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On $L_p$ spectral independence

by

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On $L_p$ spectral independence
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Abstract

If $\Omega$ is an open subset of $\mathbb{R}^d$ then we prove the $p$-independence of the spectrum for a consistent family of bounded operators on $L_p(\Omega)$ if they have a kernel $K$ satisfying $|K(x,y)| \leq \Gamma(x-y)$ with $\Gamma \in L_1(\mathbb{R}^d)$. Extensions of this theorem are given for abstract measure spaces and for Lie groups with polynomial growth. As a result, many elliptic operators on a Lie group with polynomial growth and on a Riemannian manifold with uniform subexponentially volume growth have a $p$-independent spectrum.

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1 Introduction

If for all \( p \in [1, \infty] \) there is a ‘natural’ family of operators \( H_p \) in \( L_p(\mathcal{X}) \) we consider the problem whether the spectrum \( \sigma(H_p) \) of \( H_p \) is independent of \( p \). In this paper we present several sufficient conditions.

The \( p \)-independence of the connected component of the resolvent which contains a left half-plane has been proved by Arendt [Are] for operators on \( L_p(\Omega) \) which are generators of consistent \( C_0 \)-semigroups having a kernel satisfying Gaussian upper bounds. Here \( \Omega \) is an open subset of \( \mathbb{R}^d \). The connectedness restriction in [Are] has been removed by Kunstmann [Kun1] and Liskevich-Vogt [LiV]. The Gaussian bounds condition got weakened to Poisson bounds together with a technical commutator estimate by Hieber-Schröhe [HiS] and recently Kunstmann [Kun2] again weakened the assumption to a kernel estimate on the kernel \( K \) of a common consistent resolvent operator of the form \( |K(x; y)| \leq \Gamma(x - y) \), with \( \Gamma \) a function in \( L_1(\mathbb{R}^d) \cap L_{p_0}(\mathbb{R}^d) \) together with some fast decay on \( \Gamma \). One of the results of this paper is that the condition \( \Gamma \in L_1(\mathbb{R}^d) \) suffices. This answers a problem posed in [Kun2].

The measure spaces are allowed to be more general than open subsets in \( \mathbb{R}^d \), in particular they can be subsets of groups with polynomial growth or Riemannian manifolds satisfying a volume condition. As an example we extend results of Sturm [Stu] on certain Riemannian manifold to non-symmetric operators.

It is impossible to give a full account on previous results on \( L_p \)-spectral independence of consistent operators. Other positive conditions, with exception of the previous mentioned ones, are in [Sim], [HeV], [ScV], [Dav1], [Dav2], [Sem], [LeS], [Hull], [Hu12], [GiM] and references cited therein.

If \( \mathcal{X} \) and \( \mathcal{Y} \) are Banach spaces which are both continuously embedded in one Hausdorff topological vector space and \( \mathcal{X} \cap \mathcal{Y} \) is weakly or weakly\(^*\) dense in \( \mathcal{X} \) and \( \mathcal{Y} \) then two operators \( E \in \mathcal{L}(\mathcal{X}) \) and \( F \in \mathcal{L}(\mathcal{Y}) \) are called consistent if \( E(\mathcal{X} \cap \mathcal{Y}) \subset \mathcal{Y}, F(\mathcal{X} \cap \mathcal{Y}) \subset \mathcal{X} \) and \( E \phi = F \phi \) for all \( \phi \in \mathcal{X} \cap \mathcal{Y} \).

Let \( (X, \mu) \) be a \( \sigma \)-finite measure space. If \( T^{(p)} \) is a continuous operator on \( L_p \) for all \( p \in [1, \infty] \) then \( T^{(p)}, 1 \leq p \leq \infty, \) is called a consistent family of continuous operators on \( L_p \) if \( T^{(p)} \) is consistent with \( T^{(q)} \) for all \( p, q \in [1, \infty] \).

An integrable kernel is a measurable function \( K : X \times X \to \mathbb{C} \) such that

\[
\sup_{x \in X} \int_X dy |K(x; y)| < \infty \quad \text{and} \quad \sup_{y \in X} \int_X dx |K(x; y)| < \infty.
\]

If \( p \in [1, \infty] \) then we say that an operator \( T \in \mathcal{L}(L_p) \) has a kernel \( K \) if \( K : X \times X \to \mathbb{C} \) is a measurable function and

\[
(T \phi)(x) = \int_Y dy K(x; y) \phi(y) \quad (\text{a.e. } x \in X)
\]

for all \( \phi \in L_p \).

With the aid of an integrable kernel one can define a consistent family of continuous operators.
Lemma 1.1 If $K : X \times X \to \mathbb{C}$ is an integrable kernel then there exists a unique family $T(p)$, $1 \leq p \leq \infty$, of consistent continuous operators on $L_p$ such that $T(p)$ has $K$ as kernel for all $p \in [1, \infty]$. Moreover,

$$\|T(p)\|_{p \to r} \leq \max\left( \sup_{x \in X} \int_X dy |K(x; y)|, \sup_{y \in X} \int_X dx |K(x; y)| \right)$$

(1)

for all $p \in [1, \infty]$.

Proof Since $K$ is an integrable kernel one can define for all $p \in \{1, \infty\}$ the continuous operator $T(p)$ on $L_p$ by

$$(T(p)\varphi)(x) = \int_Y dy K(x; y) \varphi(y) \quad \text{(a.e. } x \in X).$$

Then (1) is valid for $p \in \{1, \infty\}$ and the lemma follows by interpolation. \(\square\)

If $E$ is a (possibly unbounded) operator in a Banach space $X$ then the resolvent $\rho(E)$ consists of all $\lambda \in \mathbb{C}$ such that $\lambda I - E$ has a bounded inverse. If $\lambda \in \rho(E)$ then we set $R(\lambda, E) = (\lambda I - E)^{-1}$. The spectrum $\sigma(E)$ of $E$ is the complement of $\rho(E)$.

The first main theorem of this paper is the following.

Theorem 1.2 Let $X$ be a non-empty set with quasi-distance $d$ and Borel measure $\mu$ such that $X$ is $\sigma$-compact, $\mu(\{x\}) = 0$, $\mu(B(x; \rho)) > 0$ for all $x \in X$ and $\rho > 0$, and

$$\sup_{x \in X} \int_X dy \mu(y) e^{-\varepsilon d(x;y)} < \infty$$

(2)

for all $\varepsilon > 0$, where $B(x; \rho) = \{y \in X : d(x; y) < \rho\}$.

