Beyond Performance/Cost Tradeoffs in Motion Control: A Multirate Feedforward Design With Application to a Dual-Stage Wafer System

Jurgen van Zundert<sup>1</sup>, Tom Oomen<sup>2</sup>, Senior Member, IEEE, Jan Verhaegh, Wouter Aangenent, Duarte J. Antunes<sup>3</sup>, Member, IEEE, and W. P. M. H. Heemels<sup>4</sup>, Fellow, IEEE

Abstract—Motion systems with multiple control loops often run at a single sampling rate for simplicity of implementation and controller design. The achievable performance in terms of position accuracy is determined by the data acquisition hardware, such as sensors, actuators, and analog-to-digital/digital-to-analog converters, which is typically limited due to economic cost considerations. The aim of this paper is to develop a multirate approach to go beyond this traditional performance/cost tradeoff, i.e., to use different sampling rates in different control loops to optimize the performance of the overall system. The approach appropriately deals with the inherent time-varying behavior that is introduced by multirate sampling. A multirate feedforward control design framework is presented to optimize the tracking of a dual-stage multirate system. The application of the proposed approach to an industrial dual-stage wafer system demonstrates the advantages of multirate control, both in simulations and experiments.

Index Terms—Dual-stage system, experiments, feedforward design, multirate control, performance/costtradeoff, wafer stage application.

I. INTRODUCTION

MULTIVARIABLE control systems, including those in motion systems, are often implemented digitally since it offers flexibility and directly connects to the digital supervisory layers. The digital implementation requires analog-to-digital and digital-to-analog conversion. For motion systems, these processes are often executed using fixed, single-rate sampling schemes [1], [2], i.e., homogeneous for all loops, since for linear time-invariant (LTI) systems it enables controller design using well-developed design approaches. In particular, it allows the use of frequency-domain techniques such as Bode plots and Nyquist diagrams [3], which find application in various areas of controller design, including feedback control [3], [4], feedforward control [5], and iterative learning control (ILC) [6].

Fixed, single-rate sampling is preferred from a controller design point of view, but not from a performance versus cost point of view. As an example, consider systems with multiple control loops, where only one limits the overall performance. The performance of a control loop can be increased by increasing the sampling frequency of that loop. For single-rate implementations, this implies that if the performance of one of the loops is increased, the sampling frequency of all loops needs to be increased. Obviously, such an approach is expensive in terms of the required hardware, such as sensors, actuators, and analog-to-digital/digital-to-analog converters, since all loops are affected while only one is limiting performance.

From a performance versus cost point of view, flexible sampling is preferred over fixed sampling (see also Fig. 1). Examples of flexible sampling include multirate control [7]–[16], sparse control [17], and nonequidistant sampling [18], [19]. Indeed, a multirate approach is more natural for multiloop systems with different performance requirements, but also for systems with different time scales such as thermomechanical systems [20]. Sparse control and nonequidistant sampling are used in e.g., systems with limited resources and optimal resource allocation [18], [21].

Flexible sampling has a large potential, but its deployment is hampered by a lack of control design techniques. This is mainly caused by the fact that flexible sampling introduces time-varying behavior [1, Sec. 3.3]. In particular, a flexible sampling of an LTI system yields a linear periodically time-varying (LPTV) system. Due to the time variance, the frequency-domain control design techniques mentioned earlier are not (directly) applicable. Frequency-domain design for linear time-varying systems is investigated in [22]–[26] and linear time-varying feedforward design is investigated in [19] and [27], but at present, there is no systematic control design framework available.

Although flexible sampling has the potential to go beyond the traditional performance/cost tradeoff for fixed sampling, as shown in Fig. 1, at present, its deployment is hampered by a lack of control design techniques for such sampling schemes. In this paper, a framework to exploit multirate feedforward
controller design is presented to overcome this restriction, and thereby go beyond the traditional performance/cost tradeoff. Application of the framework focuses on precision motion systems. In particular, the framework is demonstrated on an experimental dual-stage system, as standard in, e.g., wafer stages [28, Ch. 9].

The main contribution of this paper is a framework to exploit multirate control for performance improvement. The following subcontributions are identified: 1) multirate controller design based on multirate system descriptions, including time variance; 2) controller optimization addressing nonperfect models; 3) performance improvement by exploiting time variance; 4) application of the design framework in simulation; and 5) experimental validation on a dual-stage system.

Initial results on simulation level can be found in [29] and related work on minimizing intersample behavior in digital control systems can be found in [1], [25], and [30]. This paper contains substantial original contributions including Contribution (I), Contribution (II), and Contribution (V). Related work on wafer stage control design includes feedback control [31], [32], feedforward control [33], linear parameter varying control [34], and sparse control [17]. In this paper, previously unexplored freedom in sampling is exploited, which makes the approach complementary to other approaches.

This paper is organized as follows. In Section II, the main problem that is considered to improve the performance/cost tradeoff through multirate control is presented. In Section III, the multirate control system is modeled. The multirate controller design is presented in Section IV. Furthermore, the performance is further improved by exploiting properties of time-varying systems. The controller design is applied to an experimental setup resembling a dual-stage wafer system. The experimental setup is detailed in Section V. Simulation results are presented in Section VI and experimental results are presented in Section VII. Conclusions are given in Section VIII.

II. PROBLEM DEFINITION

In this paper, a framework is presented to enhance the performance/cost tradeoff through multirate control. In this section, the main problem is presented.

A. Application Motivation—Dual-Stage Motion Systems With Large Differences in Performance Requirements

In many motion control applications, a high positioning accuracy is required over a large range. For such systems, a single-stage design may not suffice due to the large dynamic range. To achieve high precision over a large range, a dual-stage system can be used.

A dual-stage system, as illustrated in Fig. 2, consists of two subsystems: a short stroke (SS) with a high positioning accuracy (and limited range) connected to a long stroke (LoS) with a large range (and limited positioning accuracy). If designed properly, the dual-stage system is able to cover a large range with high positioning accuracy. Clearly, there is a large difference between the performance requirements of the two subsystems.

An example of a dual-stage system is a wafer stage in lithography machines [28, Ch. 9]. Wafer stages require an accuracy up to nanometer level over a range of 1 m ([35], [28, Sec. 9.3.1]), resulting in a large dynamic range of $O(10^9)$. Therefore, wafer stages are typically constructed as dual-stage systems. More details on the wafer stage application are presented in Section V.

B. Performance/Cost Perspective on Multivariable Systems With Large Differences in Performance Requirements

In view of the performance/cost tradeoff in Fig. 1, different (control) requirements for the subsystems of the dual-stage design provide an excellent opportunity to exploit multirate control to go beyond performance/cost tradeoffs in motion control.

The considered multirate control architecture is shown in Fig. 3 where a high sampling frequency $f_h$ is used for the short stroke $G_{SS,h}$ in Fig. 1) and a low sampling frequency $f_l$ is used for the long stroke $G_{LoS,1}$ to reduce cost in Fig. 1). The short-stroke system $G_{SS,h}$ tracks reference trajectory $\rho_{SS,h}$. The long-stroke system $G_{LoS,1}$ tracks the position of $G_{SS,h}$ to ensure the short stroke is within range and reaction forces are limited. The downsampler $D_F$ facilitates the sampling rate conversion. The control design of both subsystems consists of feedback control ($C_{FB}$), feedforward control ($C_{FF}$), and input shaping ($C_{\psi}$).

