An approximation for the effect of working overtime

van der Wal, J.

Published: 01/01/1996

Citation for published version (APA):
Memorandum COSOR 96-38

An approximation for the effect of working overtime

J. van der Wal

Eindhoven, December 1996
The Netherlands
AN APPROXIMATION FOR THE EFFECT OF WORKING OVERTIME

Jan van der Wal, October 14, 1996

ABSTRACT

This paper presents a two-moment approximation for the amount of work at the beginning of each period in a production or service system with overtime by exploiting the similarity with the D|G|1 queue. The approach is based on the one presented in De Kok [1] for the normal G|G|1 queue. The results from the approximation are compared with simulation results. This indicates that for practical purposes, such as discussion about increasing the basic capacity or working overtime, the approximation is perfect.

1. INTRODUCTION

Consider a production system with a capacity of C units of work per period. In the sequel we will think of a period as a day. At the beginning of every day just before the start of production a stochastic amount of work arrives (or is released) with distribution G. Let us denote the amount of work arriving at the beginning of day \( n \) by \( X_n \). Let further \( W_n \) denote the remaining amount of work at the end of day \( n \). Then we have

\[
W_{n+1} = \max(W_n + X_{n+1} - C, 0),
\]

(1)

which we recognize as Lindley's relation for the waiting time in a D|G|1 queue.

Based on De Kok [1] who gives an excellent two-moment approximation for the waiting time in a G|G|1 queue we will derive a very similar approximation for a system where depending on the amount of work in the system it is decided to work overtime or not.

We will consider a simple overtime rule that can be characterized by two parameters, \( D \) and \( \alpha \), as follows:

Overtime rule

If \( W_n + X_{n+1} \geq D \) then increase the capacity on day \( n+1 \) from \( C \) to \( \alpha C \).

We will assume \( \alpha > 1 \) (although the approach can be used for \( \alpha < 1 \) as well) and \( D > \alpha C \). Think for instance of \( C = 1 \) (the capacity of an 8 hour working day), \( D = 3 \) (more than 3 days of work at the beginning of the day) and \( \alpha = 1.25 \) (2 hours of overtime).
Then (1) changes to

\[ W_{n+1} = \begin{cases} 
0 & \text{if } W_n + X_{n+1} \leq C \\
W_n + X_{n+1} - C & \text{if } C < W_n + X_{n+1} < D \\
W_n + X_{n+1} - \alpha C & \text{if } W_n + X_{n+1} \geq D 
\end{cases} \]  

(2)

The two-moment approximation gives the amount of work at the beginning and end of a day and the number of days on which one has to work overtime. The algorithm is given in Section 2. Section 3 discusses some fairly standard integrals used in the computations. The results are compared with simulation in Section 4. Section 5 shows how the approximation can be used for discussions about how often and when to work overtime. Section 6 shows how the approximations can be used to compare the effects of overtime and additional capacity.

2. THE APPROXIMATION

Following De Kok the approximation is based on the first two moments of \( X_n \) only. It is very unlikely that in practice the distribution of \( X \) is known. At the very best one has good approximations for the first two moments, further denoted by \( b^{(1)} \) and \( b^{(2)} \).

In the approach we have to derive an approximation for \( W_{n+1} \) based on the first two moments of \( W_n + X_{n+1} \). Therefore we have to fit distributions to random variables with known mean and variance. If the coefficient of variation is less than 1 then \( W_n + X_{n+1} \) is approximated by a mixture of Erlang distributions with the same scale parameter \( \mu \) and \( k \) or \( k+1 \) phases. If the coefficient of variation is larger than 1 then we use a hyperexponential distribution with balanced means. (cf. Tijms [2], pp. 398, 399).

The Erlang distribution with \( k \) phases and scale parameter \( \mu \) will be denoted by \( F_{\mu,k} \).

Fitting

Consider a r.v. \( Y \) with mean \( b \), variance \( \sigma^2 \) and coefficient of variation \( c = \sigma / b \).

i) If \( c \leq 1 \) then the distribution \( F_Y \) is approximated by \( pF_{\mu,k} + (1-p)F_{\mu,k+1} \), with \( k \) such that \( 1/(k+1) < c^2 \leq 1/k \) and

\[
p = \frac{1}{1+c^2} \left[ (k+1)c^2 - ((k+1)(1+c^2) - (k+1)^2c^2) \right]^{1/2},
\]

and

\[
\mu = (k+1-p)/b.
\]

ii) If \( c > 1 \) then \( F_Y \) is approximated by \( p_1 F_{\mu_1,1} + p_2 F_{\mu_2,1} \), with

\[
p_1 = \frac{1}{2} \left[ 1 + \left( \frac{c^2-1}{c^2+1} \right)^{1/2} \right], \quad p_2 = 1 - p_1.
\]
and

\[ \mu_1 = 2p_1 / b , \quad \mu_2 = 2p_2 / b . \]

