An Evaluation of Different Contributions to Flame Stretch for Stationary Premixed Flames

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The concept of flame stretch is extended to study stationary premixed flames with a finite thickness. It is shown that the analysis results in additional contributions to the stretch rate due to changes in the flame thickness and due to density variations along the flame. Extended expressions are derived that describe the effect of stretch on variations in scalar quantities, such as the enthalpy. These expressions are used to determine local variations in the flame temperature, and it is shown that known results are recovered when a number of approximations are introduced. The extended stretch formalism might be useful to analyze and quantify the different flame stretch contributions and their effects in numerical flame studies. Finally, the different contributions to the total stretch rate and the effects thereof on the flame stabilization are numerically computed for the flame tip of a two-dimensional Bunsen flame as illustration. © 1997 by The Combustion Institute

INTRODUCTION

It is well known that deviations from pure one-dimensional flow and transport in premixed flames, such as flame curvature and nonuniform flow along the flame, might lead to local variations in flame temperature and mass burning rate. These stretch effects were first studied by Karlovitz [1] to describe flame extinction. Subsequently, Lewis and von Elbe [2] used flame stretch to study flame stabilization. Markstein [3] investigated the influence of stretch on flame front instability. Since these early publications, significant progress has been made in the understanding of flame stretch, and in particular, the structure and propagation of stretched flames have been studied in numerous papers.

The generally accepted definition of the flame stretch $K$, first given by Williams [4], reads

$$K = \frac{1}{A} \frac{dA}{dt},$$

i.e., $K$ is the fractional area change of a small area $A$ in the flame surface, which moves in this surface with a tangential velocity equal to the local tangential fluid velocity. This definition of flame stretch only holds at the flame surface. Practical expressions for $K$, based on kinematic considerations, are given by Buckmaster [5] and Matalon [6].

In the analysis of stretched flames, relations between flame temperature and burning velocity on the one hand and flame stretch on the other hand are derived. A rigorous mathematical analysis of stretched flames, based on matched asymptotic expansions, is given by Matalon and Matkowsky [7]. In [8], Buckmaster derived a relation between the burning velocity and stretch for two examples of stretched flames. An integral analysis of flame stretch is presented by Chung and Law [9].

The significance of stretch for premixed flames is controversial. In [5], Buckmaster gives a critical assessment of flame stretch. He claims that flame behavior does not solely depend on flame stretch because of the following reasons. First, heat transfer in a flame cannot only depend on the local value of flame stretch defined at the flame surface, but it must also...
depend on the stretch in the preheat zone. Second, also, variations in the preheat zone thickness are of importance for the heat transfer rate. Finally, the burning velocity does not only depend on flame stretch, but also on the Lewis number Le of the flame. This last observation is confirmed by Law [10], who asserts that flame stretch can only influence flame response in combination with preferential diffusion effects.

As the existing theory is not directly applicable to numerical flame studies, we approach the problem in an alternative way. Regarding Buckmaster’s objections, we define our flame stretch in the flame region between the burnt and unburnt gases, comprising the reaction zone and the preheat and reactant diffusion zones. In this flame region, we can identify iso-lines of a suitably chosen variable Y. The new flame stretch is then defined in terms of the mass flux along these lines. It will turn out that flame thickness variations are included in our stretch definition. The method is restricted to stationary 2D flames for the time being.

We believe that our definition of flame stretch has a sound physical basis because it is based on the mass conservation equation and no serious assumptions are made. On the other hand, the existing theory of flame stretch is always based on assumptions such as the existence of a single reaction with one Lewis number or the applicability of perturbation theory. The classical theory is therefore of limited value for numerical combustion studies with complex transport and complex chemistry models. Numerical flame studies are our primary interest, and it is our aim to apply the extended theory to study the importance of flame stretch on the mechanisms of flame stabilization. We like to emphasize that it is not our purpose to give a final assessment of flame stretch, but instead to present a new approach which could be used in the analysis of stretched flames. As an example, we apply our stretch definition to the integral analysis of Chung and Law [9].

The contents of the paper is the following. In the second section, our flame stretch definition is introduced, starting from the mass conservation equation. Our definition of flame stretch contains the strain rate of the tangential velocity along the flame surface, as in other studies [1, 3, 7, 9]. Moreover, additional terms due to density variations along the flame surface and flame thickness variations are also incorporated in our definition. The influence of the stretch field on the behavior of some scalar variables is studied in the third section. In particular, we have applied our stretch definition to the enthalpy equation in order to investigate the relation between the local flame temperature and the stretch field. As an illustration, in the fourth section, we have computed the different contributions to the flame stretch rate for the flame tip of a two-dimensional Bunsen flame numerically. Finally, some conclusions are formulated in the last section.

THE NEW FLAME STRETCH DEFINITION

Consider a 2D stationary flame “front” in a premixed gas mixture, defined in terms of a given scalar field Y, which might be the temperature or the mass fraction of any species in the flame for the time being. We assume that flame front “contours” correspond to iso-contours of Y (see Fig. 1), and that the unburned and burned boundaries of the flame front are given by the contours \( Y = Y_u \) and \( Y = Y_b \), re-
respectively. An orthogonal coordinate system 
$(\xi, \eta)$ is introduced, with axes locally adapted
to the contours of $Y$, i.e., the unit vector 
normal to the contours is given by

$$
e_\eta = \frac{\nabla Y}{|\nabla Y|} = \frac{1}{Y_L} \left( \begin{array}{c} Y_x \\ Y_y \end{array} \right)$$

(2)

where $Y_x = \partial Y/\partial x$, $Y_y = \partial Y/\partial y$, and
$Y_L = \sqrt{Y_x^2 + Y_y^2}$ and the unit vector along the
contours is defined as

$$
e_\xi = \frac{1}{Y_L} \left( \begin{array}{c} Y_y \\ -Y_x \end{array} \right).$$

(3)

Note that the $e_\xi$ unit vector can also be defined
in the opposite direction. In this paper, we
choose $Y_x > 0$. It is something to realize that
diffusive transport of the scalar quantity $Y$ is
always directed in the $e_\eta$ direction, i.e., perpen-
dicular to the local iso-contours of $Y$. How-
ever, the local velocity vector $v$ is generally not
in the $e_\eta$ direction. This means that variations
in the convective transport in the $e_\xi$ direction
might introduce local distortions in the behavior
of $Y$ through the flame, compared to the
behavior in cases where the convective and
diffusive transport directions are parallel (such
as in perfectly flat, cylindrical, or spherical
flames). Following this reasoning, we define
the generalized stretch rate in terms of the
convective transport contributions in the $e_\xi$
direction. To compute these convective transport
terms in the $e_\xi$ direction, the conservation
equation of mass will be analyzed.