For design of the long-stroke controllers, the interest is in the position error between the two stages during exposure, i.e., during the scanning motion, to limit reaction forces to the short stroke. This error measured at the highest possible sampling frequency $f_h$ is denoted $\varepsilon_{SS}$ and not available for real-time control, but typically available afterward for performance
evaluation. The sampling frequencies are related by
\[ f_s = F_h f_h = F_1 f_1, \quad f_h = F_1 f_1 \] (1)
where \( F_1 \geq F_h \geq 1 \), \( F := (F_1 / F_h) \), with \( F_h, F_1, F \in \mathbb{N} \). In this paper, finite-time signals are considered of which the signal lengths are related as
\[ N_s = F_h N_h = F_1 N_1, \quad N_h = F N_1 \] (2)
as directly follows from (1).

Remark 1: The assumption of integer sampling rate factors in (1) is imposed for ease of notation, but can easily be relaxed if the factor is a rational number. The proposed approach is not applicable for irrational factors, although these can often be closely approximated with rational factors.

C. Problem Formulation—Framework for Exploiting Multirate Sampling for Enhanced Control Performance

In this paper, the following problem is considered.

Main Problem: Given the multirate control configuration in Fig. 3 with sampling frequencies admitting (1), a given finite-time reference trajectory \( \rho_{SS,h} \in \mathbb{R}^{N_h} \) for \( \rho_{SS,h} \), models \( G_{SS}, G_{LoS,h} \), \( G_{LoS,1} \) at sampling frequency \( f_s \), and controllers \( C_{FF,SS,h}, C_{FF,SS,1}, C_{FB,SS,h}, C_{FB,SS,1} \), \( C_{FF,LoS,1}, C_{FF,LoS,1} \), \( C_{FF,SS,1}, C_{FF,SS,1} \).\( \text{.,} \) Moreover, it considers \( \varepsilon_s \) rather than \( \varepsilon_{LoS,1} \) and thereby takes intersample behavior into account, which is an important aspect in multirate control [25]. Note that (3) is posed in terms of finite-time signals, rather than infinite-time signals, since, in practice, tasks have a finite length.

The presented framework allows to recover single-rate control as a special case of multirate control by setting \( F_1 = F_h \). In Sections VI and VII, multirate control is compared with single-rate control.

D. Notation

Matrix variables are underlined, with \( I_{m \times n} \) the \( n \times n \) identity matrix, \( \Omega_{m \times n} \) the \( m \times n \) zero matrix, \( J_{n} \) the \( n \times 1 \) ones vector with all elements 1, and \( \varepsilon_s \) the \( n \times 1 \) unit vector with the first element 1 and others 0. Vector \( a \in \mathbb{R}^{N} \), \( N \in \mathbb{N} \), is given by \( a = [a[0] a[1] \ldots a[N − 1]] \), with transpose \((\cdot)^T\) and \( \|a\|_2^2 = \sum a^2 \). The Kronecker product is denoted \( \otimes \) and \( \text{diag}(\{\cdot\}) \) denotes a diagonal matrix with diagonal entries \((\cdot)\). The floor operator is given by \( \lfloor x \rfloor = \max \{m \in \mathbb{Z} | m \leq x\} \). The discrete-time delay operator is denoted as \( z^{-1} \).

III. MULTIRATE CONTROL SYSTEM

In this section, the model-based multirate controller design is presented, which constitutes Contribution (I). In Section III-A, the time-varying aspects of multirate systems are modeled. In Section III-B, these models are used to describe the multirate control diagram in Fig. 3. Based on these results, the multirate controller is presented in Section IV.

A. Modeling Multirate Systems—Time-Varying Aspects

In this section, building blocks to model the multirate system in Fig. 3 are presented. The system is modeled over the finite-time length considered in the main problem in Section II-C.

Consider a causal, single-input, single-output, discrete-time, LTI system \( H \) with Markov parameters \( h(k) \in \mathbb{R}, k = 0, 1, \ldots, N − 1 \). The mapping from the finite-time input \( a \in \mathbb{R}^{N} \) to the finite-time output \( \beta \in \mathbb{R}^{N} \) is given by \( H \in \mathbb{R}^{N \times N} \) via

\[
\beta = Ha. \quad (4)
\]

\[
\begin{bmatrix}
\beta[0] \\
\beta[1] \\
\beta[2] \\
\vdots \\
\beta[N − 1]
\end{bmatrix} = \begin{bmatrix}
h(0) & 0 & 0 & \cdots & 0 \\
h(1) & h(0) & 0 & \cdots & 0 \\
h(2) & h(1) & h(0) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
h(N − 1) & h(N − 2) & h(N − 3) & \cdots & h(0)
\end{bmatrix} \times \begin{bmatrix}
a[0] \\
a[1] \\
a[2] \\
\vdots \\
a[N − 1]
\end{bmatrix}. \quad (5)
\]

Since \( a, \beta \) have the same sampling frequency, \( H \) is square. Moreover, since \( H \) is causal and time-invariant, \( H \) is lower triangular and Toeplitz, respectively [1].

---

(Images and diagrams mentioned but not included in the text.)
The multirate system in Fig. 3 involves different sampling frequencies. The conversions between different sampling frequencies are given as follows (see also [26, Sect. 4.1.1] and [25, Definition 5]). Let \( \mathbf{a} \in \mathbb{R}^{FN} \), \( F, N \in \mathbb{N} \), then the downsampling operator \( D_F : \mathbb{R}^{FN} \to \mathbb{R}^N \) with factor \( F \) yields \( \beta = D_F(\mathbf{a}) \in \mathbb{R}^N \) where

\[
\beta[k] = a[Fk], \quad k = 0, 1, \ldots, N - 1. \tag{6}
\]

Let \( \mathbf{a} \in \mathbb{R}^N \), \( N \in \mathbb{N} \), then the upsampling operator \( S_u,F : \mathbb{R}^N \to \mathbb{R}^{FN} \) with factor \( F \in \mathbb{N} \) yields \( \beta = S_u,F(\mathbf{a}) \in \mathbb{R}^{FN} \) where

\[
\beta[k] = \begin{cases} a \left[ \frac{k}{F} \right], & k = 0, 2F, 2F, \ldots, (N - 1)F \vspace{1mm} \\ 0, & \text{otherwise} \end{cases} \tag{7}
\]

The upsampling operator inserts zeros in between the values of the low-rate signal to create a high-rate signal. The interpolation is performed using a zero-order-hold interpolator. In terms of discrete-time transfer functions, the zero-order-hold interpolator with factor \( F \) is defined by

\[
\mathcal{I}_{ZOH,F} = \sum_{j=0}^{F-1} z^{-j}. \tag{8}
\]