Let $T(p)$, $1 \leq p \leq \infty$, be a consistent family of operators on $L_p(X; \mu)$ with a kernel $K$ satisfying

$$c_r = \sup_{x \in X} \int_X dy \mu(y) |K(x; y)|^r < \infty$$

(3)

and

$$\tilde{c}_r = \sup_{y \in X} \int_X dx \mu(x) |K(x; y)|^r < \infty$$

(4)

for all $r \in \{1, p_0\}$, where $p_0 > 1$ is fixed. Moreover, suppose that

$$\lim_{\rho \to \infty} \sup_{x \in X} \int_{B(x, \rho)} dy \mu(y) |K(x; y)| = 0$$

(5)

and

$$\lim_{\rho \to \infty} \sup_{y \in X} \int_{B(y, \rho)} dx \mu(x) |K(x; y)| = 0$$

(6)

Then $\sigma(T(p)) = \sigma(T(2))$ for all $p \in [1, \infty]$.

The theorem has an immediate consequence for semigroup generators.
Corollary 1.3 Suppose that $X$, $d$ an $\mu$ are as in Theorem 1.2. For all $p \in [1, \infty]$ let $H_p$ be a (possibly unbounded) operator in $L_p(X; d\mu)$. Let $\lambda \in \mathbb{C}$ and assume that $\lambda \in \rho(H_p)$ for all $p \in [1, \infty]$ and that the family $R(\lambda, H_p)$, $1 \leq p \leq \infty$, is consistent with a kernel $K$ satisfying (3), (4), (5) and (6). Then $\sigma(H_p) = \sigma(H_2)$ for all $p \in [1, \infty]$.

If $X$ is an open subset of $\mathbb{R}^d$ with the induced metric and Lebesgue measure then the conditions can be simplified considerably and the following is a special case.

Theorem 1.4 Let $X$ be a non-empty open subset of $\mathbb{R}^d$ with the usual induced metric and Lebesgue measure. Let $T^{(p)}$, $1 \leq p \leq \infty$, be a consistent family of operators on $L_p(X)$ with an (integrable) kernel $K$. Suppose there exists a $\Gamma \in L_1(\mathbb{R}^d)$ such that

$$|K(x; y)| \leq \Gamma(x - y)$$

for all $x, y \in X$. Then $\sigma(T^{(p)}) = \sigma(T^{(2)})$ for all $p \in [1, \infty]$.

In [Kun2] Example 4.8 there is an example of a consistent family of operators on $L_p(\Omega)$, where $\Omega = (0, \infty)$, with a kernel $K$ which satisfies (3) and (4) only for $r = 1$, but the spectrum is $p$-dependent. Therefore the assumptions in Theorem 1.2 are rather weak.

In Section 2 we prove Theorem 1.2 by the aid of a refinement of ideas in [Kun2] and in Section 3 we prove Theorem 1.4 in the wider setting of Lie groups with polynomial growth. Finally, in Section 4 we present various examples on Lie groups with polynomial growth and on Riemannian manifolds with Ricci curvature bounded below and uniformly subexponential volume growth.

2 Abstract spaces

In this section we prove Theorem 1.2. The proof consists of a perturbation of the original kernel and operators, which we carry out in several steps.

For all $\delta \geq 0$ define $K_\delta: X \times X \to \mathbb{C}$ by

$$K_\delta(x; y) = e^{-\delta d(x; y)} K(x; y)$$

and let $T^{(p)}_\delta$, $1 \leq p \leq \infty$, be the consistent family of continuous operators on $L_p$ associated to the kernel $K_\delta$ as in Lemma 1.1. Then $\|T^{(p)}_\delta - T^{(p)}\|_{p \to p} \leq \max(c_1, \tilde{c}_1)$.

Lemma 2.1 If $p \in [1, \infty]$ then $\lim_{\delta \to 0} \|T^{(p)}_\delta - T^{(p)}\|_{p \to p} = 0$.

Proof Let $\varepsilon > 0$. There exists a $\rho > 0$ such that

$$\sup_{x \in X} \int_{B(x; \rho)^c} dy |K(x; y)| < \varepsilon \quad \text{and} \quad \sup_{y \in X} \int_{B(y; \rho)^c} dx |K(x; y)| < \varepsilon$$
by (5) and (6). Then
\begin{align*}
\int_X dy (1 - e^{-\delta d(x;y)}) |K(x;y)| &\leq \int_{B(x;\rho)} dy (1 - e^{-\delta d(x;y)}) |K(x;y)| + \int_{B(x;\rho)} dy |K(x;y)| \\
&\leq \int_{B(x;\rho)} dy (1 - e^{-\delta \rho}) |K(x;y)| + \varepsilon \\
&\leq (1 - e^{-\delta \rho}) c_1 + \varepsilon
\end{align*}
uniformly for all \( x \in X \) by (3). Similarly,
\begin{align*}
\int_X dx (1 - e^{-\delta d(x;y)}) |K(x;y)| &\leq (1 - e^{-\delta \rho}) \max(c_1, \tilde{c}_1) + \varepsilon
\end{align*}
Hence \( \|T^{(p)}_\delta - T^{(p)}\|_{p \to p} \leq (1 - e^{-\delta \rho}) \max(c_1, \tilde{c}_1) + \varepsilon \) for all \( p \in [1, \infty) \) by Lemma 1.1, and
the lemma follows. \( \square \)

Let \( c_0 \geq 1 \) be such that \( d(x;z) \leq c_0 (d(x;y) + d(y;z)) \) for all \( x,y,z \in X \) and define \( \Phi = \{ x \mapsto d(x;y) \land N : y \in X, N \in \mathbb{N} \} \). Then \( \Phi \subset L_\infty(X) \). Moreover, \( |\psi(x) - \psi(y)| \leq c_0 d(x;y) \) for all \( x,y \in X \) and \( \psi \in \Phi \). For \( \psi \in \Phi, \delta \geq 0 \) and \( \varepsilon \geq 0 \) define \( K_{\delta,\varepsilon,\psi} : X \times X \to \mathbb{C} \) by
\begin{align*}
K_{\delta,\varepsilon,\psi}(x;y) &= e^{\varepsilon \psi(x)} K_\delta(x;y) e^{-\varepsilon \psi(y)}
\end{align*}
Let \( T^{(p)}_{\delta,\varepsilon,\psi}, 1 \leq p \leq \infty \), be the consistent family of continuous operators on \( L_p \) associated to the kernel \( K_{\delta,\varepsilon,\psi} \) as in Lemma 1.1.