Consider the continuity equation $\nabla \cdot (\rho v) = 0$ in the $(\xi, \eta)$-coordinate system of Fig. 1.

$$
\frac{1}{h_\xi h_\eta} \left( \frac{\partial}{\partial \xi} (\rho v_\xi h_\eta) + \frac{\partial}{\partial \eta} (\rho v_\eta h_\xi) \right) = 0,
$$

(4)

where $\rho$ is the mass density and $v = v_\xi e_\xi + v_\eta e_\eta$ is the velocity vector. Furthermore, $h_\xi$ and $h_\eta$ are the scale factors of the $(\xi, \eta)$-coor-
dinate system, defined by $h_\xi = |\partial \tau/\partial \xi|$ and
$h_\eta = |\partial \tau/\partial \eta|$. The scalar stretch field $K(\xi, \eta)$
is now defined as the term in Eq. 4 which gives
rise to transport in the $e_\xi$ direction divided by $\rho$, i.e.,

$$
K = \frac{1}{\rho h_\xi h_\eta} \frac{\partial}{\partial \xi} (\rho v_\xi h_\eta),
$$

(5)

defined in the zone for $\eta$ between $\eta_a$ and $\eta_b$, where $Y$ equals $Y_a$ and $Y_b$, respectively.

Equation 5 for $K$ can be further elaborated.
It is clear that $K$ can be written as

$$
K = \frac{1}{\rho h_\xi} \frac{\partial}{\partial \xi} (\rho v_\xi) + v_\xi \nabla \cdot e_\xi.
$$

(6)

Let $\alpha$ denote the angle between $e_\xi$ and the
positive $x$ axis; then it is easy to see that

$$
\nabla \cdot e_\xi = \frac{d\alpha}{ds_\xi} = C_\xi,
$$

(7)

with $s_\xi$ the arclength along the contour $\xi = \text{Const}$ and $C_\xi$ the curvature of this contour.
The radius of curvature $\hat{R}_\xi$ of the contour
$\xi = \text{Const}$ is defined by

$$
\hat{R}_\xi = |R_\xi| \quad \text{with } R_\xi = \frac{1}{C_\xi}.
$$

(8)

Using 7 and 8, Eq. 6 can be rewritten as

$$
K = \frac{1}{\rho h_\xi} \frac{\partial}{\partial \xi} (\rho v_\xi) + \frac{v_\xi}{R_\xi}.
$$

(9)

This equation for the flame stretch field is an
extension of the expression

$$
K_a = \frac{1}{h_\xi} \frac{\partial v_\xi}{\partial \xi},
$$

(10)

which can be derived from the conventional
definition of stretch in Eq. 1; see, e.g., [5, 6].
The first term in Eq. 9 arises in case of a
nonuniform mass flux along a flame "contour"
$\eta = \text{Const}$. It should be noted that density
variations along a flame "contour" are ne-
glected in the analysis of other authors. The
first term in Eq. 9 then reduces to $K_a$. How-
ever, in real flames, the contours $\eta = \text{Const}$
generally do not coincide with isotherms be-
cause stretch induces local differences in flame
temperature. The second term on the right-
hand side of Eq. 9 can be viewed as a curvature contribution due to local variations in flame front thickness. This term is not present in the analysis of others.

That the variation in flame thickness indeed gives a contribution to the stretch rate as presented in Eq. 9 can be shown by considering Fig. 2. In this figure, we analyze the special case of a uniform flow through a hypothetical flame with flat \( \eta = \text{Const} \) contours and circular \( \xi = \text{Const} \) contours, so that we find a zero stretch rate \( K_s = (1/h_\xi)(\partial v_\xi/\partial \xi) = 0 \) when formula 10 is used. We now estimate the fractional area change \( K = (1/A)(dA/dt) \) of a piece of a flame contour \( A \) with constant \( \eta \), caused by differences in convective and diffusive transport in the flame. In Fig. 2, it is observed that the flame thickness is not constant, so that \( R_\xi \) is finite. In Fig. 2a, we define two transport cells. The convective cell has an initial area \( A_1 \), a final area \( A_2 \), and walls along the local stream tube. The diffusive transport cell has initial area \( A_3 \), final area \( A_2 \), and walls along the contours with constant \( \xi \). Diffusive transport takes place along the contours with constant \( \xi \), so that there is no area change in the diffusive transport cell; thus, \( A_2 - A_3 = 0 \). We may therefore consider the area change of the convective transport cell in Fig. 2 only. As can be seen in the enlargement in Fig. 2b, for small angles \( \epsilon \), we may write \( \epsilon = dh/A = h_\eta d\eta/R_\xi \). Furthermore, it is observed that \( dH/A = v_\eta/v_\xi \). Using these relations gives

\[
\frac{1}{A} \frac{dA}{dt} = \frac{v_\eta}{h_\eta d\eta} \frac{dA}{A} = \frac{v_\xi}{h_\eta d\eta} \frac{dh}{A} = \frac{v_\xi}{R_\xi}.
\]

(11)

This shows that the last term in Eq. 9 is indeed related to stretch effects due to flame thickness variations.