The zero-order-hold interpolator is used in combination with the upsampling operator for upsampling. The resulting zero-order-hold upsampling is defined by \( \mathcal{H}_F := \mathcal{I}_{ZOH,F} S_u,F \), i.e., let \( \mathbf{a} \in \mathbb{R}^N \), \( N \in \mathbb{N} \), then \( \mathcal{H}_F \) with factor \( F \in \mathbb{N} \) yields \( \beta = \mathcal{H}_F(\mathbf{a}) \in \mathbb{R}^{FN} \) where

\[
\beta[k] = a \left[ \frac{k}{F} \right], \quad k = 0, 1, \ldots, (N - 1)F. \tag{9}
\]

The system description and controller design are based on finite-time descriptions. The finite-time description of the downsampling operator \( D_F \) with factor \( F \in \mathbb{N} \) is given by

\[
D_F = L_N \circ \mathbf{e}_F^T \in \mathbb{R}^{N \times FN} \tag{10}
\]

i.e., let \( \mathbf{a} \in \mathbb{R}^{FN} \), \( N \in \mathbb{N} \) and let \( \beta \in \mathbb{R}^N \) be given by (6), then \( \beta = D_F(\mathbf{a}) \) with \( D_F \) in (10). The finite-time description of the zero-order-hold upsampling operator \( \mathcal{H}_F \) with factor \( F \in \mathbb{N} \) is given by

\[
\mathcal{H}_F = L_N \otimes 1_F \in \mathbb{R}^{FN \times N} \tag{11}
\]

i.e., let \( \mathbf{a} \in \mathbb{R}^N \), \( N \in \mathbb{N} \) and let \( \beta \in \mathbb{R}^{FN} \) be given by (9), then \( \beta = \mathcal{H}_F(\mathbf{a}) \) with \( \mathcal{H}_F \) in (11). Examples of \( D_F \) and \( \mathcal{H}_F \) are provided in Example 2.

**Example 2 (Downsampler and Upsampler):** Let \( F = 2 \), \( N = 3 \), then \( D_F \) in (10) and \( \mathcal{H}_F \) in (11) are given by

\[
D_F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathcal{H}_F = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}
\]

Let \( \mathbf{a} = [1 \ 2 \ 3 \ 4 \ 5 \ 6]^T \), then \( \beta = D_F(\mathbf{a}) = D_F(\mathbf{a}) = [1 \ 3 \ 5]^T \) and \( \gamma := \mathcal{H}_F(\beta) = \mathcal{H}_F(\beta) = [1 \ 1 \ 3 \ 3 \ 5 \ 5]^T \). Note that \( \gamma = \mathcal{H}_F D_F \mathbf{a} \neq \mathbf{a} \), since

\[
\mathcal{H}_F D_F = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \neq \mathbf{I}_6. \tag{13}
\]

Example 2 shows that down/up sampling affects the signal. More generally, using the Kronecker mixed-product property

\[
(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}) \tag{14}
\]

it can be shown that

\[
D_F \mathcal{H}_F = \mathbf{I}_N, \quad \mathcal{H}_F D_F = \mathbf{I}_N \otimes (1_F e_F^T) \neq \mathbf{I}_{FN}. \tag{15}
\]

A key observation is that up/down sampling \( D_F \mathcal{H}_F \) has no effect on the signal, whereas down/up sampling \( \mathcal{H}_F D_F \) does affect the signal. In fact, \( \mathcal{H}_F D_F \) is block Toeplitz with block size \( F \), see also Example 2, and hence the down/up sampling operation is not LTI, but LPTV with period \( F \). An important consequence is that if an input-output operation involves any sampling rate lower than the input sampling rate, then the operation is LPTV. Indeed, this is the case for the multirate control diagram in Fig. 3, which is thus LPTV. The presented finite-time descriptions enable to exactly describe this time-varying multirate system.

In Section III-B, the multirate control diagram is presented, based on the finite-time descriptions presented in this section.

**Remark 3:** A more general definition of the downsampler \( D_F \) in (6) is obtained by considering \( \mathbf{a} \in \mathbb{R}^M \), \( \beta \in \mathbb{R}^\lfloor (M/F) \rfloor \), \( F, M \in \mathbb{N} \). For ease of notation, it is assumed that \( M = FN \).

### B. Multirate Control Diagram

The full control diagram of the architecture in Fig. 3 is shown in Fig. 4 and includes the modeling of systems \( G_{SS,h} \) and \( G_{LoS,l} \). The systems are modeled through \( G_{SS,*} \) and \( G_{LoS,*} \) operating at the extremely high-rate \( f_\ast \), which approximate the underlying continuous-time systems \( G_{SS} \) and \( G_{LoS} \), respectively. Here, \( \mathbf{e}_S, \mathbf{e}_a \) are the continuous-time hold (digital-to-analog) and sampling (analog-to-digital converter).

Recall that signals at rate \( f_\ast \) are not available for real-time feedback control. However, this approach enables the evaluation of the tracking error \( \varepsilon_\ast \) at rate \( f_\ast \).

To determine the optimal controllers, the relation between \( C_{FF,LoS,l}, C_{WF,LoS,l}, \) and \( \varepsilon_\ast \) is required. The dependence of finite-time \( \varepsilon_\ast \) on \( \mathbf{R}_{SS,h}, \mathbf{R}_{FF,LoS,l} \), \( \mathbf{R}_{WF,LoS,l} \) is given by Lemma 4.

**Lemma 4:** Given the finite-time descriptions in Section III-A, \( \varepsilon_\ast \) in Fig. 4 is given by

\[
\varepsilon_\ast = \psi_{SS,*} - \Delta \begin{bmatrix} 1_F & \mathbf{R}_{FF,LoS,l} \\ \mathbf{R}_{WF,LoS,l} \end{bmatrix}. \tag{16}
\]
The short-stroke loop (top) runs at high rate $f_h$ and the long-stroke loop (bottom) at low rate $f_l$. The interconnection is provided through downsampled $D_F$. Error $\varepsilon_*$ is an approximation of the continuous-time signal $e$ at extreme high-rate $f_h$. Solid lines: continuous-time signals. Dotted lines: extreme high sampling rates $f_h$. Dashed-dotted lines: low sampling rates $f_l$. Dashed lines: low sampling rates $f_l$. Both subsystems $(G)$ are controlled through feedback ($C_{FF}$), feedforward ($C_{FF}$), and input shaping filters are parameterized in terms of basis functions. The objective is to minimize $\varepsilon_*$ through design of $C_{FF,LoS,1}$ and $C_{\psi,LoS,1}$ such that the long stroke tracks the short stroke. In this configuration, $C_{FF,LoS,1}$ and $C_{\psi,LoS,1}$ are implemented at the low rate, i.e., $f_c = f_l$.

### Definition 5: $C_{FF,LoS,1}$ and $C_{\psi,LoS,1}$ in Fig. 4 are given by

$$C_{FF,LoS,1}(\theta_{FF}) = \sum_{i=0}^{n_{\theta_{FF}}-1} \theta_{FF}[i] \left( \frac{f(z^{-1})}{z} \right)^{i+1} \tag{21}$$

$$C_{\psi,LoS,1}(\theta_{\psi}) = 1 + \sum_{i=0}^{n_{\theta_{\psi}}-1} \theta_{\psi}[i] \left( \frac{f(z^{-1})}{z} \right)^{i+1} \tag{22}$$

with design parameters $\theta_{FF}$ and $\theta_{\psi}$.