Lemma 2.2 If \( \delta > 0 \) then
\begin{align*}
\limsup_{\varepsilon \to 0} \|T^{(p)}_\delta - T^{(p)}_{\delta,\varepsilon,\psi}\|_{p \to p} = 0
\end{align*}
for all \( p \in [1, \infty) \).

\textbf{Proof} Fix \( \delta > 0 \). Then the kernel of the operator \( T^{(p)}_\delta - T^{(p)}_{\delta,\varepsilon,\psi} \) equals
\begin{align*}
(x,y) \mapsto K_\delta(x;y) - K_{\delta,\varepsilon,\psi}(x;y) &= e^{-\delta d(x;y)} (1 - e^{(\psi(x) - \psi(y))}) K(x;y)
\end{align*}
Since
\begin{align*}
|1 - e^{(\psi(x) - \psi(y))}| &\leq \varepsilon |\psi(x) - \psi(y)| e^{\varepsilon |\psi(x) - \psi(y)|} \\
&\leq c_0 \varepsilon d(x;y) e^{c_0 \varepsilon d(x;y)} \leq 2c_0 \delta^{-1} \varepsilon e^{2^{-1} \delta d(x;y)} e^{c_0 \varepsilon d(x;y)}
\end{align*}
it follows that
\begin{align*}
|K_\delta(x;y) - K_{\delta,\varepsilon,\psi}(x;y)| &\leq 2c_0 \delta^{-1} \varepsilon |K(x;y)|
\end{align*}
for all \( x,y \in X \) if \( 2c_0 \varepsilon < \delta \). So \( \|T^{(p)}_\delta - T^{(p)}_{\delta,\varepsilon,\psi}\|_{p \to p} \leq 2c_0 \delta^{-1} \varepsilon \max(c_1, \tilde{c}_1) \) uniformly for all \( \varepsilon \in [0, (2c_0)^{-1} \delta) \), \( \psi \in \Phi \) and \( p \in [1, \infty] \) and the lemma follows. \( \square \)
Lemma 2.3 If \( n \geq (p_0 - 1)^{-1} + 1, \delta \geq 0, \varepsilon \in [0, c_0^{-1} \delta], \psi \in \Phi \) and \( p \in [1, \infty] \) then \( (T^{(p)}_{\delta, \varepsilon, \psi})^n \) maps \( L_q \) continuously into \( L_s \) for all \( 1 \leq q \leq s \leq \infty \) and \( \| (T^{(p)}_{\delta, \varepsilon, \psi})^n \|_{p \rightarrow p} \leq (\max(c_1, c_1 p_0, c_1^2 p_0))^{n} \).

Proof It follows from (3) and (4) that \( T^{(p)}_{\delta, \varepsilon, \psi} \) maps \( L_q \) continuously into \( L_s \) for all \( 1 \leq q \leq s \leq \infty \) and \( \| T^{(p)}_{\delta, \varepsilon, \psi} \|_{p \rightarrow p} \leq (\max(c_1, c_1 p_0, c_1^2 p_0))^{n} \).

Moreover, \( T^{(p)}_{\delta, \varepsilon, \psi} \) maps \( L_q \) continuously into \( L_q \) for all \( q \in [1, \infty] \), with norm bounded by \( \max(c_1, c_1) \). Then the lemma is an easy consequence of the Riesz–Thorin interpolation theorem.

Before we start with the general proof of the \( p \)-independence of the spectrum we first consider whether or not 0 is in the spectrum.

Lemma 2.4 If \( p \in [1, \infty) \) then \( 0 \in \sigma(T^{(p)}) \) for all \( p \in [1, \infty] \).

Proof By duality we only need to consider the case \( p < \infty \). Suppose \( p \in [1, \infty) \) and \( 0 \in \rho(T^{(p)}) \). If \( n \in \mathbb{N} \) is such that \( n \geq (p_0 - 1)^{-1} + 1 \) then it follows from Lemma 2.3, with \( \delta = \varepsilon = 0 \), that \( (T^{(p)})^n \) maps \( L_p \) continuously into \( L_{\infty} \). But since \( 0 \in \rho((T^{(p)}))^n \) it then follows that \( L_p \subset L_{\infty} \) and the embedding is continuous. Therefore \( L_1 \subset L_{\infty} \) and the embedding is continuous by duality and interpolation. So there is an \( M > 0 \) such that \( \| \varphi \|_{\infty} \leq M \| \varphi \|_1 \) for all \( \varphi \in L_1 \).

Let \( x \in X \). Then \( \mu(B(x; \rho)) > 0 \) for all \( \rho > 0 \) and \( \lim_{\rho \rightarrow 0} \mu(B(x; \rho)) = \mu(\{x\}) = 0 \). So if \( \rho > 0 \) is such that \( \mu(B(x; \rho)) < M^{-1} \) and one takes \( \varphi = (\mu(B(x; \rho)))^{-1} B(x; \rho) \) then \( \varphi \in L_1, \| \varphi \|_1 = 1 \) but \( \| \varphi \|_{\infty} > M \). Hence \( 0 \notin \rho(T^{(p)}) \). \( \square \)

Now we are able to prove the first theorem of the introduction.

Proof of Theorem 1.2 Let \( p \in [1, \infty] \) and \( \lambda \in \rho(T^{(p)}) \). Then \( \lambda \neq 0 \) by Lemma 2.4. Since inversion is continuous and \( \lambda I - T^{(p)} \) is invertible, it follows from Lemma 2.1 that there is a \( \delta_0 > 0 \) such that \( \lambda \in \rho(T^{(p)}_{\delta}) \) and \( \| R(\lambda, T^{(p)}_{\delta}) \|_{p \rightarrow p} \leq 2 \| R(\lambda, T^{(p)}) \|_{p \rightarrow p} \) for all \( \delta \in [0, \delta_0) \). The main difficulty of the proof is to show that \( \lambda \in \rho(T^{(p)}_{\delta}) \) for all \( q \in [1, \infty] \) and \( \delta \in (0, \delta_0) \).