In Fig. 3, we present the general case with nonuniform flow, flame curvature, and flame thickness variation. In this figure, we introduce the convective transport cell \( ABFE \) and the diffusive transport cell \( CDFE \). We will show that the stretch rate \( K(\xi, \eta) \) defined in Eq. 9 can be interpreted as the total fractional area change of a part \( AB \) of an arbitrary flame contour with constant \( \eta \), due to convection in \( ABFE \) and diffusion in \( CDFE \). Note that we speak of the area of \( AB \), while strictly speaking, we mean the length of \( AB \). First consider the convective cell \( ABFE \). The line segments \( AE \) and \( BF \) are streamlines and \( AB \) and \( EF \) are flame contours a distance \( h_\eta d\eta = v_\eta dt \) apart. Due to conservation of mass in \( ABFE \), it is clear that \( (\rho v_\eta)(\eta + d\eta) = (\rho v_\eta)(\eta) \), from which we can conclude that

\[
A(\eta + d\eta) = \left( 1 - \frac{d\eta}{\rho v_\eta} \right) A(\eta) + \Theta(d\eta^2) A(\eta).
\]

(12)

Substitution of \( d\eta = v_\eta dt/h_\eta \) into (12) and letting \( dt \to 0 \) gives the following expression for the fractional area change of \( AB \) due to convection in \( ABFE \):

\[
\left( \frac{1}{A} \frac{dA}{dt} \right)_c = - \frac{1}{\rho h_\eta} \frac{\partial}{\partial \eta} (\rho v_\eta).
\]

(13)

In the diffusive cell \( CDFE \), the parametrisations of the line segments \( CD \) and \( EF \) are
related by \( r(\xi, \eta) = r(\xi, \eta + d\eta) - v_\eta(\xi, \eta + d\eta) dt \). Thus, substitution of \( a = -v_\eta(\xi, \eta + d\eta) dt \) into Eq. 55 of Appendix A gives the following relation between the areas \( A(\eta) \) of CD and \( A(\eta + d\eta) \) of EF:

\[
A(\eta) = \left( 1 - \frac{v_\eta dt}{h_\xi \eta} \frac{\partial h_\xi}{\partial \eta} + \Theta(dt^2) \right) A(\eta + d\eta) \tag{14}
\]

The fractional area change due to diffusion in CDFE can be easily derived from Eq. 14 whenever \( dt \to 0 \):

\[
\left( \frac{1}{A} \frac{dA}{dt} \right)_d = -\frac{v_\eta}{h_\xi \eta} \frac{\partial h_\xi}{\partial \eta}. \tag{15}
\]

Adding the two fractional area changes of Eqs. 13 and 15 then gives, for the total fractional area change,

\[
\frac{1}{A} \frac{dA}{dt} = \frac{1}{\rho h_\xi \eta} \left( \frac{\partial}{\partial \xi} (\rho v_\eta h_\xi) \right) = \frac{1}{\rho h_\xi \eta} \left( \frac{\partial}{\partial \xi} (\rho v_\eta h_\eta) \right) = K(\xi, \eta). \tag{16}
\]

This analysis shows that our definition of stretch indeed includes those terms which give rise to flame area variations for convective and diffusive transport. Equation 9 defines a scalar stretch field not only on an infinitely thin flame sheet, but in the whole region for \( Y(x, y) \) between \( Y_u \) and \( Y_s \), and the separate terms can be computed from the numerical solution of the flow field \( v \) and the scalar fields \( \rho \) and \( Y \). Explicit expressions for the different terms of \( K(\xi, \eta) \) are derived in terms of \( v, \rho, \) and \( Y \) in a 2D Cartesian coordinate system in Appendix B.

A question not yet answered is: Which scalar field in the flame should be used to define the flame front? It is important to realise that the results for \( K \) depend on the choice of \( Y \). In most cases, there is no unique choice possible. The only mathematical restriction is that \( Y \) has to be a monotonous field without local extrema in order to be able to introduce a well-defined \( (\xi, \eta) \)-coordinate system. This eliminates the possibility to use the mass fraction of intermediate species in the flame for \( Y \). Also, it seems to be unwise to use the temperature for \( Y \) because local temperature variations along the flame contours induced by stretch effects may be of interest. Any other choice, i.e., the mass fraction of one of the main species, is possible. In our case, it seems most obvious to use the mass fraction of fuel for \( Y \). The position of the flame contours then depends on the amount of fuel already consumed in the combustion process.

**FLAME STRETCH AND THE CONSERVATION EQUATIONS**

It is well known that stretch has important effects on the local behavior of flames through local variations in scalar quantities, such as the (flame) temperature. In this section, it will be shown that the terms in Eq. 9 which arise from a nonuniform flow and from flame thickness variations are precisely those contributions which induce such variations in the behavior of \( Y \) and in the other scalar fields of the flame. As an example, the effect of stretch on the flame temperature will be studied, and it will be shown that the theory gives identical results as the analysis of Chung et al. [9] when a number of assumptions and approximations are introduced. For that purpose, we will study the conservation equation of the scalar quantity \( Y \) in the coordinate system of Fig. 1.
We start from the mass conservation equation 4 again. The total amount of mass which enters or leaves the diffusive cell (hatched area in Fig. 1) at the boundaries $\eta = \eta_u$ or $\eta = \eta_b$ can be obtained by integration of Eq. 9 over this cell:

$$
\int_{\xi_1}^{\xi_2} \int_{\eta_u}^{\eta_b} \frac{1}{h_\xi h_\eta} \left[ \frac{\partial}{\partial \xi} \left( \rho v_\eta h_\xi \right) + \frac{\partial}{\partial \eta} \left( \rho v_\xi h_\eta \right) \right] \times |J| d\eta d\xi = 0 \tag{17}
$$

where $|J| = h_\xi h_\eta$ denotes the Jacobian of the transformation $(x, y) \rightarrow (\xi, \eta)$. Substitution of Eq. 5 into Eq. 17 gives

$$
\int_{\xi_1}^{\xi_2} \left[ \left( \rho v_\eta h_\xi \right)_b - \left( \rho v_\eta h_\xi \right)_u \right] d\xi = -\int_{\xi_1}^{\xi_2} \rho K(\xi, \eta) h_\xi h_\eta d\eta, \tag{18}
$$

which holds for arbitrary $\xi_1$ and $\xi_2$, and therefore,

$$
\left( \rho v_\eta h_\xi \right)_b - \left( \rho v_\eta h_\xi \right)_u = -\int_{\eta_u}^{\eta_b} \rho K(\xi, \eta) h_\xi h_\eta d\eta. \tag{19}
$$

The coefficients $h_{\xi, u}$ and $h_{\xi, b}$ are measures for the area of the unburned and burned boundaries of the diffusive cell through which the scalar variable $Y$ diffuses. Differences in $h_{\xi, u}$ and $h_{\xi, b}$ are accounted for by the curvature term

$$
\frac{\rho v_\eta}{h_\xi h_\eta} \frac{\partial h_\xi}{\partial \eta} = \rho v_\eta \nabla \cdot e_\eta = \frac{\rho v_\eta}{R_\eta}, \tag{20}
$$

with $R_\eta$ being the radius of curvature of the $\eta = \text{Const}$ contours (see Eq. 4). Note that the mass fluxes at the unburned and burned cell boundaries are equal, i.e., $\left( \rho v_\eta h_\xi \right)_u = \left( \rho v_\eta h_\xi \right)_b$, when the local stretch rate is zero.