### Theorem 6: Given Definition 5, the finite-time descriptions of $C_{FF,LoS,1}$ and $C_{\psi,LoS,1}$ are given by

$$C_{FF,LoS,1}(\theta_{FF}) = \sum_{i=0}^{n_{\theta_{FF}}-1} \theta_{FF}[i] \left( \frac{f(z^{-1})}{z} \right)^{i+1} \tag{23}$$

$$C_{\psi,LoS,1}(\theta_{\psi}) = 1 + \sum_{i=0}^{n_{\theta_{\psi}}-1} \theta_{\psi}[i] \left( \frac{f(z^{-1})}{z} \right)^{i+1} \tag{22}$$

with design parameters $\theta_{FF}$ and $\theta_{\psi}$, respectively.

### A. Controller Parameterization

To address arbitrary reference trajectories, the feedforward and input shaping filters are parameterized in terms of basis functions (see [37], [38]). Basis functions decouple the parameters from the reference trajectory, allowing variations in the reference trajectories without affecting the parameters. This is in contrast to standard learning approaches [6] in which a command signal for one specific reference trajectory is learned.

Inspired by [39], controllers $C_{FF,LoS,1}$, $C_{\psi,LoS,1}$ are parameterized in terms of difference operators according to Definition 5. Note that $C_{FF,LoS,1}(\mathbf{0}) = 0$ and $C_{\psi,LoS,1}(\mathbf{0}) = 1$ such that if the parameters are zero, only feedback control is used.

$$\Phi_{\psi,1} = D_F T_{\psi,h} L_{n_1,1} \otimes \frac{L_{n_1,1}}{L_{n_1,1,1}} \mathbb{R}_{\psi,1} \tag{25}$$

where $x$ refers to FF or $\psi$. 

$$\mathbb{R}_{\psi,1} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ -1 & -2 & -3 & \ldots & -n_x \\ 0 & 1 & 3 & \ldots & * \\ 0 & 0 & -1 & \ldots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & (-1)^{n_x} \end{bmatrix} \text{diag} \{f_1^1, \ldots, f_1^n\} \tag{27}$$
Proof: See Appendix B. □

Combining Theorem 6 with Lemma 4 reveals an affine dependence of $\bar{z}_*\epsilon$ on $\theta_{FF}$ and $\theta_{V}$ as made explicit in Lemma 7.

Lemma 7: Error $\bar{z}_*$ is given by

$$\bar{z}_* = b - A\Phi \theta$$

with

$$b = \psi_{SS, *} - \Phi_{FF, *}(H_{F, SI} + \psi_{FB, SI})D_{F, W_{SS, h}}$$

and

$$\Phi = \begin{bmatrix} \Phi_{FF, 1} & 0 \\ 0 & \Phi_{V, 1} \end{bmatrix}$$

$$\theta = \begin{bmatrix} \theta_{FF} \\ \theta_{V} \end{bmatrix}$$

Proof: See Appendix C. □

Lemma 7 provides the dependence of $\bar{z}_*$ on the controller parameters $\theta$. In Section IV-B, the parameters $\theta$ are optimized.

B. Controller Optimization

The optimal parameters for the control objective in (3) are given by the solution of the optimization problem

$$\min_\theta \|z_*\|^2_2 \text{ subject to } z_* = b - A\Phi \theta.$$  

If $A\Phi$ is full rank, the solution to this quadratic optimization problem is given by the least-squares solution $\theta = \theta_0$, with

$$\theta_0 = ((A\Phi)^\top (A\Phi))^{-1} (A\Phi)^\top b.$$  

For perfect models, solution (33) provides the optimal solution.

In practice, there are always model mismatches for which the parameters are iteratively learned through an approach that closely resembles norm-optimal ILC [6] based on the models and data of previous executions. A key observation is that the models are time varying, which is in sharp contrast to standard learning techniques. One execution of the learning approach is referred to as a trial or task and indicated with subscript $j$. The parameters $\theta_{j+1}$ for the next trial are determined as those minimizing the performance criterion in Definition 8 [6] based on measured data from trial $j$.

Definition 8 (Performance Criterion): The performance criterion for trial $j+1$, $j = 0, 1, 2, \ldots$ is given by

$$J(\theta_{j+1}) = \|\bar{z}_{j+1,*}\|^2_2 + \|\bar{z}_{j+1,I}\|^2_2 + \|\bar{z}_{j+1,J} - \bar{z}_{j,I}\|^2_2.$$  

where $\|\cdot\|^2_2 = (\cdot)^\top W(\cdot)$, with $W_\epsilon \in \mathbb{R}^{N_x \times N_x}$ positive definite, $W_{\epsilon} \in \mathbb{R}^{2N_x \times 2N_x}$ semipositive definite, and

$$\bar{z}_{j+1,*} = \bar{z}_{j,*} - A\Phi(\theta_{j+1} - \theta_j)$$

and

$$\bar{z}_{j+1,I} = \Theta \theta_j.$$  

Performance criterion (34) can be used to address several control goals. For example, for $W_\epsilon = I_{N_x}$ and $W_\epsilon = W_{\epsilon} = 0_{2N_x}$, the control goal in (3) is addressed, i.e., minimizing $\|z_*\|^2_2$. The optimal parameters for the general criterion are given by Theorem 9.

Theorem 9 (Iterative Solution): The parameters $\theta_{j+1}$, $j = 0, 1, 2, \ldots$, that minimize $J(\theta_{j+1})$ in Definition 8 are given by

$$\theta_{j+1} = \theta_0 + L\bar{z}_{j,*}$$

with

$$L = ((A\Phi)^\top W_{\epsilon}(A\Phi) + \Phi^\top (W_{\epsilon} + W_{\Delta}\dot{z})\Phi)^{-1}$$

and

$$Q = ((A\Phi)^\top W_{\epsilon}(A\Phi) + \Phi^\top (W_{\epsilon} + W_{\Delta}\dot{z})\Phi).$$

Theorem 9 directly follows from substitution of (35) and (36) in (34) and equating $\nabla J(\theta_{j+1}) = 0$ (see also [37]). Note that $W_{\epsilon}$, $W_{\Delta}$, and $W_{\Delta}$ should be chosen such that the inverse in (38) and (39) exists. A step-by-step procedure for the iterative algorithm is provided in Algorithm 10, where (33) provides initial parameters based on models only.

Algorithm 10 (Iterative Tuning Procedure): Calculate $Q$, $L$ using (38), (39), set $j = 0$ and determine $\theta_0$ in (33). Then, perform the following sequence of steps.

1. Execute task $j$ and record data $\bar{z}_{j,*}$.
2. Determine $\theta_{j+1}$ through (37).
3. Set $j \rightarrow j + 1$ and repeat from step 1 until satisfactory convergence in $\theta_j$ or a user-defined maximum number of trials is reached.