Let \( \delta \in (0, \delta_0) \). Again since inversion is continuous it follows now from Lemma 2.2 that there is an \( \varepsilon_0 > 0 \) with \( \varepsilon_0 < c_0^{-1} \delta \), such that \( \lambda \in \rho(T^{(p)}_{\delta, \varepsilon}) \) and \( \| R(\lambda, T^{(p)}_{\delta, \varepsilon}) \|_{p \rightarrow p} \leq 4 \| R(\lambda, T^{(p)}) \|_{p \rightarrow p} \) for all \( \psi \in \Phi \) and \( \varepsilon \in [0, \varepsilon_0] \).

Since \( \lambda \neq 0 \) one has the resolvent identity

\[
R(\lambda, T^{(p)}_{\delta, \varepsilon, \psi}) = \lambda^{-2n} (T^{(p)}_{\delta, \varepsilon, \psi})^n R(\lambda, T^{(p)}_{\delta, \varepsilon, \psi}) (T^{(p)}_{\delta, \varepsilon, \psi})^n + \sum_{k=0}^{2n-1} \lambda^{-k-1} (T^{(p)}_{\delta, \varepsilon, \psi})^k
\]

where \( n \) is the smallest natural number such that \( n \geq (p_0 - 1)^{-1} + 1 \). Consider the first term. It follows from Lemma 2.3 that

\[
R_{p, \delta, \varepsilon, \psi} = (T^{(p)}_{\delta, \varepsilon, \psi})^n R(\lambda, T^{(p)}_{\delta, \varepsilon, \psi}) (T^{(p)}_{\delta, \varepsilon, \psi})^n
\]

5
maps $L_1$ continuously into $L_\infty$ and the norm is bounded by

$$M = 4\|R(\lambda, T^{(p)})\|_{p\to p}(\max(c_1, \hat{c}_1, \epsilon_{r_0}^{1/p_0}, \hat{\epsilon}_{r_0}^{1/p_0}))^{2n}$$

for all $\psi \in \Phi$ and $\epsilon \in [0, \epsilon_0]$. Hence the Dunford–Pettis theorem (see, for example [ArB] Theorem 1.3) implies that the operator $R_{p, \delta, \epsilon, \psi}$ has a bounded kernel $K_{\delta, \epsilon, \psi}$ such that $\|K_{\delta, \epsilon, \psi}\|_\infty \leq M$. If $K_\delta = K_{\delta, 0, \psi}$ and $R_{p, \delta} = R_{p, \delta, 0, \psi}$, which are clearly independent of $\psi$, then

$$R_{p, \delta, \epsilon, \psi} = U_{\epsilon, \psi} R_{p, \delta} U_{\epsilon, \psi}^{-1},$$

where $U_{\epsilon, \psi}$ is the bounded invertible multiplication operator with the function $e^{\epsilon \psi}$ on the $L_q$-spaces. Here we use that $\psi$ is bounded. Hence

$$K_{\delta, \epsilon, \psi}(x; y) = e^{\epsilon \psi(x)} K_\delta(x; y) e^{-\epsilon \psi(y)} \quad \text{(a.e. } (x, y) \in X \times X)$$

for all $\psi \in \Phi$ and $\epsilon \in [0, \epsilon_0]$. So

$$|K_\delta(x; y)| \leq M e^{-\epsilon_0 (\psi(x) - \psi(y))} \quad \text{(a.e. } (x, y) \in X \times X)$$

for all $\psi \in \Phi$ and

$$|K_\delta(x; y)| \leq M \inf_{\psi \in \Phi} e^{-\epsilon_0 (\psi(x) - \psi(y))} \leq M e^{-\epsilon_0 d(x; y)}$$

for a.e. $(x, y) \in X \times X$ by [AIE] Lemma 3.4. Then Lemma 1.1 and Condition (2) implies that $R_{p, \delta}$ is consistent with a bounded operator on $L_q$ for any $q \in [1, \infty]$. Obviously, the second term in (7) is also consistent with a bounded operator on $L_q$ for any $q \in [1, \infty]$, in particular if $\epsilon = 0$. Hence, if $q \in [1, \infty]$ then $R(\lambda, T^{(p)}_\delta)$ is consistent with a bounded operator $E$ on $L_q$. Then $(\lambda I - T^{(q)}_\delta)E \varphi = E(\lambda I - T^{(q)}_\delta)\varphi$ for all $\varphi \in L_p \cap L_q$ and, by density, for all $\varphi \in L_q$. Therefore $\lambda \in \rho(T^{(q)}_\delta)$ for all $q \in [1, \infty]$.

Finally let $q \in [1, \infty]$. Then we just proved that $\lambda \in \rho(T^{(q)}_\delta)$ for all $\delta \in (0, \delta_0)$. Therefore

$$1 = \|R(\lambda, T^{(q)}_\delta) (\lambda I - T^{(q)}_\delta)\|_{q \to q} \leq \|R(\lambda, T^{(q)}_\delta)\|_{q \to q} (|\lambda| + \max(c_1, \hat{c}_1))$$

and

$$\|R(\lambda, T^{(q)}_\delta)\|_{q \to q}^{-1} \leq |\lambda| + \max(c_1, \hat{c}_1)$$

uniformly for all $\delta \in (0, \delta_0)$. But $\lim_{\delta \to 0} \|T^{(q)}_\delta - T^{(q)}\|_{q \to q} = 0$ and inversion is continuous. So if $\delta \in (0, \delta_0)$ is such that $\|T^{(q)}_\delta - T^{(q)}\|_{q \to q} < |\lambda| + \max(c_1, \hat{c}_1)$ then one can consider $T^{(q)}_\delta$ as a perturbation of $T^{(q)}$ and it follows that $\lambda \in \rho(T^{(q)}_\delta)$. This completes the proof of Theorem 1.2. \qed