We will consider two iterative schemes:

**Primitive iteration step**

Let \( w_n^{(1)} \) and \( w_n^{(2)} \) be the approximations for the first two moments of \( W_n \). Then compute the approximations \( y_n^{(1)} \) and \( y_n^{(2)} \) for the first two moments of \( W_n + X_{n+1} \) according to

\[
y_n^{(1)} = w_n^{(1)} + b^{(1)} \quad \text{and} \quad y_n^{(2)} = w_n^{(2)} + b^{(2)} + 2w_n^{(1)}b^{(1)}. \tag{3}
\]

Based on the approximations for \( y_n^{(i)} \), fit a mixture of Erlang distributions or a hyperexponential distribution. Then use (2) to obtain the first two moments of \( W_{n+1} \).

A more sophisticated and more accurate approximation distinguishes between \( W_n > 0 \) and \( W_n = 0 \). Therefore we introduce an approximation \( \beta_n \) for \( P[ W_n > 0 ] \) and we define

\[ u_n^{(i)} = w_n^{(i)} / \beta_n . \]

Then we get the

**Modified iteration step**

With probability \( \beta_n \)

\[ y_{n+1}^{(1)} = u_n^{(1)} + b^{(1)} \quad \text{and} \quad y_{n+1}^{(2)} = u_n^{(2)} + b^{(2)} + 2u_n^{(1)}b^{(1)} \]

and with probability \( 1 - \beta_n \)

\[ y_{n+1}^{(1)} = b^{(1)} \quad \text{and} \quad y_{n+1}^{(2)} = b^{(2)}. \]

3. **SOME INTEGRALS**

The iterative procedure involves the computation of the first two moments of the amount of work left at the end of the period for the overtime rule specified in (2).

Let us assume that the amount of work at the beginning of the period after the new arrivals, denoted by \( Y \), is \( F_{\mu,k} \) distributed. And let \( \gamma(Y) \) denote the probability that we have to work overtime, \( T(Y) \) the amount of work left at the end of the period and \( \beta(Y) \) the probability that \( T(Y) \) is positive.

Then we have

\[
\beta(Y) := P[ Y > C ] = \int_C^\infty dF_{\mu,k}(x) , \tag{4}
\]

\[
\gamma(Y) := P[ Y > D ] = \int_D^\infty dF_{\mu,k}(x) . \tag{5}
\]
\[ ET(Y) = \int_C^{\infty} (x - C) \, dF_{\mu,k}(x) + \int_D^{\infty} (x - \alpha C) \, dF_{\mu,k}(x) \]
\[ = \int_C^{\infty} (x - C) \, dF_{\mu,k}(x) - \int_D^{\infty} (\alpha - 1) C \, dF_{\mu,k}(x) , \]

and

\[ ET^2(Y) = \int_C^{\infty} (x - C)^2 \, dF_{\mu,k}(x) + \int_D^{\infty} (x - \alpha C)^2 \, dF_{\mu,k}(x) \]
\[ = \int_C^{\infty} (x - C)^2 \, dF_{\mu,k}(x) + \int_D^{\infty} \left[ (\alpha^2 - 1) C^2 - 2(\alpha - 1) Cx \right] dF_{\mu,k}(x) . \]

It is useful to introduce \( B(\mu,k,l,A) \) as a compact notation for these integrals:

\[ B(\mu,k,l,A) = \int_A^{\infty} x^l \, dF_{\mu,k}(x) . \] (8)

Then

\[ \beta(Y) = B(\mu,k,0,C) , \] (9)

and

\[ \gamma(Y) = B(\mu,k,0,D) , \] (10)

The \( B(\mu,k,l,A) \) can be obtained as finite sums from

\[ B(\mu,k,l,A) = \int_A^{\infty} x^l \, \mu x^{k-1} e^{-\mu x} \, dx \] (11)
\[ = \frac{(l+k-1)!}{\mu^{l+1}(k-1)!} \int_A^{\infty} \mu x^{l+k-1} \, e^{-\mu x} \, dx \]
\[ = \frac{(l+k-1)!}{\mu^{l+1}(k-1)!} \sum_{m=0}^{l+k-1} \frac{\mu^m}{m!} A^m e^{-\mu A} . \]