Algorithm 10 provides the iterative tuning solution for the time-varying multirate system with controller design at the low rate. In Section IV-C, the controllers are explicitly designed and implemented at the high rate to enhance the performance/cost tradeoff in Fig. 1.

C. Performance Enhancement—High-Rate Control

In Section IV-B, the optimal controller for the multirate system in Fig. 4 is presented. In this section, the performance of the multirate system is further improved by modifying the controller implementation and design, which constitutes Contribution (III). The results of Section IV-B are recovered as a special case.

In contrast to time-invariant systems, time-varying systems do not generally commute, i.e., interchanging the order affects the output. One key advantage of the proposed approach is that this property can be directly exploited to enhance the performance/cost tradeoff in Fig. 1. In Fig. 4, both the feedforward controller and input shaper of the long stroke are implemented at the low rate $f_l$. In this section, these controllers are implemented at high rate $f_h$ as shown in Fig. 5(a). This implementation has the potential to improve the performance since $\psi_{SS, h}$ contains more information than $\rho_{SS, f_l} = D_{F, W_{SS, h}}$. This also follows from the noble identity $D_{F, H(z_{\dot{z}})} H(z_{\dot{z}}) \equiv D_{F, H}$, with $H$ a discrete-time system rational in $z$ [36, Sec. 4.2]. Indeed, since the frequency response of $C_{FF, Lo, Sl}$, $h$ is independent of that of $C_{FF, Lo, Sl}$, there is more design freedom as illustrated in Fig. 5(b).

The additional cost of the high-rate implementation is negligible since it only involves a different controller design.
in software, without affecting hardware. In particular, it uses sensor information of the short stroke loop at high rate, which is also required for feedback control of the short stroke. The new design does not require sensor information of the long-stroke loop at a higher rate. The actuation of the long-stroke loop remains at low rate.

The parameterization of the controllers at high rate is similar to that in Definition 5 and provided by Definition 11.

Definition 11: $C_{\psi, LoS, h}$ and $C_{\psi, LoS, h}$ in Fig. 5 are given by

\[
C_{\psi, LoS, h}(\theta_{FF}) = \frac{n_{\psi} - 1}{\eta} \sum_{i=0}^{n_{x}-1} \theta_{FF}[i] \left( \frac{f_{h}(z-1)}{z} \right)_{i+1} \quad (40)
\]

\[
C_{\psi, LoS, h}(\theta_{FF}) = 1 + \sum_{i=0}^{n_{x}-1} \theta_{FF}[i] \left( \frac{f_{h}(z-1)}{z} \right)_{i+1} \quad (41)
\]

The finite-time descriptions for this parameterization are provided in Lemma 12. Using these results, the iterative approach outlined in Algorithm 10 is directly applicable.

Lemma 12: Given Definition 11, the finite-time descriptions (23), (24), and (30) change to

\[
L_{\psi, LoS, h} = D_{F} C_{\psi, LoS, h} \frac{y_{SS, h}}{y_{SS, h}} = D_{F} \Phi_{\psi, h} \theta_{FF} \quad (42)
\]

\[
L_{\psi, LoS, h} = D_{F} C_{\psi, LoS, h} \frac{y_{SS, h}}{y_{SS, h}} = D_{F} \Phi_{\psi, h} \theta_{FF}
\]

\[
\Phi = \begin{bmatrix}
D_{F} \Phi_{\psi, h} \\
0 \\
D_{F} \Phi_{\psi, h}
\end{bmatrix}
\quad (44)
\]

with

\[
\Phi_{x,h} = T_{ySS,h} \begin{bmatrix}
\frac{L_{n_{x}+1}}{(n_{x}+1) \times (n_{x}+1)} \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \\
1 \

where $x$ refers to FF or $\psi$.

Proof: See Appendix D.

The controller design and implementation at high rate complete the multirate controller design. Next, the advantages of multirate control over single-rate control are demonstrated in both simulation and experiments.

V. EXPERIMENTAL SETUP—A DUAL-STAGE WAFER STAGE SYSTEM

In the remainder of this paper, the multirate control design framework presented in Section IV is validated on a dual-stage system, both in simulations and experiments. In this section, the wafer stage system is introduced in more detail and the experimental setup of the dual-stage system is presented.

A. Wafer Stages—Key Components in Lithography Machines

Wafer stages are key components in wafer scanners. Wafer scanners are state-of-the-art lithography machines for the automated production of integrated circuits. In Fig. 6, a schematic illustration of a wafer scanner system is depicted. Ultraviolet light from a light source passes through a reticle, which contains a blueprint of the integrated circuits to be manufactured. The reticle is clamped atop the reticle stage, which performs a scanning motion. The resulting image of the reticle is scaled down by a lens system and projected onto the light-sensitive layers of a wafer. The wafer is clamped on the wafer stage and performs a synchronized scanning motion with the reticle stage.

During the scanning process, the wafer stage and reticle stage track reference signals with nanometer positioning accuracy. In this paper, the focus is on the control of the wafer stage, which has more stringent performance requirements than the reticle stage [40].

B. Experimental Setup

The experimental setup is shown in Fig. 7 and consists of two stages: a long stroke (LoS) and a short stroke (SS). Both stages can translate in one horizontal direction and are air guided. Each stage is actuated through a Lorentz actuator attached to the force frame. The position of each stage is measured through 1-nm resolution optical encoders attached to the metrology frame, which is separated from the force frame to reduce interaction. The total stroke is 16.0 mm.
The sampling rate of $f_s$ is $f_s = 10080$ Hz. The identified frequency response functions of both stages are shown in Fig. 8. The stages are modeled as freely moving masses with one sample I/O delay

$$G_{x,s} = \frac{z^{-1}(z+1)}{2m_s f_s^2(z-1)^2}$$

(47)

with masses $m_{SS} = 4.70$ kg and $m_{LoS} = 4.33$ kg. The sampling rate factors $F_h$, $F_l$ are varied and provided when relevant.

Reference trajectory $\rho_{SS,h}$ consists of a forward and backward movement with a total duration of 0.25 s and is shown in Fig. 9. The point-to-point profile is representative for the application in terms of distance, maximum acceleration, and so on. A fourth-order profile is used to guarantee a smooth signal with limited high-frequency content to avoid excitation of higher order dynamics (see also Fig. 8 and [41]).

Experiments show that the measurement noise on both $\psi_{SS,*}$ and $\psi_{LoS,*}$ has a variance of $(45 \text{ nm})^2$. This value is used during simulation to mimic experimental conditions.

C. Controller Design

The fixed feedback controllers $C_{FB,SS,h}$ and $C_{FB,LoS,l}$ both consist of a lead filter, weak integrator, and second-order lowpass filter based on loop-shaping techniques [5]. The controllers stabilize their respective closed-loop systems and yield a bandwidth (first 0-dB crossing of the open loop) of 100 Hz for both loops. The feedforward controller and input shaper for the short stroke are given by

$$C_{FF,SS,h} = m_{SS} f_h^2 (z-1)^2$$

(48)

$$C_{\psi,SS,h} = G_{SS,h} C_{FF,SS,h}.$$  

(49)

Hence, $C_{FF,SS,h}$ generates mass feedforward and the combination results in $\epsilon_{SS,h} = 0$, if $G_{SS,h}$ is exact.