Now consider the conservation equation for the scalar quantity $Y$:

$$
\nabla \cdot (\rho vY) - \nabla \cdot (\rho D_Y \nabla Y) = S_Y, \tag{21}
$$

with $D_Y$ and $S_Y$ the diffusion coefficient and the (chemical) source term, generally depending on the other field variables in the flame. In the coordinate system of Fig. 1, we find

$$
\frac{1}{h_\xi h_\eta} \frac{\partial}{\partial \xi} \left( \rho v_\xi h_\xi Y \right) - \frac{1}{h_\xi h_\eta} \frac{\partial}{\partial \eta} \left( \rho D_Y h_\xi \frac{\partial Y}{\partial \eta} \right) = -S_Y = -\frac{1}{h_\xi h_\eta} \frac{\partial}{\partial \xi} \left( \rho v_\xi h_\xi Y \right) = -\rho KY \tag{22}
$$

where we used that the diffusive transport is directed in the $e_\eta$ direction, so that $\partial Y / \partial \xi = 0$. The left-hand side of Eq. 22 is a quasi-“1D” conservation equation in the $e_\eta$ direction. All distortions from 1D behavior, being the transport contributions in the $e_\xi$ direction, are gathered on the right-hand side of Eq. 22. Integrating this equation over the diffusive cell of Fig. 1, analogous to Eq. 17, then gives

$$
\left( \rho v_\eta h_\xi Y \right)_b - \left( \rho v_\eta h_\xi Y \right)_u - \int_{\eta_u}^{\eta_b} S_Y h_\xi h_\eta d\eta = -\int_{\eta_u}^{\eta_b} \rho K(\xi, \eta) Y h_\xi h_\eta d\eta. \tag{23}
$$

To arrive at Eq. 23, we used that the diffusive fluxes vanish in the unburned and burned mixture.

Let us now consider the variation $\Delta Y = Y_b - Y_u$ of $Y$ through the flame, and compare this variation with $\Delta Y^0 = Y_b^0 - Y_u^0$ in the case of stretchless flames. The undistorted value $\Delta Y^0$ follows from Eq. 23:

$$
\left( \rho h_\xi \right)_u s_L \Delta Y^0 - \int_{\eta_u}^{\eta_b} S_Y^0 h_\xi h_\eta d\eta = 0 \tag{24}
$$

where we used that the mass flux $\left( \rho v_\eta h_\xi \right)_b = \left( \rho v_\eta h_\xi \right)_u$ is equal to the adiabatic mass burning rate $\left( \rho h_\xi \right)_u s_L$ for $K = 0$, $s_L$ being the laminar burning velocity. Furthermore, $S_Y^0$ denotes the undistorted chemical source term. In case $K \neq 0$, we find from Eq. 23

$$
\Delta Y = \frac{s_L}{v_{\eta, u}} \Delta Y^0 + \Delta S + \frac{1}{\left( \rho v_\eta h_\xi \right)_u} \int_{\eta_u}^{\eta_b} \rho K \times (\xi, \eta) \left[Y_b - Y(\eta)\right] h_\xi h_\eta d\eta. \tag{25}
$$

In Eq. 25, we used the mass balance equation 19 and Eq. 24. Furthermore, we introduced

$$
\Delta S = \frac{1}{\left( \rho v_\eta h_\xi \right)_u} \int_{\eta_u}^{\eta_b} (S_Y - S_Y^0) h_\xi h_\eta d\eta. \tag{26}
$$
Equation 25 indicates that a nonuniform flow and flame thickness variations may change the local value of $A_Y$ in the flame through the terms in $K(\xi, \eta)$ we considered in the previous section.

We will now derive Eq. 25 in terms of the enthalpy $H$, being the mass weighted average of the species enthalpies $H_i$:

$$H = \sum_{i=1}^{N} H_i Y_i, \quad H_i = H_i^0 + \int_{\tau_0}^{T} c_p, i(\tau) \, d\tau,$$

(27)

with $Y_i$ the mass fractions of the $N$ species, in order to compute the local variations in the flame temperature. We start from the stationary enthalpy conservation equation:

$$\nabla \cdot (\rho v H) = -\nabla \cdot q,$$

(28)

with

$$q = -\lambda \nabla T + \sum_{i=1}^{N} \rho Y_i H_i V_i,$$

(29)

$V_i$ being the diffusion velocity of species $i$. After introducing the constant Lewis numbers $L_{e_i} = (\lambda/\rho D_{im} c_p)$, $D_{im}$ being the diffusion coefficient of species $i$ in the mixture, the equation can be written as [13]

$$\nabla \cdot (\rho v H) - \nabla \cdot \left( \frac{\lambda}{c_p} \nabla H \right) = \sum_{i=1}^{N} (1 - L_{e_i}) \nabla \cdot (\rho D_{im} H_i \nabla Y_i).$$

(30)

When Eq. 5 is used, Eq. 30 evolves to

$$\nabla \cdot (\rho v_{\eta} H e_{\eta}) - \nabla \cdot \left( \frac{\lambda}{c_p} |\nabla H| e_{\eta} \right) = -\rho KH - Q.$$

(31)

The left-hand side is again a "1D" conservation equation in the $e_{\eta}$ direction. Distortions from local 1D behavior are gathered on the right-hand side. Most convective enthalpy fluxes are now combined in the $\rho KH$ term. The convective flux $(\rho v_{\xi}/h_{\xi}) (\partial H/\partial \xi)$ and the diffusive enthalpy fluxes along the flame contours, due to terms arising from the factors $\partial Y_i^2/\partial \xi^2$ and $\partial T^2/\partial \xi^2$ in Eq. 30, are combined in the term $Q$. It must be realized that $\partial Y_i/\partial \xi$, $\partial T/\partial \xi \neq 0$ in general as the iso-contours of the scalar quantities are not parallel. Local variations in, e.g., the temperature along flame contours would not be possible otherwise.