From (6) and (7) we get

**Lemma**

The first two moments of \( T(Y) \) can be expressed as follows

\[ ET(Y) = B(\mu,k,1,C) - \mu B(\mu,k,0,C) - (\alpha - 1) \mu B(\mu,k,0,D) , \]
\[ ET^2(Y) = B(\mu,k,1,C) - \mu B(\mu,k,0,C) - (\alpha - 1) \mu B(\mu,k,0,D) , \] (12)

and
\[
ET^2(Y) = B(\mu, k, 2, C) - 2C B(\mu, k, 1, C) + C^2 B(\mu, k, 0, C) \\
- 2(\alpha - 1)C B(\mu, k, 1, D) + (\alpha^2 - 1) C^2 B(\mu, k, 0, D).
\] (13)

(In general \(Y\) will be a mixture of Erlang or exponential distributions, in which case each moment of \(T(Y)\) requires two calculations of the type (12) or (13) instead of one.)

4. RESULTS

In this section we compare the results of the standard and modified approximations with simulation. Further we will discuss the value of these results in relation to the relevant timescale, a year say (250 periods).

We have tested the approximations on a set of 96 examples. In all examples the basic capacity \(C\) is equal to 1. The average amount of work arriving in a period \(b_1\) varies, and takes the values 0.8, 0.9, 0.95. So without overtime \(b_1\) is just the utilization of the server. The coefficient of variation of the work arriving in a period varies: \(c_2 = 0.25, 0.5, 1, 2\). The overwork capacity \(\alpha C\) is either 1.125 or 1.25, corresponding to 1 or 2 hours overtime on a normal 8 hour working day. Overtime starts at the threshold \(D\) which varies, taking the values 2, 3, 5 and 10.

We use the indices S, M and P to refer to Simulation, Modified and Primitive iteration, respectively.

Accurate simulation results require very long runs. We have run the simulations for 1 million periods. (If a period is a day, this means 4000 years!) Then the standard deviation in the mean service time becomes less than 1 percent, and still nearly 1 percent if \(\rho = 0.95\) and \(D = 10\).

The distribution of \(X_n\) was assumed to be Erlang, exponential or hyperexponential, depending on \(c^2\). We have run the approximations until the difference between certain computed values in two successive iterations was sufficiently small, meaning less then \(10^{-7}\). For the primitive version we used the relative difference in first and second moments of \(W_n\), for the modified version \(\beta_n\).

The following 8 tables present the two most important performance results:
- The mean amount of work at the end of the period, denoted by \(W_S\), \(W_M\) and \(W_P\).
- The overtime capacity, equal to \(P [W+B > D ](\alpha - 1)C\), denoted by \(C_S\), \(C_M\) and \(C_P\). Note that multiplication with 4, for \(\alpha = 1.25\), or 8, for \(\alpha = 1.125\), gives the fraction of the periods with overtime.
\[ c^2 = 0.25, \quad \alpha = 1.125 \]

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( D )</th>
<th>( W_S )</th>
<th>( W_M )</th>
<th>( W_P )</th>
<th>( C_S )</th>
<th>( C_M )</th>
<th>( C_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>2</td>
<td>0.20</td>
<td>0.20</td>
<td>0.20</td>
<td>0.007</td>
<td>0.007</td>
<td>0.006</td>
</tr>
<tr>
<td>0.80</td>
<td>3</td>
<td>0.24</td>
<td>0.24</td>
<td>0.24</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>0.80</td>
<td>5</td>
<td>0.26</td>
<td>0.25</td>
<td>0.25</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.80</td>
<td>10</td>
<td>0.26</td>
<td>0.25</td>
<td>0.25</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.90</td>
<td>2</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
<td>0.021</td>
<td>0.022</td>
<td>0.021</td>
</tr>
<tr>
<td>0.90</td>
<td>3</td>
<td>0.61</td>
<td>0.61</td>
<td>0.60</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
</tr>
<tr>
<td>0.90</td>
<td>5</td>
<td>0.77</td>
<td>0.77</td>
<td>0.76</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>0.90</td>
<td>10</td>
<td>0.84</td>
<td>0.83</td>
<td>0.81</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.95</td>
<td>2</td>
<td>0.68</td>
<td>0.67</td>
<td>0.67</td>
<td>0.035</td>
<td>0.036</td>
<td>0.036</td>
</tr>
<tr>
<td>0.95</td>
<td>3</td>
<td>0.95</td>
<td>0.94</td>
<td>0.94</td>
<td>0.019</td>
<td>0.019</td>
<td>0.018</td>
</tr>
<tr>
<td>0.95</td>
<td>5</td>
<td>1.40</td>
<td>1.39</td>
<td>1.36</td>
<td>0.007</td>
<td>0.006</td>
<td>0.006</td>
</tr>
<tr>
<td>0.95</td>
<td>10</td>
<td>1.92</td>
<td>1.93</td>
<td>1.90</td>
<td>0.001</td>
<td>0.000</td>
<td>0.001</td>
</tr>
</tbody>
</table>

\[ c^2 = 0.5, \quad \alpha = 1.125 \]