The design of the long-stroke feedforward controller and input shaper aims to minimize $\| \epsilon \|^2_2$ by setting the weights in Definition 8 to

$$W_e = L_{N_e}, \quad W_{\epsilon}, W_{\Delta \epsilon} = \Omega_{2N_l \times 2N_l}.$$  

(50)

Note that these settings also facilitate fast convergence of the iterative procedure in Algorithm 10.
Fig. 10. Simulation results in (a) and (c), and experimental results in (b) and (d) for the four control configurations in Table I. As is expected, single-rate high (○) outperforms single-rate low (▲). The performance of multirate low (●) is similar to that of single-rate low (▲). The performance of multirate high (●) is close to the performance of single-rate high (○). The results demonstrate the advantages of multirate control. Indeed, a high level of performance is achievable with multirate control for limited cost since one of the feedback control loops is evaluated at a lower rate. (a) Simulation results for varying $\tau_{\text{FF}}$ show that, due to more design freedom in terms of parameters $n_{\text{FF}}$, the performance increases ($J$ decreases) for increasing cost (increasing buffer length $\tau_{\text{FF}}$). The results shown are for fixed $C_{\psi,\text{LoS}} = 1$ and varying $n_{\text{FF}}$. (b) Experimental validation of the simulation in (a). The results are in line with the simulation results. (c) Simulation results for varying $\tau_{\psi}$ show that larger cost (larger buffer length $\tau_{\psi}$) yields better performance (lower $J$). The results shown are for mass feedforward ($n_{\text{FF}} = 2$ and $\theta_{\text{FF}}[0] = 0$) and varying $n_{\psi}$. (d) Experimental validation of the simulation in (c). The results are in line with the simulation results.

The performance/cost tradeoff curves for the configurations in Table I are shown in Fig. 10(a) and (c). Both figures show increasing performance (decreasing $J$) for increasing cost (increasing $\tau$) and excellent performance through multirate control with design at high rate. As a direct consequence of a higher sampling rate, single-rate high outperforms single-rate low. Multirate control is a tradeoff between these two and, hence, the performance is somewhere in between. The performance improvement of multirate low is limited compared to single-rate low. In contrast, the performance of multirate high is close to that of single-rate high. The results show that multirate control can achieve high performance with limited cost when designed and implemented at the high rate. Indeed, the noise introduces trial-varying disturbances, which cannot be compensated through the iterative tuning algorithm and thereby limits the achievable performance.

### C. Results

The performance/cost tradeoff curves for the configurations in Table I are shown in Fig. 10(a) and (c). Both figures show the enhancement of the performance/cost tradeoff through multirate control as illustrated in Fig. 1. In particular, both figures show increasing performance (decreasing $J$) for increasing cost (increasing $\tau$) and excellent performance through multirate control with design at high rate. As a direct consequence of a higher sampling rate, single-rate high outperforms single-rate low. Multirate control is a tradeoff between these two and, hence, the performance is somewhere in between. The performance improvement of multirate low is limited compared to single-rate low. In contrast, the performance of multirate high is close to that of single-rate high. The results show that multirate control can achieve high performance with limited cost when designed and implemented at the high rate. Indeed, the long-stroke feedback control loop remains executed at the low rate.

The results in Fig. 10(a) show the importance of adding the acceleration profile as basis function in terms of performance improvement, as is also apparent from the frequency response functions in Fig. 8 and identified models in (47). Indeed, especially for low frequencies, the stages behave as

the controller parameterization on low rate (Definition 5) and high rate (Definition 11), the number of parameters $n_{\text{FF}}$ and $n_{\psi}$ alone does not provide a fair comparison between the controllers. Therefore, the controller buffer lengths

$$
\tau_{\text{FF}} := \frac{n_{\text{FF}}}{f_c}, \quad \tau_{\psi} := \frac{n_{\psi}}{f_c}
$$

(51)

are defined, where $f_c$ is the sampling rate of the optimized controllers (see Table I). These buffer lengths are an indication for the implementation cost of the controller.

### B. Simulation Setup

For comparison with the experimental results in Section VII, measurement noise is added to $w_{\text{SS},*}$ and $w_{\text{LoS},*}$. The noise is modeled as zero mean, Gaussian white noise with variance $\sigma^2 = (45 \text{ nm})^2$ based on experimental data (see also Section V-B).

In the simulation, the models are exact and hence the initial parameters $\theta_0$ in (33) provide the optimal solution. Note that

<table>
<thead>
<tr>
<th>Label</th>
<th>Symbol</th>
<th>$f_\ast$ [Hz]</th>
<th>$f_\delta$ [Hz]</th>
<th>$f_1$ [Hz]</th>
<th>$f_c$ [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-rate low</td>
<td>▲</td>
<td>10080</td>
<td>1008</td>
<td>1008</td>
<td>1008</td>
</tr>
<tr>
<td>Multirate high</td>
<td>■</td>
<td>10080</td>
<td>2016</td>
<td>1008</td>
<td>2016</td>
</tr>
<tr>
<td>Multirate low</td>
<td>◇</td>
<td>10080</td>
<td>2016</td>
<td>1008</td>
<td>1008</td>
</tr>
</tbody>
</table>

### TABLE I

**FOUR DIFFERENT CONTROL CONFIGURATIONS THAT ARE EVALUATED.**
Fig. 11. Time-domain simulation results for single-rate high with $n_{FF} = 2$, $n_\psi = 0$. The results show the importance of mass feedforward. (a) Feedforward signal $\nu_{FF,LoS,1}$. (b) Error signal $\xi$. 

The simulation results demonstrate the potential of multirate control, especially when the controllers are designed and implemented at the high rate. Next, the results are experimentally validated.

VII. EXPERIMENTAL RESULTS

In this section, the simulation results of Section VI are experimentally validated on the setup described in Section V. The results experimentally validate the advantages of multirate control and constitute Contribution (V).

Fig. 12. Experimental results of the performance criterion over trials for $\tau_{FF} = 1 \text{ ms}$, $\tau_\psi = 0$ with single-rate high (•), multirate high (○), multirate low (◇), and single-rate low (△). The results show that all control configurations converge in one trial up to the level of trial-varying disturbances for which the iterative tuning algorithm cannot compensate.

A. Application of Iterative Tuning

In contrast to simulation, the models do not exactly describe the system in experiments. Therefore, the iterative tuning procedure in Algorithm 10 is invoked to iteratively update the parameters based on measured data. The convergence of the iterative tuning algorithm is shown in Fig. 12 for the various control configurations in Table I with a fixed buffer length $\tau_{FF} = 1 \text{ ms}$ ($\tau_\psi = 0$).

The results in Fig. 12 show fast convergence (one trial) of the iterative algorithm as desired. Note that the deviations over the trials are caused by trial-varying disturbances for which the
algorithm cannot compensate. In the remainder, five trials are used and only the results of the fifth trial are shown.