The proof of Theorem 1.2 allows to add weights to the operator and measure, under a small addition decay assumption on the kernel. Theorems of this type for generators of semigroups with Gaussian kernel bounds are given in [Dav1], Section 5 and [Kun1], Theorem 1.2.
Theorem 2.5 Let $X$, $d$ and $\mu$ be as in Theorem 1.2 and $T^{(p)}$, $1 \leq p \leq \infty$, a consistent family of continuous operators on $L_p(X; \mu)$ with kernel $K$. Suppose that there is an $\eta > 0$ such that $(x,y) \mapsto e^{\eta d(x,y)}K(x,y)$ instead of $K$ satisfies Conditions (3) and (4). Let $w: X \to (0, \infty)$ be a locally bounded measurable function such that for all $\varepsilon > 0$ there is a $C_\varepsilon > 0$ such that $w(x)w(y)^{-1} \leq C_\varepsilon e^{\varepsilon d(x,y)}$ for all $x, y \in X$. Then for all $p \in [0, \infty]$ there exists an operator $T^{(p)}_w \in \mathcal{L}(L_p(X; w\mu))$ which is consistent with $T^{(p)}$. Moreover,

$$\sigma(T^{(p)}_w) = \sigma(\hat{T}^{(p)}_w) = \sigma(T^{(2)})$$

for all $p \in [1, \infty]$, where $\hat{T}^{(p)}_w$, $1 \leq p \leq \infty$, denotes the consistent family of continuous operators on $L_p(X; \mu)$ with kernel $\hat{K}$ given by

$$\hat{K}(x,y) = w(x)K(x,y)w(y)^{-1}.$$ 

Proof If $\varphi \in L_1(X; \mu) \cap L_1(X; w\mu)$ then

$$\int d\mu(x)w(x)\int d\mu(y)\left|\varphi(x)\right| \leq \int d\mu(x)\int d\mu(y)\left|K(x,y)\varphi(y)\right| \leq C_\eta \sup_{y \in X} \int d\mu(x)e^{\eta d(x,y)}\left|K(x,y)\right|.$$ (8)

So $T^{(1)}$ maps $L_1(X; \mu) \cap L_1(X; w\mu)$ continuously into $L_1(X; \mu)$. Obviously, $T^{(\infty)}$ maps $L_\infty(X; \mu) = L_\infty(X; w\mu)$ continuously into $L_\infty(X; w\mu)$. So the first part of the theorem follows by interpolation.

Since $T^{(p)}_w = T^{(\infty)}$ obviously $\sigma(T^{(p)}_w) = \sigma(T^{(\infty)}) = \sigma(T^{(1)})$. The kernel $\hat{K}$ satisfies the Conditions (3), (4), (5) and (6). Therefore $\sigma(\hat{T}^{(p)}_w) = \sigma(\hat{T}^{(1)})$ for all $p \in [1, \infty]$ by Theorem 1.2.

Next, for $p \in [1, \infty]$, define the multiplication operator $W: L_p(X; w\mu) \to L_p(X; \mu)$ by $W\varphi = w\varphi$. Then $\hat{T}^{(p)}_w\varphi = W T^{(p)}_w W^{-1}\varphi$ for all $\varphi \in L_p(X; \mu) \cap L_p(X; w\mu)$ and therefore, by density, for all $\varphi \in L_p(X; \mu)$. Since $W$ is isometric, it follows that $\sigma(T^{(p)}_w) = \sigma(\hat{T}^{(p)}_w) = \sigma(\hat{T}^{(1)})$ for all $p \in [1, \infty]$.

Now take $p = 1$ and suppose that $\lambda \in \rho(T^{(1)})$. Then $\lambda \neq 0$. Since $(x,y) \mapsto e^{\eta d(x,y)}K(x,y)$ is an integrable kernel, one can argue as in the proof of Lemma 2.2, but without the need to make a detour for small positive $\delta$, and then argue as in the proof of Theorem 1.2, to deduce that the operator $\lambda^{-2n}(T^{(1)})^n R(\lambda) (T^{(1)})^n$ has a kernel $K_0$. Moreover, there are $\varepsilon_0 > 0$ and $M > 0$ such that $|K_0(x,y)| \leq M e^{-\varepsilon_0 d(x,y)}$ for all $x, y \in X$. Let $K'(x,y) = w(x)K_0(x,y)w(y)^{-1}$. Then $|K'(x,y)| \leq C_{\varepsilon_0/2}M e^{-\varepsilon_0 d(x,y)}$ for all $x, y \in X$ and $K'$ is an integrable kernel. Let $S \in \mathcal{L}(L_1(X; \mu))$ be the operator with kernel $K'$. Consider the operator $E = W^{-1}SW + \sum_{k=0}^{2n-1} \lambda^{-k-1}(T^{(1)})^k \in \mathcal{L}(L_1(X; w\mu))$. It is straightforward to verify that $E\varphi = R(\lambda) T^{(1)}\varphi$ for all $\varphi \in L_1(X; \mu) \cap L_1(X; w\mu)$. Hence $\lambda \in \rho(T^{(1)})$ and $\rho(T^{(1)}) \subseteq \rho(T^{(1)})$. Similarly, $\rho(T^{(1)}) = \rho(T^{(1)}) \subseteq \rho(T^{(1)})$. Therefore $\sigma(T^{(1)}) = \sigma(T^{(1)})$. It follows that $\sigma(T^{(p)}_w) = \sigma(T^{(1)})$ for all $p \in [1, \infty]$. Since $w^{1/p}$ satisfies the same properties as $w$, one has $\sigma(T^{(p)}_w) = \sigma(T^{(1)})$ for all $p \in [1, \infty]$ and the proof of the theorem is complete. \qed
This theorem will be useful to handle the volume factor in the kernels on Riemannian manifolds (see Example 4.2).

3 Lie groups with polynomial growth

Let $G$ be a Lie group with (left-) Haar measure $dg$ and let $d$ be a connected left-, or right-, invariant distance on $G$. If $|B(g;\rho)|$ denotes the measure of the ball $B(g;\rho) = \{ h \in G : d(g;h) < \rho \}$ for all $g \in G$ and $\rho > 0$ then $G$ is called to have polynomial growth if there are $c \geq 1$ and $N \in \mathbb{N}$ such that $|B(e;\rho)| \leq c\rho^N$ for all $\rho \geq 1$, where $e$ is the identity element of $G$. It follows from [VSC] Proposition III.4.2 that this definition is indeed independent of the distance $d$. The Haar measure on a group with polynomial growth is also right-invariant. Hence Condition (2) is satisfied if $X = G$ has polynomial growth.

Theorem 1.4 is a special case of the following theorem.

Theorem 3.1 Let $X$ be a non-empty open subset of a Lie group $G$ with polynomial growth and $T^{(p)}$, $1 \leq p \leq \infty$, be a consistent family of operators on $L_p(X)$ with an (integrable) kernel $K$. Suppose there exists a $\Gamma \in L_1(G)$ such that

$$|K(g;h)| \leq \Gamma(gh^{-1})$$

for all $g,h \in X$. Then $\sigma(T^{(p)}) = \sigma(T^{(2)})$ for all $p \in [1,\infty]$.