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( D )</th>
<th>( W_S )</th>
<th>( W_M )</th>
<th>( W_P )</th>
<th>( C_S )</th>
<th>( C_M )</th>
<th>( C_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>2</td>
<td>0.41</td>
<td>0.41</td>
<td>0.41</td>
<td>0.020</td>
<td>0.020</td>
<td>0.020</td>
</tr>
<tr>
<td>0.80</td>
<td>3</td>
<td>0.50</td>
<td>0.49</td>
<td>0.49</td>
<td>0.007</td>
<td>0.008</td>
<td>0.007</td>
</tr>
<tr>
<td>0.80</td>
<td>5</td>
<td>0.58</td>
<td>0.58</td>
<td>0.57</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>0.80</td>
<td>10</td>
<td>0.61</td>
<td>0.60</td>
<td>0.59</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>0.90</td>
<td>2</td>
<td>0.83</td>
<td>0.82</td>
<td>0.82</td>
<td>0.040</td>
<td>0.041</td>
<td>0.041</td>
</tr>
<tr>
<td>0.90</td>
<td>3</td>
<td>1.02</td>
<td>1.00</td>
<td>1.00</td>
<td>0.023</td>
<td>0.023</td>
<td>0.023</td>
</tr>
<tr>
<td>0.90</td>
<td>5</td>
<td>1.34</td>
<td>1.33</td>
<td>1.31</td>
<td>0.009</td>
<td>0.008</td>
<td>0.008</td>
</tr>
<tr>
<td>0.90</td>
<td>10</td>
<td>1.71</td>
<td>1.71</td>
<td>1.68</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>0.95</td>
<td>2</td>
<td>1.23</td>
<td>1.20</td>
<td>1.20</td>
<td>0.054</td>
<td>0.056</td>
<td>0.056</td>
</tr>
<tr>
<td>0.95</td>
<td>3</td>
<td>1.49</td>
<td>1.46</td>
<td>1.46</td>
<td>0.037</td>
<td>0.038</td>
<td>0.036</td>
</tr>
<tr>
<td>0.95</td>
<td>5</td>
<td>2.03</td>
<td>2.00</td>
<td>1.97</td>
<td>0.019</td>
<td>0.019</td>
<td>0.019</td>
</tr>
<tr>
<td>0.95</td>
<td>10</td>
<td>3.10</td>
<td>3.04</td>
<td>2.95</td>
<td>0.006</td>
<td>0.005</td>
<td>0.006</td>
</tr>
</tbody>
</table>

\[ c^2 = 1, \quad \alpha = 1.125 \]

<table>
<thead>
<tr>
<th>( \rho )</th>
<th>( D )</th>
<th>( W_S )</th>
<th>( W_M )</th>
<th>( W_P )</th>
<th>( C_S )</th>
<th>( C_M )</th>
<th>( C_P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.80</td>
<td>2</td>
<td>0.83</td>
<td>0.83</td>
<td>0.83</td>
<td>0.037</td>
<td>0.038</td>
<td>0.038</td>
</tr>
<tr>
<td>0.80</td>
<td>3</td>
<td>0.94</td>
<td>0.94</td>
<td>0.94</td>
<td>0.022</td>
<td>0.022</td>
<td>0.022</td>
</tr>
<tr>
<td>0.80</td>
<td>5</td>
<td>1.12</td>
<td>1.11</td>
<td>1.11</td>
<td>0.008</td>
<td>0.008</td>
<td>0.008</td>
</tr>
<tr>
<td>0.80</td>
<td>10</td>
<td>1.31</td>
<td>1.31</td>
<td>1.30</td>
<td>0.001</td>
<td>0.001</td>
<td>0.001</td>
</tr>
<tr>
<td>0.90</td>
<td>2</td>
<td>1.63</td>
<td>1.62</td>
<td>1.62</td>
<td>0.058</td>
<td>0.060</td>
<td>0.059</td>
</tr>
<tr>
<td>0.90</td>
<td>3</td>
<td>1.79</td>
<td>1.78</td>
<td>1.79</td>
<td>0.042</td>
<td>0.043</td>
<td>0.043</td>
</tr>
<tr>
<td>0.90</td>
<td>5</td>
<td>2.17</td>
<td>2.15</td>
<td>2.14</td>
<td>0.024</td>
<td>0.024</td>
<td>0.024</td>
</tr>
<tr>
<td>0.90</td>
<td>10</td>
<td>2.95</td>
<td>2.91</td>
<td>2.84</td>
<td>0.007</td>
<td>0.007</td>
<td>0.007</td>
</tr>
<tr>
<td>0.95</td>
<td>2</td>
<td>2.39</td>
<td>2.38</td>
<td>2.39</td>
<td>0.070</td>
<td>0.073</td>
<td>0.072</td>
</tr>
<tr>
<td>0.95</td>
<td>3</td>
<td>2.60</td>
<td>2.57</td>
<td>2.58</td>
<td>0.056</td>
<td>0.058</td>
<td>0.057</td>
</tr>
<tr>
<td>0.95</td>
<td>5</td>
<td>3.10</td>
<td>3.07</td>
<td>3.07</td>
<td>0.038</td>
<td>0.038</td>
<td>0.037</td>
</tr>
<tr>
<td>0.95</td>
<td>10</td>
<td>4.45</td>
<td>4.35</td>
<td>4.26</td>
<td>0.017</td>
<td>0.017</td>
<td>0.017</td>
</tr>
</tbody>
</table>
The tables show that the results from the two iterative methods are excellent. The Modified method is slightly better than the Primitive one. Both the mean amount of work and the overtime capacity are quite close to the simulation results.

