) = \left(\frac{d}{d\lambda}\right)^{j-1} (\bar{p}(e_{\lambda,1})) = \\ &= \left(\frac{d}{d\lambda}\right)^{j-1} (\tilde{p}(\lambda)e_{\lambda,1}) = \sum_{\ell=0}^{j-1} \binom{j-1}{\ell} \tilde{p}^{(\ell)}(\lambda) e_{\lambda,j-\ell},\end{aligned}\tag{19}$$

where we used formula (18) and the fact that in this particular situation the operators $(\frac{d}{d\lambda})^{j-1}$ and $\bar{p}$ commute. Using formulae (18) and (19) one can easily verify whether a function $e_{\lambda,j}$ belongs to the kernel of an operator $\bar{p} \in \overline{\mathcal{R}}$. This result is stated in the next proposition, and can be considered as a generalization of a similar result in [1, p. 55] and [6, p. 18].

Proposition 5.1 *Let $p \in \mathcal{R}$, and let for $\lambda \in \mathbb{C}$ and $j \in \mathbb{N}$, $e_{\lambda,j}$ denote the Bohl function $e_{\lambda,j}(t) = t^{j-1}e^{\lambda t}$. Then*

$$\begin{aligned}e_{\lambda,j} &\in \ker(\bar{p}), \\ \iff \\ \lambda &\text{ is a zero of } \tilde{p} \text{ of multiplicity at least } j.\end{aligned}\quad \blacksquare$$

The relationship between the differential-difference operator $\bar{p}$ and the exponential polynomial $\tilde{p}$ is now used to obtain an alternative characterization of the ring $\mathcal{R}_{\mathcal{M}}$. For this purpose we introduce the ring

$$\mathcal{Q}(\mathcal{R})_{\text{anal}} := \left\{ \frac{p}{q} \in \mathcal{Q}(\mathcal{R}) \mid p, q \in \mathcal{R} \text{ and } \frac{\tilde{p}}{\tilde{q}} \text{ is analytic in } \mathbb{C} \right\}.\tag{20}$$

Theorem 5.2 *For differential-difference systems with incommensurable time-delays, modeled as AR-systems over the ring $\mathcal{R} = \mathbb{R}[s, z_1, \dots, z_k]$, with the module $\mathcal{M} = \mathcal{E}(\mathbb{R})$ as signal space, we have*

$$\mathcal{R}_{\mathcal{M}} = \mathcal{Q}(\mathcal{R})_{\text{anal}}.\tag{21}$$

In the proof of formula (21) we need the following result of De Rijcke on the structure of closed translation invariant subspaces in the Fréchet space $\mathcal{E}(\mathbb{R})$. The proof of this result can be found in [9, Theorem 3.33], and is based on earlier works of Schwartz and Kahane (see [12], [7], [8]).

Proposition 5.3 *A closed linear subspace V of $\mathcal{E}(\mathbb{R})$ is translation invariant if and only if there exists a countable set $\Sigma \subset \mathbb{C}$ and a mapping $N : \Sigma \rightarrow \mathbb{N}$ such that*

$$V = \overline{\text{span}\{e_{\lambda,j} \mid \lambda \in \Sigma, j = 1, \dots, N(\lambda)\}}.\quad \blacksquare$$

For every $p \in \mathcal{R}$, it is obvious that if $f \in \ker(\bar{p})$ and $\tau \in \mathbb{R}$, then the translation g of f , defined by $g(t) := f(t - \tau)$ ($t \in \mathbb{R}$), is also an element of $\ker(\bar{p})$. Furthermore, since the operator $\bar{p} : \mathcal{E}(\mathbb{R}) \rightarrow \mathcal{E}(\mathbb{R})$ is continuous, its kernel is closed. Therefore $\ker(\bar{p})$ is a closed translation invariant subspace of $\mathcal{E}(\mathbb{R})$, and thus Proposition 5.3 describes the structure of $\ker(\bar{p})$. This observation is used to prove that $\mathcal{R}_{\mathcal{M}} = \mathcal{Q}(\mathcal{R})_{\text{anal}}$.

Proof of Theorem 5.2: " $\mathcal{R}_{\mathcal{M}} \subset \mathcal{Q}(\mathcal{R})_{\text{anal}}$ " Let $r \in \mathcal{R}_{\mathcal{M}}$ with admissible fractional representation $\frac{p}{q}$. Then we know that $\ker(\bar{q}) \subset \ker(\bar{p})$, and we have to show that $\frac{\tilde{p}}{\tilde{q}}$ is analytic on $\mathbb{C}$.

Since $\tilde{q} \neq 0$ is analytic on $\mathbb{C}$, it has only finitely many zeros in every compact subset of $\mathbb{C}$. In particular, $\tilde{q}$ has at most countably many zeros, no accumulation point of zeros, and every zero has finite multiplicity. Therefore it suffices to show that every zero of $\tilde{q}$ is also a zero of $\tilde{p}$ (including multiplicities).