Proof For $N \in \mathbb{N}$ define $\Omega_N = \{ g \in G : |\Gamma(g)| < N \}$ and $K_N : X \times X \to \mathbb{C}$ by $K_N(g;h) = K(g;h)1_{\Omega_N}(gh^{-1})$. Then $K_N$ is an integrable kernel and $K \in L_\infty(X \times X)$. Let $T^{(p)}_N$, $1 \leq p \leq \infty$, be the consistent family of continuous operators on $L_p$ associated to the kernel $K_N$ as in Lemma 1.1. We may assume that the distance is right-invariant. Then obviously $K_N$ satisfies the Conditions (3), (4), (5) and (6). So $\sigma(T^{(p)}_N) = \sigma(T^{(2)}_N)$ for all $p \in [1,\infty]$ by Theorem 1.2.

Next,

$$\sup_{g \in X} \int_X \! \! dh \left| K_N(g;h) - K(g;h) \right| \leq \sup_{g \in G} \int_G \! \! dh \, \Gamma(gh^{-1}) \, 1_{\Omega_N}(gh^{-1}) = \int_G \! \! dh \, \Gamma(h) \, 1_{\Omega_N}(h)$$

and similarly

$$\sup_{h \in X} \int_X \! \! dg \left| K_N(g;h) - K(g;h) \right| \leq \int_G \! \! dh \, \Gamma(h) \, 1_{\Omega_N}(h)$$

for all $N \in \mathbb{N}$. Therefore $\|T^{(p)}_N - T^{(p)}\|_{p \to p} \leq \int_G \! \! dh \, \Gamma(h) \, 1_{\Omega_N}(h)$ by Lemma 1.1 and $\lim_{N \to \infty} \|T^{(p)}_N - T^{(p)}\|_{p \to p} = 0$ for all $p \in [1,\infty]$.

Now let $p,q \in [1,\infty]$ and $\lambda \in \rho(T^{(p)})$. Then $\lambda \in \rho(T^{(p)}_N)$ for large $N$ and hence $\lambda \in \rho(T^{(q)}_N)$ for large $N$. Since $\|T^{(q)}_N\|_{q \to q} \leq \|\Gamma\|_1$ for all $N \in \mathbb{N}$ it follows by the same arguments as at the end of the proof of Theorem 1.2 that $\lambda \in \rho(T^{(q)})$. 

\[\square\]
4 Examples

In this section we discuss several examples.

Example 4.1 Let $G$ be a Lie group with polynomial growth. Let $d'$ be the right-invariant distance associated with an algebraic basis $a_1, \ldots, a_{d'}$ for the Lie algebra $\mathfrak{g}$ of $G$ and $D'$ the local dimension. Let $H_2$ be an operator in $L_2(G)$ and assume that $H_2$ generates a semigroup $S^{(2)}$ on $L_2(G)$ which has a kernel $K$ satisfying $m$-th order Gaussian bounds, i.e., there are $b, c, \omega > 0$ such that

$$|K_1(g; h)| \leq c t^{-D'/m} e^{\omega t} e^{-b((|gh^{-1}|)^m t^{-1})^{1/(m-1)}}$$

uniformly for all $g, h \in G$ and $t > 0$. Here $m > 1$ and $|g'| = d'(g; e)$, where $e$ is the identity element of $G$. Then for all $t > 0$ and $p \in [1, \infty]$ the operator $S^{(p)}_t$ is consistent to a continuous operator $S^{(p)}_t$ on $L_p(G)$ and the $(S^{(p)}_t)_{t>0}$ form a continuous semigroup on $L_p(G)$ (see [DuR], Proposition 2.3). Let $H_p$ denote the generator of $S^{(p)}$. Then $-(\omega + 1) \in \rho(H_p)$ for all $p \in [1, \infty]$ and it follows from the appendix of [ElR1] that the resolvent operator $R((\omega + 1), H_p)$ has a kernel $K$. Moreover, there exist $b', c' > 0$ such that $|K(g; h)| \leq \Gamma(g h^{-1})$, where

$$\Gamma(g) = \begin{cases} 
  c' (|g'|)^{-D'} e^{-b'|g'|} & \text{if } D' > m, \\
  c' (1 + |\log |g'||) e^{-b'|g'|} & \text{if } D' = m, \\
  c' e^{-b'|g'|} & \text{if } D' < m.
\end{cases}$$

In any case, $\Gamma \in L_1(G)$. Therefore it follows from Theorem 3.1 that $\sigma(R((\omega + 1), H_p))$ is independent of $p$ and hence $\sigma(H_p) = \sigma(H_2)$ for all $p \in [1, \infty]$.

Gaussian bounds (9) have been proved for a large class of elliptic operators. For all $i \in \{1, \ldots, d'\}$ let $A_i$ denote the infinitesimal generator of the one parameter group $t \mapsto L(\exp(-ta_i))$ where $L$ is the left regular representation of $G$ in $L_2(G)$ and $\exp: \mathfrak{g} \to G$ is the exponential map. If

$$H_2 = -\sum_{i,j=1}^{d'} A_i c_{ij} A_j + \sum_{i=1}^{d'} b_i A_i + \sum_{i=1}^{d'} A_i b'_i + c_0 I$$

is an operator in divergence form with (complex) $c_{ij}, b_i, b'_i, c_0 \in L_\infty(G)$ and there is a $\mu > 0$ such that $\Re \sum_{i,j=1}^{d'} \xi_i c_{ij}(g) \xi_j \geq \mu |\xi|^2$ for all $\xi \in \mathbb{C}^{d'}$, then the semigroup generated by $H_2$ has a kernel satisfying the bounds (9) with $m = 2$ if

- the $c_{ij}$ are real (see [ElR4]), or,
- the $c_{ij}$ are right uniformly continuous (see [ElR3]).

If $m \in \mathbb{N}$ is even and

$$H_2 = \sum_{|\alpha| \leq m} c_{\alpha} A^\alpha$$


is an \( m \)-th order operator in the \( A_1, \ldots, A_d \) with domain \( D(H_2) = \bigcap_{|\alpha| \leq m} D(A^\alpha) \) and with \( c_\alpha \in L_\infty(G) \), the leading coefficients with \(|\alpha| = m \) are \( m \) times differentiable in the directions \( a_1, \ldots, a_d \) in \( L_\infty \)-sense and the leading coefficients satisfy uniformly a subcoercivity condition, then the Gaussian bounds (9) are valid (see [EIR1] Theorem 2.1).