We now continue with integrating Eq. 31 through the flame from $\eta = \eta_u$ to $\eta = \eta_b$:

$$(\rho v_{\eta} h_{\xi} H)_{\eta} - (\rho v_{\eta} h_{\xi} H)_{u} = -\int_{\eta_u}^{\eta_b} (\rho KH + Q) h_{\xi} h_{\eta} \, d\eta$$

(32)

where we used that $|\nabla H| = |\nabla Y_i| = 0$ in the (un)burnt mixture. Substituting Eq. 19 finally gives

$$H_b - H_u = \frac{1}{(\rho v_{\eta} h_{\xi})_{u}} \int_{\eta_u}^{\eta_b} [\rho K (H_b - H) - Q] \times h_{\xi} h_{\eta} \, d\eta.$$

(33)

Equation 33 is our final equation, which can be used in numerical computations to study the effects of flame stretch on local variations in the flame temperature. In the remainder of this section, we will derive an approximation of Eq. 33.

When it is assumed that the composition of the mixture at the "unburned" flame boundary $\eta = \eta_u$ is equal to the composition of the related stretchless flame with which the behavior is compared, we have $H_u = H_u^0 = H_b^0$, so that

$$H_b - H_u = H_b - H_b^0 \approx c_p (T_b - T_b^0).$$

(34)

Note that $c_p$ is taken to be constant. The last step in Eq. 34 follows from the fact that local enthalpy variations mainly introduce temperature differences; changes in the mixture composition have a smaller contribution, especially in lean mixtures. In Eq. 34, $T_b - T_b^0$ is the local deviation from the undistorted adiabatic temperature $T_b^0$.

In the weak flame stretch limit, we may assume that the dimensionless stretch rate $\tilde{K}$
= \frac{K_j \eta}{v} \frac{d\eta}{v} is small enough so that terms of order \( K^2 \) and higher are negligible in Eq. 33. The contribution of the parallel diffusive fluxes of, for instance, the temperature \( \frac{\lambda}{h} \frac{\partial T}{\partial \xi} \) compared to the parallel convective flux \( \rho v c_p T \) are small for weakly stretched flames as \( \partial T / \partial \xi \rightarrow 0 \) when \( K \rightarrow 0 \). That \( \partial T / \partial \xi \) is indeed small also follows from the numerical illustration in the next section, where it is shown that the density variations along the flame (directly coupled with the tangential temperature variations) are very small. From the above, it is reasonable to assume that terms like \( \frac{\partial}{\partial \xi} (\lambda h) \frac{\partial T}{\partial \xi} \) are small compared to \( \frac{\partial}{\partial \eta} (\rho v c_p T) \). This indicates that \( |Q| \) is negligible compared to \( |\rho KH| \). Numerical data of the flame tip, discussed extensively in the next section, have shown that the \( Q \) term gives a contribution which is 2–3 orders of magnitude smaller than the \( \rho KH \) term. We therefore neglect the \( Q \) term in the following.

For weak stretch rates, we may restrict ourselves to the lowest order \( K^0 \) contribution to calculate \( H_b - H \) on the right-hand side of Eq. 33. This means that, when integrating Eq. 31 over \( \eta \) from \( \eta \) to \( \eta_b \), we may neglect the term \( \rho KH + Q \). This gives

\[
(\rho v_{\eta} h_{\xi} H)_b - (\rho v_{\eta} h_{\xi} H) = \frac{\lambda}{c_p} \frac{h_{\xi}}{\eta} \frac{\partial H}{\partial \eta} + O(\tilde{K}).
\]

In this expression, \( H_b \) has been replaced by the dominant part \( H^0 \). The contribution arising from \( \int c_p h dT \) can be neglected in hydrocarbon flames [17]. From mass conservation, we know that

\[
(\rho v_{\eta} h_{\xi})_b = \rho v_{\eta} h_{\xi} + O(\tilde{K}),
\]

so that

\[
(H_b - H) = - \frac{\delta_f}{h} \frac{\partial H}{\partial \eta} - \frac{\delta_f}{c_p} \sum_{i=1}^{N} H_i^0 \frac{h_{\xi}}{\eta} \frac{\partial Y_i}{\partial \eta} + O(\tilde{K}).
\]

And an equivalent relation for \( Y(\eta) \). When these equations are differentiated, we find

\[
\frac{1}{h} \frac{\partial Y_i}{\partial \eta} = \frac{\delta_f}{h} \frac{H_i}{h} + O(\tilde{K})
\]

and

\[
\frac{1}{h} \frac{\partial H}{\partial \eta} = \frac{(H_b - H)}{\delta_f} \exp \left[ \frac{1}{\delta_f} \int_{\eta_b}^{\eta} h d\eta \right].
\]

Equations 34, 37, 39, and 40 are now substituted into Eq. 33. The term arising from \( \partial H / \partial \eta \) is proportional to \( H_b - H \), and is therefore of \( O(\tilde{K}^2) \) in Eq. 33 and is neglected. We thus finally have

\[
T_b - T_b^0 = - \sum_{i=1}^{N} \frac{H_i^0}{c_p} \left( \frac{1}{Le_i} - 1 \right) \times (Y_{i,b} - Y_{i,u}) Ka(Le_i) + O(\tilde{K}^2)
\]

where \( \delta_f = \lambda / \rho v_{\eta} c_p \) is constant up to \( O(\tilde{K}) \). To continue, we search expressions for \( \frac{1}{h} (\partial H / \partial \eta) \) and \( \frac{1}{\eta} (\partial Y_i / \partial \eta) \) of order \( K^0 \). Assuming that the reaction zone is thin and that the behavior of these terms is determined predominantly by the solution in the preheating zone, we may write

\[
Y(\eta) = Y_{i,u} + (Y_{i,b} - Y_{i,u}) \exp \left[ \frac{Le_i}{\delta_f} \int_{\eta_b}^{\eta} h d\eta \right]
\]

and

\[
\frac{1}{h} \frac{\partial Y_i}{\partial \eta} = \frac{Le_i}{\delta_f} \frac{H_i}{h} \exp \left[ \frac{1}{\delta_f} \int_{\eta_b}^{\eta} h d\eta \right].
\]

We thus finally have

\[
T_b - T_b^0 = - \sum_{i=1}^{N} \frac{H_i^0}{c_p} \left( \frac{1}{Le_i} - 1 \right) \times (Y_{i,b} - Y_{i,u}) Ka(Le_i) + O(\tilde{K}^2)
\]

where the Karlovitz numbers \( Ka(Le_i) \) are given by

\[
Ka(Le_i) = \frac{Le_i}{(\rho v_{\eta} h_{\xi})_u} \int_{\eta_b}^{\eta} \rho K(\xi, \eta) \exp \left[ \frac{Le_i}{\delta_f} \int_{\eta_b}^{\eta} h d\eta \right] h \eta d\eta.
\]
This expression indicates that the local stretch rate $pK$ has to be multiplied with an exponential factor in numerical studies, so that the contribution of $pK$ in the cold part of the preheating zone is damped and the predominant contribution to $Ka(Le)$ is found near the reaction zone. This demonstrates that flame stretch throughout the whole preheat zone influences the flame temperature, in agreement with Buckmaster's remarks [5].