Let λ be a zero of $\tilde{q}$ of multiplicity j . Then Proposition 5.1 implies that $\bar{q}(e_{\lambda,j}) = 0$. Since $\ker(\bar{q}) \subset \ker(\bar{p})$, we also have $\bar{p}(e_{\lambda,j}) = 0$. Applying again Proposition 5.1, we conclude that λ is a zero of $\tilde{p}$ of multiplicity at least j . In this way all zeros of $\tilde{q}$ may be treated successively, and thus $\frac{\tilde{p}}{\tilde{q}}$ is analytic on $\mathbb{C}$. Hence $\frac{p}{q} \in \mathcal{Q}(\mathcal{R})_{\text{anal}}$.

" $\mathcal{R}_{\mathcal{M}} \supset \mathcal{Q}(\mathcal{R})_{\text{anal}}$ " Let $\frac{p}{q} \in \mathcal{Q}(\mathcal{R})_{\text{anal}}$, i.e. $\frac{p}{q} \in \mathcal{Q}(\mathcal{R})$ and $\frac{\tilde{p}}{\tilde{q}}$ is analytic on $\mathbb{C}$. We show that $\frac{p}{q}$ is an admissible fractional representation. Since $\bar{q}$ is surjective, we only have to verify that $\ker(\bar{q}) \subset \ker(\bar{p})$.

First recall that $\ker(\bar{q})$ is a closed translation invariant subspace of $\mathcal{E}(\mathbb{R})$. According to Proposition 5.3, this implies that there exists a countable set $\Sigma \subset \mathbb{C}$ and a mapping $N : \Sigma \rightarrow \mathbb{N}$, such that

$$\ker(\bar{q}) = \overline{\text{span}\{e_{\lambda,j} \mid \lambda \in \Sigma, j = 1, \dots, N(\lambda)\}}.$$

Using Proposition 5.1, it is obvious that every $\lambda \in \Sigma$ is a zero of $\tilde{q}$ of multiplicity at least $N(\lambda)$. Since $\frac{\tilde{p}}{\tilde{q}}$ is analytic on $\mathbb{C}$, it follows that every $\lambda \in \Sigma$ is a zero of $\tilde{p}$ of multiplicity at least $N(\lambda)$. Again applying Proposition 5.1, we find that

$$\text{span}\{e_{\lambda,j} \mid \lambda \in \Sigma, j = 1, \dots, N(\lambda)\} \subset \ker(\bar{p}).$$

Since $\ker(\bar{p})$ is a *closed* linear subspace of $\mathcal{E}(\mathbb{R})$, we conclude that

$$\ker(\bar{q}) = \overline{\text{span}\{e_{\lambda,j} \mid \lambda \in \Sigma, j = 1, \dots, N(\lambda)\}} \subset \ker(\bar{p}). \quad \blacksquare$$

The importance of Theorem 5.2 is obvious. To verify whether a rational function $r = \frac{p}{q} \in \mathcal{Q}(\mathcal{R})$ belongs to $\mathcal{R}_{\mathcal{M}}$, it suffices to check whether the related meromorphic function $\frac{\tilde{p}}{\tilde{q}}$ is analytic on $\mathbb{C}$. So, instead of testing the condition $\ker(\bar{q}) \subset \ker(\bar{p})$, which requires the solution of a differential-difference equation, one may verify whether all zeros of the exponential polynomial $\tilde{q}$ are also zeros of $\tilde{p}$ (including multiplicities).

The problem of system equivalence for differential-difference systems with commensurable delays (i.e. the case $k = 1$, only one delay operator σ with time-delay τ), has recently been solved in [5] in a somewhat different way. Let us check whether in this situation our results are consistent with [5].

For systems with commensurable delays, $\mathcal{R} = \mathbb{R}[s, z]$ and $\mathcal{M} = \mathcal{E}(\mathbb{R})$. Also the rings $\overline{\mathcal{R}}$, $\tilde{\mathcal{R}}$, and $\mathcal{Q}(\mathcal{R})_{\text{anal}}$ are easily adapted to this situation. In [5] the ring

$$\mathcal{H} := \left\{ \frac{p}{z^n \cdot q} \in \mathcal{Q}(\mathcal{R}) \mid p \in \mathcal{R}, q \in \mathbb{R}[s], n \in \mathbb{N} \cup \{0\}, \right. \\ \left. \text{such that } \frac{\tilde{p}}{e^{-n\tau s} \tilde{q}} : \mathbb{C} \rightarrow \mathbb{C} \text{ is analytic} \right\} \quad (22)$$

was introduced, and it was shown that $\mathcal{H}$ is a Bezout ring, i.e. every finitely generated ideal of $\mathcal{H}$ is principal. Using this property, system equivalence for differential-difference systems with commensurable delays was characterized by division properties over the ring $\mathcal{H}$. So Theorem 4.9 holds with $\mathcal{R}_{\mathcal{M}}$ replaced by $\mathcal{H}$. This indicates that in this situation the rings $\mathcal{H}$ and $\mathcal{R}_{\mathcal{M}}$ should be identical. The next proposition confirms this conjecture.