5. VARIATIONS FROM YEAR TO YEAR

For discussions about using overtime or increasing the capacity the approximations for the means by the iterative methods are perfect. However, it is important to note that the amount of work in the system and the overtime capacity used vary enormously.

Table 2 gives the simulation results for 10 years. (Each simulation run is preceded by 100 days not used in the measurements.)

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Days</td>
<td>113</td>
<td>21</td>
<td>50</td>
<td>23</td>
<td>35</td>
<td>41</td>
<td>54</td>
<td>59</td>
<td>54</td>
<td>33</td>
</tr>
</tbody>
</table>

The differences between the years are striking. With this in mind it is clear that the quality of the approximations for the means are perfect.

Remark 1. Although not presented in the tables, the simulations show that also the standard deviation in the amount of work at the end of the period is approximated quite good by the iterative methods.

Remark 2. An important problem is the choice between working overtime, how long and when, and an increase of the basic capacity. The results of the approximations are
sufficiently accurate to serve as part of the quantitative input in the discussion.

6. OVERTIME VERSUS NORMAL CAPACITY

An interesting question is the efficiency of overtime compared to an increase of the basic capacity.

As an example we start with a basic capacity of $1$ and a workload of $\rho = 0.95$. Overtime capacity is assumed to be 25 percent more expensive than basic capacity. We compare the effect of an average overtime capacity of $0.04$ with an increase of the basic capacity of $0.05$ (so that the alternatives lead to the same salary costs).

For 1 and 2 hours of overtime we use the modified approximation to determine (by bisection for instance) the value for the threshold $D$ for which the average overtime capacity becomes $0.04$.

Table 3 (which was computed in seconds) gives the results for the three cases for the squared coefficients of variation $0.5, 1$ and $2$.

<table>
<thead>
<tr>
<th>$c^2$</th>
<th>$\alpha$</th>
<th>$C$</th>
<th>$W_M$</th>
<th>$D$</th>
<th>$C_M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1</td>
<td>1.05</td>
<td>2.00</td>
<td>----</td>
<td>0</td>
</tr>
<tr>
<td>0.5</td>
<td>1.125</td>
<td>1</td>
<td>1.42</td>
<td>2.84</td>
<td>0.04</td>
</tr>
<tr>
<td>0.5</td>
<td>1.25</td>
<td>1</td>
<td>1.21</td>
<td>3.53</td>
<td>0.04</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1.05</td>
<td>4.20</td>
<td>----</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1.125</td>
<td>1</td>
<td>3.01</td>
<td>4.77</td>
<td>0.04</td>
</tr>
<tr>
<td>1</td>
<td>1.25</td>
<td>1</td>
<td>2.58</td>
<td>6.14</td>
<td>0.04</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.05</td>
<td>8.69</td>
<td>----</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>1.125</td>
<td>1</td>
<td>6.30</td>
<td>8.86</td>
<td>0.04</td>
</tr>
<tr>
<td>2</td>
<td>1.25</td>
<td>1</td>
<td>5.33</td>
<td>11.40</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 3: Overtime versus normal capacity, basic utilization $\rho = 0.95$

6. CONCLUSIONS

We have presented a simple, accurate and very fast two-moment approximation for the effect of working overtime in a production system with work arriving once a day which was described as a D[G]1 queueing system.
References