If \( a_1, \ldots, a_d \) is a vector space basis for \( g, n \in \mathbb{N} \) and \( H_2 \) is the divergence form operator

\[
H_2 = \sum_{|\alpha|, |\beta| \leq n} (A^\alpha)^* c_{\alpha, \beta} A^\beta,
\]

where \( c_{\alpha, \beta} \in L_\infty(G) \) and

\[
\text{Re} \sum_{|\alpha| = |\beta| = n} (\psi_\alpha, c_{\alpha, \beta} \psi_\beta) \geq \mu \sum_{|\alpha| = n} ||\psi_\alpha||^2
\]

for some \( \mu > 0 \), uniformly for all \( \psi_\alpha \in L_2(G) \), then the kernel of the semigroup generated by \( H_2 \) on \( L_2(G) \) satisfies Gaussian bounds of order \( m = 2n \) by [EIR2] Theorem 1.1.

The Gaussian bounds (9) with \( m = 2 \) are also valid for the Schrödinger operator \( H \) for a spinless particle of mass \( m \) in an external magnetic field \( B \) where \( B \) is a polynomial. Then the Hamiltonian is given by \( H = \frac{1}{2m} (\vec{p} - \frac{e}{\hbar} \vec{A})^2 \) where \( \vec{p} = -i\hbar \vec{\nabla} \) and \( \vec{A} \) is a polynomial vector potential satisfying \( B = \vec{\nabla} \times \vec{A} \) (see [EIP]).

The next example extends the results of Sturm [Stu] for the \( p \)-indpendence of the spectrum of a pure second-order symmetric operator to non-symmetric operators with lower order terms.

**Example 4.2** Let \((M, g)\) be a \( d \)-dimensional complete Riemannian manifold (without boundary) with Ricci curvature bounded by below. Let \( A \) be a measurable section of \( \text{End}(T_M) \), i.e., \( A_x \) is a (real) endomorphism of \( T_x \) for all \( x \in M \) and the map \( x \mapsto (x, A_x) \) from \( M \) into \( \text{End}(T_M) \) is measurable. Assume there exists a \( \mu > 0 \) such that

\[
g(X, A_x X) \geq \mu |X|^2 \quad \text{and} \quad |A_x X| \leq \mu^{-1} |X|
\]

uniformly for all \( x \in M \) and \( X \in T_x \). Next let \( \mathcal{X} \) and \( \mathcal{Y} \) be two real vector fields on \( M \) such that \( x \mapsto (x, \mathcal{X}_x) \) and \( x \mapsto (x, \mathcal{Y}_x) \) from \( M \) into \( T_M \) are measurable. Further let \( m: M \to [0, \infty) \) be a uniformly positive bounded measurable function on \( M \) and \( c_0 \in L_\infty(M) \) real. Suppose that \( \sup_{x \in M} |\mathcal{X}_x| + |\mathcal{Y}_x| < \infty \). Consider the divergence form operator

\[
H_2 \varphi = -m^{-1} \text{div}(m A \nabla \varphi + \varphi \mathcal{X}) + \mathcal{Y} \varphi + c_0 \varphi
\]

in \( L_2(M ; d\mu) \), where \( \mu \) is the measure associated with \( g \). It follows from [Sal] Theorem 6.1 that the semigroup \( S^{(2)} \) generated by \( H_2 \) has a kernel \( K \) and, moreover, there are \( b, c, \omega > 0 \) such that

\[
0 \leq K_t(x; y) \leq c \left( \mu(B(x; t^{1/2})) \mu(B(y; t^{1/2})) \right)^{-1/2} e^{\omega t} e^{-b d(x; y)^2 t^{-1}}
\]

uniformly for all \( t > 0 \) and \( x, y \in M \), where \( d \) is the metric associated to \( g \) and \( B(x; r) = \{ y \in M : d(x; y) < r \} \). We may assume that \( \omega = -4 \).
Now suppose that the volume of \((M, g)\) grows uniformly subexponentially, i.e., for all \(\varepsilon > 0\) there exists a \(C_\varepsilon \geq 1\) such that
\[
\mu(B(x; r)) \leq C_\varepsilon e^{\varepsilon r} \mu(B(x; 1))
\] (10)
uniformly for all \(x \in M\) and \(r > 0\). We shall prove that the semigroup \(S^{(2)}\) on \(L^2(M; d\mu)\) is consistent with a continuous semigroup \(S^{(p)}\) on \(L_p(M; d\mu)\) and that \(\sigma(H_2) = \sigma(H_p)\) for all \(p \in [1, \infty]\), where \(H_p\) is the generator of \(S^{(p)}\).

Unfortunately we have to make a detour via a weighted space. Define \(w: M \to (0, \infty)\) by \(w(x) = \mu(B(x; 1))\). Then
\[
w(x) \leq \mu(B(y; 1 + d(x; y))) \leq C_\varepsilon e^\varepsilon e^{d(x; y)} w(y)
\] (11)
for all \(x, y \in M\) and \(\varepsilon > 0\). For all \(t > 0\) define \(\bar{K}_t: M \times M \to \mathbb{R}\) by
\[
\bar{K}_t(x; y) = K_t(x; y) w(y).
\]
Moreover, let \(\nu = w^{-1} \mu\). We need several \(L_p\)-estimates.

**Lemma 4.3** For all \(r \in [1, \infty)\) there exists a \(\bar{c} > 0\) such that
\[
\sup_{x \in X} \int_X d\nu(y) |e^{\eta d(x; y)} \bar{K}_t(x; y)|^r \leq \bar{c} (1 + t^{-2d} r^{-1}) e^{-rt}
\]
and
\[
\sup_{y \in X} \int_X d\nu(x) |e^{\eta d(x; y)} \bar{K}_t(x; y)|^r \leq \bar{c} (1 + t^{-2d} r^{-1}) e^{-rt}
\]
uniformly for all \(t > 0\), where \(\eta = b^{1/2}\).