In most analytical studies so far, a one-step reaction is used. To show that we arrive at the same result as Chung, we finally consider only one chemical species (e.g., fuel) with one Lewis number. Further, assuming that $H^0_f(Y_{fu,u} - Y_{fu,b})/(c_p T^0_b) = 1$ (as Chung also does), finally gives

$$\frac{T_b}{T_b^0} = 1 + Ka \left( \frac{1}{Le} - 1 \right) + O(Ka^2)$$

for Eq. 41. Note that Eq. 43 can also be found from Eq. 41 in case of multispecies combustion, when $\rho K_{He} = (\rho K_{He})_u$ is constant in the range $\eta_u < \eta < \eta_b$, so that the integral Eq. 42 reduces to the usual definition for the Karlovitz number $Ka(Le) = Ka(1) = K_\delta/\nu_{\eta,u}$. The effective Lewis number $Le$, taking into account the combined effect of all species which have an impact on diffusive energy transport in the flame, is then found by equating Eqs. 41 and 43:

$$\left( \frac{1}{Le} - 1 \right) = \sum_{i=1}^{N} \left( \frac{1}{Le_i} - 1 \right) (Y_{i,u} - Y_{i,b}) \frac{H^0_f}{c_p T^0_b}.$$  \hfill (44)

The result 43 is equal to the result of Chung. The analysis shows that Eq. 25 gives identical results as the analysis of others when the appropriate assumptions are used.

ILLUSTRATION: FLAME STRETCH IN THE TIP OF A BUNSEN FLAME

As an illustration, we will compare the typical order of magnitude of the different contributions to the stretch rate $K$ in this section. The contributions are determined for the tip of a stationary premixed Bunsen flame on a multiple-slit burner. This study gives an indication of the importance of the corresponding terms on the flame behavior and flame stabilization of the tip. The burner slits have a width of 4.0 mm and a burner wall thickness of 0.4 mm. A stoichiometric methane/air mixture enters the computational domain with a parabolic velocity profile and a maximum velocity of 0.8 m/s. The stoichiometric methane/air flame on this burner is computed using the 2D flame code developed by de Lange and Mallens [11, 14]. This code uses a one-step chemistry model; the stream function/vorticity formulation for the flow field and the combustion equations are discretized on an adaptive locally refined grid. This code is validated extensively for 2D atmospheric premixed methane/air flames [11, 14] and is suitable to model the global flame structure, flow field, and flame stabilization [16] accurately. The mass fraction of fuel $Y_{fu}$ is used as the scalar quantity $Y$ to define the stretch rate. For $K$, we may write

$$K(\xi, \eta) = K_a + K_b + K_c$$

where $K_a$ is the usual stretch rate (given by Eq. 10). Furthermore, $K_b$ and $K_c$ are given by

$$K_b = \frac{v_{\xi}}{\rho h_{\xi}} \frac{\partial \rho}{\partial \xi}$$

and

$$K_c = \frac{u_{\xi}}{R_{\xi}}$$

arising from density variations along the iso-contours and flame thickness variations. False-color plots of $K_a$, $K_b$, $K_c$, and $K$ in the flame tip are presented in Fig. 4a, b, c, and d, respectively. A number of iso-contours of $Y_{fu}$ ($\eta = \text{constant}$) and the stream function are also presented in the figures. Note that the maximum values for $|K_a|$, $|K_b|$, $|K_c|$, and $|K|$ are about 17,000, 700, 4000, and 17,000 sec$^{-1}$ respectively. This means that $K_b$ is negligible in the flame tip. However, the contribution of $K_c$ to $K$ is significant.

The behavior of $K$ on the central symmetry axis ($\xi = 0$) is shown in Fig. 5. Note that $K = K_a$ because $K_b = K_c = 0$ on this axis for symmetry reasons. From Eq. 59, it is easy to see that $K = K_a = u_x - vY_{xx}$ for $\xi = 0$. Fur-
Fig. 4. False-color plots of the different stretch field contributions in the flame tip on a 2D multiple-slit burner. Lines: η contours (fuel mass fraction $Y_{fu}$) and contours of the stream function $\phi$. a: $K_a$, b: $K_b$, c: $K_c$, d: $K_d$. 
thermore, for the flame front curvature, we can deduce that \( \frac{1}{R_\eta} = \overline{\nabla}_{xx} \) for \( \xi = 0 \), so that \( K \) can also be written as

\[
K = K_d = u_x - \frac{\nu}{R_\eta}
\]  

for the central axis. The different contributions \( u_x \) and \( \nu \overline{\nabla}_{xx} \) to \( K \) are also presented in Fig. 5. It is clearly seen that, although the flow divergence term \( u_x \) is not negligible, the curvature part has the most significant contribution, which is also concluded by other authors, such as Law [10] and Poinsot [15]. It should, however, be noted that \( R_\eta \) shows large variations through the flame. This indicates that the simple formula \( K = -\frac{\nu}{R_f} \) is valuable to approximate the stretch rate in a flame tip when the correct value for the flame tip radius \( R_f \) is used.

The stretch rate \( K \) near the upstream boundary of the flame tip, where the tip radius \( R_\eta \) becomes smaller than the grid spacing, is not shown in Figs. 4 and 5. The gradients are small there, and the curvature terms are not reliable in this region. It should be noted that this is not a problem because the exponential function in Eq. 42 for the Karlovitz number \( K_a \) dampens the contribution of the upstream part of \( \rho K \). To show this, both \( \rho K \) and the integrand \( \rho K \exp\left[\left(\frac{\nu}{2T_b}\right)\int_{\eta}^{\eta+\delta_\eta} d\eta\right] \) of Eq. 42 for \( K_a(1) \) are presented in Fig. 6. Note that this figure indicates that the exponential damping of the preheating part of the stretch rate is essential.