Proposition 5.4 *For delay-differential systems with commensurable delays we have*

$$\mathcal{R}_{\mathcal{M}} = \mathcal{Q}(\mathcal{R})_{\text{anal}} = \mathcal{H}.$$

Proof: Since $\mathcal{R}_{\mathcal{M}} = \mathcal{Q}(\mathcal{R})_{\text{anal}}$ (Theorem 5.2), and according to definition (22), $\mathcal{H} \subset \mathcal{Q}(\mathcal{R})_{\text{anal}}$, we only have to show that $\mathcal{Q}(\mathcal{R})_{\text{anal}} \subset \mathcal{H}$.

Let $\frac{p}{q} \in \mathcal{Q}(\mathcal{R})_{\text{anal}}$ and assume without loss of generality that $p, q \in \mathcal{R}$ are coprime. This implies that in $\mathbb{C}^2$ the polynomials p and q have a finite number of common zeros. If p or q is univariate, this is trivial; otherwise coprimeness of p and q guarantees that their resultant is a nonzero univariate polynomial (see e.g. [3, p. 153, 159]), and one may proceed as in the univariate case. Since $\frac{\tilde{p}}{\tilde{q}}$ is analytic on $\mathbb{C}$, all zeros of $\tilde{q}$ are also zeros of $\tilde{p}$ (including multiplicities), and we conclude that $\tilde{q}$ has finitely many zeros. (If $\tilde{p}$ and $\tilde{q}$ have infinitely many common zeros in $\mathbb{C}$, p and q have an infinite number of common zeros in $\mathbb{C}^2$, and p and q are not coprime). Using [5, Proposition 2.6], we conclude that q is of the form $q = z^n \cdot q^*$, with $n \in \mathbb{N} \cup \{0\}$ and $q^* \in \mathbb{R}[s] \setminus \{0\}$, and thus $\frac{p}{q} \in \mathcal{H}$. ■

Example 5.5 The behaviors in $\mathcal{E}(\mathbb{R})^3$ of the differential-difference systems with time-delay $\tau \in \mathbb{R}^+$, described by

$$P = \begin{pmatrix} s & s^2 + sz & 0 \\ 0 & z & 1 \end{pmatrix}, \text{ and}$$

$$Q = \begin{pmatrix} z - 1 & (z - 1)s + z^2 & 1 \\ sz & zs^2 + (z^2 + z)s & s \end{pmatrix},$$

are equivalent because $Q = UP$, with

$$U = \begin{pmatrix} \frac{z-1}{s} & 1 \\ z & s \end{pmatrix}.$$

First we show that U is a matrix over $\mathcal{R}_{\mathcal{M}}$. Let $u_{11} = \frac{p}{q}$ with $p = z - 1$ and $q = s$. Then $\frac{\tilde{p}}{\tilde{q}} = \frac{e^{-\tau s} - 1}{s}$ is analytic in $\mathbb{C}$ because 0 is a zero of $\tilde{p} = e^{-\tau s} - 1$. This implies that $u_{11} \in \mathcal{Q}(\mathcal{R})_{\text{anal}} = \mathcal{R}_{\mathcal{M}}$, which was also shown in a different way in Example 3.6. Furthermore $\det(U) = -1$, so U is invertible over $\mathcal{R}_{\mathcal{M}}$, and system equivalence follows with Theorem 4.9. Note that in this example (which is in fact the same as Example 2.5) there does not exist a matrix over $\mathcal{R}$, that transforms P into Q , i.e. P is not a right divisor of Q over $\mathcal{R}$.

6 Conclusions

In this paper we solved the problem of system equivalence for AR-systems over an arbitrary integral domain $\mathcal{R}$. For this type of systems, the signal space is assumed to be a module $\mathcal{M}$ over $\mathcal{R}$. After introduction of a ring extension $\mathcal{R}_{\mathcal{M}}$ of $\mathcal{R}$, explicitly depending on the module $\mathcal{M}$, system equivalence was characterized by division properties on the system-defining matrices over the ring $\mathcal{R}_{\mathcal{M}}$.

The theory is applicable to differential-difference systems; in this case the ring $\mathcal{R}_{\mathcal{M}}$ takes a special form and is described by the pole-zero cancellation properties of two exponential polynomials that are obtained by the formal Laplace transformation of the corresponding delay-differential operators. This alternative characterization facilitates the verification of system equivalence considerably. Further research is necessary to characterize the ring $\mathcal{R}_{\mathcal{M}}$ in other applications.

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