B. Results

The experimental results for the simulations in Fig. 10(a) and (c) are shown in Fig. 10(b) and (d), respectively. The results are in line with the simulation results and the conclusions in Section VI, i.e., higher performance (lower J) for increasing number of parameters (increasing τ), and excellent performance for multirate control with control design at high rate (multirate high).

Time-domain signals for several parameterizations with multirate high are shown in Fig. 13. Clearly, mass feedforward rate (multirate high).

The following identity, known as the push-through rule, is expressed in (54).

\[ \psi_{SS,*} = G_{SS,*}H_{F_{h}}(I_{FF,SS,h} + C_{FB,SS,h}C_{SS,h})\rho_{SS,h} \]

Using Fig. 4 and (52), \( \psi_{SS,*} \) is expressed in \( \rho_{SS,h} \)

\[ \psi_{SS,*} = G_{SS,*}H_{F_{h}}(I_{FF,SS,h} + C_{FB,SS,h}C_{SS,h})\rho_{SS,h} \]

\[ = (I_{N_{h}} + G_{SS,*}H_{F_{h}}C_{FB,SS,h}D_{F_{h}})G_{SS,*}H_{F_{h}}^{-1}(I_{FF,SS,h} + C_{FB,SS,h}C_{SS,h})\rho_{SS,h} \]

\[ = G_{SS,*}H_{F_{h}}(I_{N_{h}} + C_{FB,SS,h}D_{F_{h}}G_{SS,*}H_{F_{h}}^{-1})(I_{FF,SS,h} + C_{FB,SS,h}C_{SS,h})\rho_{SS,h} \]

\[ = G_{SS,*}H_{F_{h}}(I_{FF,SS,h} + C_{FB,SS,h}C_{SS,h})\rho_{SS,h} \]

\[ = \psi_{SS,*} \]

\[ \psi_{LoS,*} = G_{LoS,*}H_{F_{l}}(I_{FF,LoS,l} + C_{FB,LoS,l}I_{LoS,l}) \]

\[ = (I_{N_{l}} + G_{LoS,*}H_{F_{l}}C_{FB,LoS,l})H_{F_{l}}^{-1}(I_{FF,LoS,l} + C_{FB,LoS,l}I_{LoS,l}) \]

\[ = G_{LoS,*}H_{F_{l}}(I_{N_{l}} + C_{FB,LoS,l}I_{FF,LoS,l})G_{LoS,*}H_{F_{l}}^{-1}(I_{FF,LoS,l} + C_{FB,LoS,l}I_{LoS,l}) \]

\[ = G_{LoS,*}H_{F_{l}}(I_{FF,LoS,l} + C_{FB,LoS,l}I_{LoS,l}) \]

\[ \psi_{LoS,*} \]

with \( \sum_{LoS,l} \) in (20) and \( \psi_{LoS,*} \)

\[ \sum_{LoS,l} = \psi_{SS,*} - \psi_{LoS,*} \]

APPENDIX B

PROOF THEOREM 6

It is shown that for the parameterization

\[ C_i(z) = \sum_{i=0}^{n-1} \theta[i] (\frac{f_i(z-1)}{z})^{i+1} \]
it holds

$$\hat{z}_1 = C_1 \mathcal{D}_F \rho_{SS,h} = \Phi \theta.$$  \hspace{1cm} (63)  

Relations (23) and (24) directly follow from this result.

Parameterization (62) can equivalently be written as a finite impulse response (FIR) structure of order $n_a = n + 1$

$$C_l(\theta) = \sum_{i=0}^{n_a-1} \theta[i] \left( \frac{f_i(z-1)}{z} \right)^{i+1} = \sum_{i=0}^{n_a-1} a[i] z^{-i}. \hspace{1cm} (64)$$

By equating coefficients, it directly follows that the relation between the parameters is given by $\theta = R \hat{\theta}$ with $R \in \mathbb{R}^{n_a \times n}$ as given in (27). Note that $R_l$ is the product of a truncated transposed (lower triangular Cholesky factor of the) Pascal matrix of order $n_a$, with a diagonal scaling matrix depending on $f_l$.

The finite-time description of $C_1$ in terms of $\alpha$ is given by

$$C_1 = \begin{bmatrix} 0 & 0 & ... & 0 \\ a[0] & 0 & ... & 0 \\ a[1] & a[0] & ... & 0 \\ ... & ... & ... & ... \\ a[n_a-1] & ... & a[0] & 0 \\ 0 & a[n_a-1] & ... & 0 \\ 0 & 0 & a[n_a-1] & ... \\ & & & & \end{bmatrix}.$$  \hspace{1cm} (65)  

Using the Kronecker mixed-product property rule (14), the order of $C_l$ and $\mathcal{D}_F$, see (10), is interchanged

$$C_l \mathcal{D}_F = (C_l \otimes I)(L \otimes \mathcal{D}_F) \hspace{1cm} (66)$$

$$= (C_l \otimes I) \otimes (L \otimes \mathcal{D}_F) \hspace{1cm} (67)$$

$$= (C_l \otimes I)(L \otimes \mathcal{D}_F) \hspace{1cm} (68)$$

$$= (C_l \otimes I)(L \otimes \mathcal{D}_F) \hspace{1cm} (69)$$

$$= (C_l \otimes I)(L \otimes \mathcal{D}_F) \hspace{1cm} (70)$$

Note that $C_l \otimes (L \otimes \mathcal{D}_F)$ is a lower triangular matrix and that $\rho_{SS,h} = T_{SS,h} \rho_{Nh}$, with $T_{SS,h}$ in (26) is also a lower triangular matrix.

Next, the commutative property of lower triangular matrices is exploited to express $\hat{z}_1$ in $\theta$. To this end, the Kronecker product rule and the relation $\alpha = R \hat{\theta}$ are used

$$\hat{z}_1 = C_1 \mathcal{D}_F \rho_{SS,h} = \Phi \theta.$$  \hspace{1cm} (71)  

which concludes the proof of (62). Relations (23) and (24) directly follow from this result.

**APPENDIX C**

**PROOF LEMMA 7**

Substitution of (23) and (24) in (16) and using (18) yields

$$\hat{z}_* = \psi_{SS,*} - A \left[ \begin{bmatrix} \Phi_{FF}, \theta_{FF} \\ \Phi_{SS,h} \end{bmatrix} \right]$$  \hspace{1cm} (80)  

$$= \psi_{SS,*} - A \left[ \begin{bmatrix} \Phi_{FF}, \theta_{FF} \\ \Phi_{SS,h} \end{bmatrix} \right]$$  \hspace{1cm} (81)  

$$= \psi_{SS,*} - A \left[ \begin{bmatrix} \Phi_{FF}, \theta_{FF} \end{bmatrix} \right] - A \left[ \begin{bmatrix} \Phi_{FF,1} \theta_{FF} \\ \Phi_{SS,h} \end{bmatrix} \right]$$  \hspace{1cm} (82)  

$$= \psi_{SS,*} - A \left[ \begin{bmatrix} \Phi_{FF,1} \theta_{FF} \end{bmatrix} \right] - A \left[ \begin{bmatrix} \Phi_{FF,1} \theta_{FF} \end{bmatrix} \right]$$  \hspace{1cm} (83)  

$$= \psi_{SS,*} - A \left[ \begin{bmatrix} \Phi_{FF,1} \theta_{FF} \end{bmatrix} \right]$$  \hspace{1cm} (84)  

$$= b - A \Phi \theta.$$  \hspace{1cm} (85)  

with $b, \Phi, \theta$ as given in Lemma 7.