For the Karlovitz number, we find \( K_a(1) \approx -0.3 \) in the tip with Eq. 42. The computed value for \( K_a \) does not appear to be very sensitive to variations of the Lewis number \( Le \), from 1. The contribution of the curvature part \( \nu \overline{\nabla}_{xx} \) is again dominant.

To quantify the importance of flame stretch on the flame temperature and the tip stabilization in our case, we computed the effective Lewis number \( Le \) from Eq. 44 for the case of our one-step chemical model. For stoichiometric methane/air flames, a Lewis number \( Le = 0.92 \) is found. Combining this with \( K_a \approx 0.3 \) leads to \( T_b/T_b^0 \approx 0.97 \) with Eq. 43. For skeletal chemistry, we find \( Le = 0.98 \) for \( \phi = 1 \), which indicates an even smaller variation in the flame temperature when a more complex chemical model is used. A direct analysis of the local flame temperature data in the numerically computed tip shows that \( T_b - T_b^0 \approx -80 \) K, so that \( T_b/T_b^0 \approx 0.96 \). This result is in good agreement with the value found with Eq. 43, and this analysis shows that temperature variations due to flame stretch are small.

The effect of \( K_a \) on the local mass burning rate, expressed by [9]

\[
\frac{\rho_b s_{L,b}}{\rho_b s_{L,b}^0} = 1 - \frac{K_a}{Le} + \frac{T_a}{2T_b} \frac{Le}{K_a} \left[ \frac{1}{Le} - 1 \right],
\]

with \( T_a \approx 16,000 \) K being the effective activation temperature, is also expected to be small because the computed values for \( K_a \) and \( Le \) approximately give \( \rho_b s_{L,b}/\rho_b s_{L,b}^0 \approx 1.3 \). For an adiabatic burning velocity of \( s_{L,b}^0 \approx 0.40 \) m/s,
we then find \( \rho b s_{L, b} \approx 0.55 \text{ kg/(m}^2\text{ s)} \). The mass flux \( \rho v_n \) on the center line is presented in Fig.7. From this figure, we clearly observe that \( \rho v_n \) approaches the above computed value for the mass burning rate \( \rho b s_{L, b} \) near the reaction zone, although the mass burning rate in the upstream part of the flame is still quite large, \( \rho v_n \approx 1.2 \text{ kg/(m}^2\text{ s)} \). It is therefore concluded that flame stretch has a very small contribution on the stabilization process in the flame tip. It is striking to observe that, although the contribution of the flow straining term \( u_x \) in \( K \) is not large, it is primarily responsible for the tip behavior of this flame. It is the \( u_x \) term which, induced by hydrodynamic effects, is responsible for the steady decrease of \( \rho v_n \) through the tip.

CONCLUSIONS

The flame stretch concept has been extended to the case of 2D stationary flames with a finite flame front thickness. Additional terms due to density variations along the flame and flame thickness variations appear. The generalized formalism is applied to study the local variations in scalar quantities, such as the enthalpy and flame temperature. On this basis, it appears to be possible to derive generalized equations for these variations, starting from the conservation equations. Furthermore, these generalized equations reduce to known expressions when a number of approximations is introduced.

Finally, the order of magnitude of the separate terms is computed for the tip of a Bunsen flame as an illustration. The contribution to the stretch rate arising from density variations along the flame contours appears to be negligible in this case, while the term caused by flame thickness variations has a nonnegligible contribution. It is shown that the Karlovitz number is not large, and that the effective Lewis number is close to 1, so that flame stretch has a small effect on the local temperature and mass burning rate for the particular flame studied. The stabilization of the tip seems to be dominated by hydrodynamic effects, in particular by the flow straining term \( u_x \), which induces a steady decrease of the mass flow rate through the tip. The importance of flame stretch may be different in other flame geometries.

We believe that the proposed method is a valuable tool to quantify the contribution of different effects on the local flame behavior in numerical studies, such as the stabilization of flames on burners. The theory will be applied to the study of the stabilization and blow-off of inverted flames and flames on Bunsen-type burners in the near future.

REFERENCES

APPENDIX A: AREA CHANGE OF FLAME FRONT CONTOURS

Consider two flame front contours \( C_1: Y(x, y) = \eta_1 \) and \( C_2: Y(x, y) = \eta_2 \) \((\eta_1 < \eta_2)\) with parametrisations \( r = r(\xi, \eta_1) \) and \( r = r(\xi, \eta_2) \), respectively. Let \( A(\eta_1) \) and \( A(\eta_2) \) denote the areas (lengths) of infinitesimal segments on \( C_1 \) and \( C_2 \), respectively; then

\[
A(\eta_1) = \left| \frac{\partial r}{\partial \xi}(\xi, \eta_1) \right| d\xi,
\]

\[
A(\eta_2) = \left| \frac{\partial r}{\partial \xi}(\xi, \eta_2) \right| d\xi. \tag{50}
\]

In this Appendix, we study the relation between \( A(\eta_1) \) and \( A(\eta_2) \). Suppose \( r(\xi, \eta_1) = r(\xi, \eta_2) + a(\xi) \), with \( a(\xi) \) an arbitrary displacement vector. Substitution of this relation into the second equation of 50 gives

\[
A(\eta_2) = \left( \left| \frac{\partial r}{\partial \xi}(\xi, \eta_1) \right|^2 + 2 \left( \frac{\partial r}{\partial \xi}(\xi, \eta_1), \frac{da}{d\xi}(\xi) \right) + \left| \frac{da}{d\xi} \right|^2 \right)^{1/2} d\xi. \tag{51}
\]

Introducing the unit vectors \( e_\xi = (1/h_\xi)(\partial r/\partial \xi) \) and \( e_\eta = (1/h_\eta)(\partial r/\partial \eta) \) with \( h_\xi = |\partial r/\partial \xi| \) and \( h_\eta = |\partial r/\partial \eta| \) the corresponding scale factors, Eq. 51 can also be written as

\[
A(\eta_2) = \left( 1 + \frac{2}{h_\xi} \left( e_\xi, \frac{da}{d\xi} \right) + \frac{1}{h_\eta^2} \left| \frac{da}{d\xi} \right|^2 \right)^{1/2} \times A(\eta_1). \tag{52}
\]

where \( e_\xi \) and \( h_\xi \) have to be evaluated at \( \eta = \eta_1 \). Anticipating that \( |a(\xi)| \) and \( |(da/d\xi)| \) are small, we only elaborate the inner product \( (e_\xi, (da/d\xi)) \). Let \( a = a_\xi e_\xi + a_\eta e_\eta \) in the \((\xi, \eta)\)-coordinate system; then it is easy to verify that

\[
\left( e_\xi, \frac{da}{d\xi} \right) = \frac{da_\xi}{d\xi} + a_\eta \left( e_\xi, \frac{de_\eta}{d\xi} \right). \tag{53}
\]

Furthermore, from the identity \( h_\xi (\partial h_\xi/\partial \eta) = (1/2)(\partial/\partial \eta)((\partial r/\partial \xi), (\partial r/\partial \xi)) \), one can easily derive the following expression for the last term in 53:

\[
\left( e_\xi, \frac{de_\eta}{d\xi} \right) = \frac{1}{h_\eta} \frac{\partial h_\xi}{\partial \eta}. \tag{54}
\]

Combining 52–54 then finally results in

\[
A(\eta_2) = \left( 1 + \frac{2}{h_\xi} \left( \frac{da_\xi}{d\xi} + a_\eta \frac{\partial h_\xi}{\partial \eta} \right) + \frac{1}{h_\eta^2} \left| \frac{da}{d\xi} \right|^2 \right)^{1/2} A(\eta_1). \tag{55}
\]

This latter expression is used in the second section to derive the formula for the stretch rate \( K(\xi, \eta) \).

APPENDIX B: EVALUATION OF STRETCH RATE CONTRIBUTIONS IN FLAME COMPUTATIONS

Explicit expressions for the different contributions to the generalized stretch rate \( K \) and for the conventional stretch rate \( K_a \) will be presented in this Appendix. For \( K \), we may write

\[
K = K_a + K_b + K_c \tag{56}
\]

where \( K_a, K_b, \) and \( K_c \) are given by Eqs. 10 and 46–47.
To be able to compare the different contributions to $K$ in the case of a flame, computed numerically in the Cartesian $(x, y)$-coordinate system, we express the different terms of Eq. 56 in the $v$ and $Y$ variables computed in this coordinate system. We already saw that the local unit vectors $e_n$ and $e_\xi$ are given by Eqs. 2 and 3. For the flow vector $v$, we have $v = u e_x + v e_y = v_\xi e_\xi + v_n e_n$ so that

$$v_\xi = e_\xi \cdot v = \bar{Y}_x u - \bar{Y}_y v$$
$$v_n = e_n \cdot v = \bar{Y}_x u + \bar{Y}_y v$$  \hspace{1cm} (57)$$

where $\bar{Y}_x = Y_x/Y_L$ and $\bar{Y}_y = Y_y/Y_L$. For the vector $\mathbf{v}$, we may write $\mathbf{v} = e_\xi (\partial \mathbf{v} / \partial \xi) + e_n (\partial \mathbf{v} / \partial n) (\partial / \partial n)$, leading to

$$\frac{1}{h_\xi} \frac{\partial}{\partial \xi} = e_\xi \cdot \mathbf{v} = \bar{Y}_y \frac{\partial}{\partial x} - \bar{Y}_x \frac{\partial}{\partial y}$$
$$\frac{1}{h_n} \frac{\partial}{\partial n} = e_n \cdot \mathbf{v} = \bar{Y}_x \frac{\partial}{\partial x} + \bar{Y}_y \frac{\partial}{\partial y}.$$  \hspace{1cm} (58)$$

With the use of these relations, we find for $K_a$

$$K_a = \frac{1}{h_\xi} \frac{\partial v_\xi}{\partial \xi}$$
$$= \left[ \bar{Y}_y \frac{\partial (\bar{Y}_x u - \bar{Y}_y v)}{\partial x} - \bar{Y}_x \frac{\partial (\bar{Y}_x u - \bar{Y}_y v)}{\partial y} \right]$$
$$= \left[ \bar{Y}_x u + \bar{Y}_y v \right]$$
$$\times \left[ -\bar{Y}_y^2 \bar{Y}_{xx} + 2 \bar{Y}_x \bar{Y}_y \bar{Y}_{xy} - \bar{Y}_x^2 \bar{Y}_{yy} \right]$$
$$+ \left[ u_x \bar{Y}_y^2 - v_x \bar{Y}_x \bar{Y}_y - u_y \bar{Y}_x \bar{Y}_y + v_y \bar{Y}_x^2 \right]$$  \hspace{1cm} (59)$$

where, e.g., $\bar{Y}_{xx} = Y_{xx}/Y_L$. For $K_b$, we obtain

$$K_b = \frac{v_\xi}{\rho h_\xi} \frac{\partial \rho}{\partial \xi} = \left[ \bar{Y}_y u - \bar{Y}_x v \right] \left[ \bar{Y}_y \frac{\partial \rho}{\partial x} - \bar{Y}_x \frac{\partial \rho}{\partial y} \right]$$  \hspace{1cm} (60)$$

and for $K_c$,

$$K_c = v_\xi e_n \cdot \frac{1}{h_n} \frac{\partial e_\xi}{\partial n}$$
$$= \left[ \bar{Y}_y u - \bar{Y}_x v \right] \frac{1}{h_n} \left[ \bar{Y}_x \frac{\partial \bar{Y}_y}{\partial n} - \bar{Y}_y \frac{\partial \bar{Y}_x}{\partial n} \right]$$
$$= \left[ \bar{Y}_y u - \bar{Y}_x v \right] \left[ \bar{Y}_x \bar{Y}_y (\bar{Y}_{yy} - \bar{Y}_{xx}) \right] + \left[ \bar{Y}_x \bar{Y}_y (\bar{Y}_{xx} - \bar{Y}_{yy}) \right].$$  \hspace{1cm} (61)$$

These expressions can be evaluated in the computational orthogonal $(x, y)$-coordinate system. It should be stressed, however, that it is preferable to evaluate the derivatives in the $x$ and $y$ directions identically as they are calculated in the flame computation. We use a conservative finite volume method in combination with an exponential fitting scheme to evaluate the fluxes [12] in the numerical flame computations. Therefore, we also apply the exponential fitting scheme to determine the derivatives.