**APPENDIX D**

**PROOF LEMMA 12**

It is shown that for the general parameterization

$$C_h(\theta) = \sum_{i=0}^{n_a-1} \theta[i] \left( \frac{f_i(z-1)}{z} \right)^{i+1} \hspace{1cm} (86)$$

it holds

$$\hat{z}_1 = D_F C_h \psi_{SS,h} = D_F \Phi_h \theta.$$  \hspace{1cm} (87)  

Relations (42) and (43) directly follow from this result.

The proof is similar to that of Theorem 6. First, $C_h$ is expressed in terms of FIR parameters $\alpha$

$$C_h(\theta) = \sum_{i=0}^{n_a-1} \theta[i] \left( \frac{f_i(z-1)}{z} \right)^{i+1} = \sum_{i=0}^{n_a-1} a[i] z^{-i} \hspace{1cm} (88)$$

where $\alpha = R \hat{\theta}$ with $R_h \in \mathbb{R}^{n_a \times n}$ as given in (46) and $n_a = n + 1$. The finite-time description of $C_h$ in terms of $\alpha$ is given by

$$C_h = \begin{bmatrix} a[0] & 0 & 0 & ... \\ a[1] & a[0] & 0 & ... \\ ... & ... & ... & ... \\ a[n_a-1] & ... & a[1] & 0 \\ 0 & a[n_a-1] & ... & ... \\ ... & ... & ... & ... \end{bmatrix}.$$  \hspace{1cm} (89)  

with $b, \Phi, \theta$ as given in Lemma 7.
Next, it is exploited that the lower triangular matrices $C_h$ and $L_{\text{sys},h}$ commute

\[
\begin{align*}
\tilde{z}_i &= \mathcal{D}_F \tilde{C}_h L_{\text{sys},h} \\
&= \mathcal{D}_F \tilde{C}_h L_{\text{sys},h} L_{N_h} \\
&= \mathcal{D}_F \tilde{C}_h \tilde{L}_{N_h} \\
&= \mathcal{D}_F \tilde{C}_h \tilde{L}_{N_h} \begin{bmatrix} \alpha \end{bmatrix} \\
&= \mathcal{D}_F \tilde{L}_{N_h} \begin{bmatrix} \alpha \end{bmatrix} \\
&= \mathcal{D}_F \tilde{L}_{N_h} \begin{bmatrix} \alpha \end{bmatrix} \begin{bmatrix} \varphi \end{bmatrix} \\
&= \mathcal{D}_F \tilde{L}_{N_h} \begin{bmatrix} \alpha \end{bmatrix} \begin{bmatrix} \varphi \end{bmatrix} \\
&= \mathcal{D}_F \Phi_{\theta \varphi} \end{align*}
\]

Relations (42) and (43) directly follow from this result.

**ACKNOWLEDGMENT**

The authors would like to thank G. Leenknecht for his contributions to the experimental setup.

**REFERENCES**


Jurgen van Zundert received the M.Sc. degree (Hons.) in mechanical engineering from the Eindhoven University of Technology, Eindhoven, The Netherlands, in 2014, where he is currently pursuing the Ph.D. degree with the Control Systems Technology Group, Department of Mechanical Engineering.

His current research interests include feedforward motion control, multirate control, and iterative learning control.

Tom Oomen (SM’06) received the M.Sc. degree (cum laude) and the Ph.D. degree from the Eindhoven University of Technology (TU/e), Eindhoven, The Netherlands.

He held visiting positions at KTH Royal Institute of Technology, Stockholm, Sweden, and the University of Newcastle, Newcastle, NSW, Australia. He is currently an Associate Professor with the Department of Mechanical Engineering, TU/e. His current research interests include system identification, robust control, and learning control, with applications in mechatronic systems.

Dr. Oomen was a recipient of the Corus Young Talent Graduation Award, the 2015 IEEE Transactions on Control Systems Technology Outstanding Paper Award, and the 2017 IFAC Mechatronics Best Paper Award. He is an Associate Editor of the IEEE CONTROL SYSTEMS LETTERS and IFAC MECHATRONICS.

Jan Verhaegh received the M.Sc. degree (Hons.) in mechanical engineering from the Eindhoven University of Technology, Eindhoven, The Netherlands.

Since 2015, he has been a Research Scientist with the Department of Integrated Vehicle Safety, TNO, Helmond, The Netherlands. His current research interests include vehicle dynamics and (wireless) control topics related to automated and cooperative driving technologies. He is involved in projects include design of Robust and Fail-Safe Cooperative Adaptive Cruise Control in truck platooning and the development, and implementation of Active Lane Keep Assist for high-way automation.

Wouter Aangenent received the M.Sc. degree (cum laude) and the Ph.D. degree in mechanical engineering from the Eindhoven University of Technology, Eindhoven, The Netherlands, in 2004 and 2008, respectively.

In 2008, he joined the Research Department of ASML, Veldhoven, The Netherlands, where he currently leads the Mechatronics and Control Research Group.

Duarte J. Antunes (M’13) was born in Viseu, Portugal, in 1982. He received the Licenciatura in electrical and computer engineering from the Instituto Superior Técnico (IST), Lisbon, Portugal, in 2005, and the Ph.D. degree in automatic control with the Institute for Systems and Robotics, IST, in 2011.

From 2011 to 2013, he held a postdoctoral position at the Eindhoven University of Technology (TU/e). He is currently an Assistant Professor with the Department of Mechanical Engineering, TU/e. His current research interests include networked control systems, stochastic control, dynamic programming, and systems biology.

W. P. M. H. Heemels (F’16) received the M.Sc. degree (summa cum laude) in mathematics and the Ph.D. degree (summa cum laude) in control theory from the Eindhoven University of Technology (TU/e), Eindhoven, The Netherlands, in 1995 and 1999, respectively.

From 2000 to 2004, he was with the Electrical Engineering Department, TU/e. From 2004 to 2006, he was with the Embedded Systems Institute, Eindhoven. Since 2006, he has been with the Department of Mechanical Engineering, TU/e, where he is currently a Full Professor. He held visiting professor positions at the Swiss Federal Institute of Technology, Zürich, Switzerland, in 2001, and the University of California at Santa Barbara, Santa Barbara, CA, USA, in 2008. He was with Océ, Venlo, The Netherlands, in 2004. His current research interests include hybrid and cyber-physical systems, networked and event-triggered control systems, and constrained systems including model predictive control.

Dr. Heemels was a recipient of a personal VICI grant awarded by STW (Dutch Technology Foundation). He served/serves on the Editorial Boards for Automatica, Nonlinear Analysis: Hybrid Systems, Annual Reviews in Control, and the IEEE TRANSACTIONS ON AUTOMATIC CONTROL.