**Proof** Since the Ricci curvature is bounded below there is an \(\alpha > 0\) is such that \(\text{Ric} \geq -\alpha^2 g\). Then it follows from [CGT] that
\[
\mu(B(x; r)) \leq e^{(d-1)^{1/2} \alpha r} \mu(B(x; s))
\] (12)
uniformly for all \(x \in M\) and \(0 < s < r < \infty\). Hence
\[
(\mu(B(x; r)))^{-1} \leq e^{(d-1)^{1/2} \alpha (1 + r^{-d})} w(x)^{-1}
\] (13)
for all \(r > 0\). Let \(x, y \in M\), \(t > 0\), \(n \in \mathbb{N}\) and suppose that \(d(x; y) \leq nt^{1/2}\). Then it follows from (12) that
\[
\mu(B(x; nt^{1/2})) \leq e^{(d-1)^{1/2} \alpha n} \mu(B(x; t^{1/2}))
\]
and
\[
\mu(B(x; nt^{1/2})) \leq \mu(B(y; 2nt^{1/2})) \leq e^{2(d-1)^{1/2} \alpha (2n)} \mu(B(y; t^{1/2}))
\]
if \(t \leq 1\). Moreover,
\[
\mu(B(x; nt^{1/2})) \leq C_\epsilon e^{n t^{1/2}} w(x) \leq C_\epsilon e^{n t} \mu(B(x; t^{1/2}))
\]
and similarly,

$$\mu(B(x; nt^{1/2})) \leq C_\epsilon e^{2\epsilon n^2 + 2\epsilon t} \mu(B(y; t^{1/2}))$$

for all $t \geq 1$ and $\epsilon > 0$. Since $2(d-1)^{1/2} \alpha n \leq \epsilon n^2 + \epsilon^{-1}(d-1) \alpha^2$ it follows that

$$\mu(B(x; nt^{1/2})) \leq \tilde{C}_\epsilon e^{\epsilon n^2 + \epsilon t} n^d \mu(B(x; t^{1/2}))$$

and

$$\mu(B(y; nt^{1/2})) \leq \tilde{C}_\epsilon e^{\epsilon n^2 + \epsilon t} n^d \mu(B(y; t^{1/2}))$$

for all $\epsilon > 0$, where $\tilde{C}_\epsilon = C_\epsilon \vee C_{\epsilon/2} \vee 2^d e^{\epsilon^{-1}(d-1) \alpha^2}$.

Next, note that

$$e^{-2^{-1} \beta d(x; y)^2} \leq e^{-4t} e^{2\eta d(x; y)} \leq e^{-2t}$$

for all $x, y \in M$ and $t > 0$. Therefore if $\epsilon = \eta \wedge 4^{-1} \beta \wedge 1$ then

$$\int_M d\nu(y) \left| e^{\eta d(x; y)} K_t(x; y) w(y) \right|^r$$

$$\leq c^r \sum_{n=1}^\infty \int_{B(x; nt^{1/2}) \setminus B(x; (n-1)t^{1/2})} d\mu(y) w(y)^{r-1} \cdot \left| (\mu(B(x; t^{1/2})) \mu(B(y; t^{1/2})) \right|^{-1/2} e^{-2t} e^{-\eta d(x; y)} e^{-2^{-1} \beta d(x; y)^2} t^{1/2-r}$$

$$\leq c^r \sum_{n=1}^\infty \int_{B(x; nt^{1/2}) \setminus B(x; (n-1)t^{1/2})} dp(y) C_\epsilon^{r-1} e^{(r-1)\eta d(x; y)} w(x)^{r-1} \cdot \tilde{C}_\epsilon e^{\epsilon n^2 + \epsilon t} n^d \mu(B(x; nt^{1/2}))^{-1/2} e^{-2t} e^{-\eta d(x; y)} e^{-2^{-1} \beta d(x; y)^2} t^{-1}$$

$$\leq (cC_\epsilon \tilde{C}_\epsilon e^{r}) e^{-rt} \sum_{n=1}^\infty w(x)^{r-1} \left( n^d e^{-4^{-1} \beta n^2 + \beta n} \right)^r \mu(B(x; nt^{1/2}))^{-1}$$

$$\leq (cC_\epsilon \tilde{C}_\epsilon e^{r}) e^{-rt} \sum_{n=1}^\infty \left( n^d e^{-4^{-1} \beta n^2 + \beta n} \right)^r$$

for all $x \in M$ and $t > 0$, where we used (13) in the last step. This proves the first inequality of the lemma and the second one follows similarly. \( \Box \)

We continue with Example 4.2.

By the previous lemma and Lemma 1.1 one can define for all $t > 0$ a family $S_t^{(p)}$, $1 \leq p \leq \infty$, of consistent continuous operators on $L_p(M; d\nu)$ with kernel $\overline{K}_t$. Moreover, by (11) and the first part of Theorem 2.5 for all $t > 0$ and $p \in [1, \infty] \setminus \{2\}$ there exists an operator $S_t^{(p)} \in L(L(M; d\nu)) = L(L(M; d\mu))$ which is consistent with $S_t^{(p)}$. Then $\|S_t^{(p)}\|_{p \rightarrow p} \leq \tilde{C} e^{-t}$ and $\|S_t^{(p)}\|_{p \rightarrow p} \leq C_0 e^{\tilde{C} e^{-t}}$ uniformly for all $t > 0$ and $p \in [1, \infty]$ by (8), where $\tilde{C}, \eta > 0$ are as in Lemma 4.3.

Clearly $S_t^{(2)}$ is consistent with $S_t^{(2)}$ for all $t > 0$ and since $S_t^{(2)}$ is a semigroup it follows that $S_t^{(p)}$ and $\tilde{S}_t^{(p)}$ are semigroups for all $p \in [1, \infty]$. Since $S_t^{(2)}$ is a continuous semigroup it is easy to prove that that semigroups $S_t^{(p)}$ and $\tilde{S}_t^{(p)}$ are continuous for all $p \in (1, \infty)$. But
also at the endpoints \( p \in \{1, \infty\} \) the semigroups are continuous by Lemma 4.3 and [DuR], or by using the positivity of the kernels \( K_t \) and \( \tilde{K}_t \) together with [Voi].

Let \( H_p \) and \( \tilde{H}_p \) be the generators of \( S^{(p)} \) and \( \tilde{S}^{(p)} \). Then \( 0 \in \rho(H_p) \) and \( 0 \in \rho(\tilde{H}_p) \) for all \( p \in [1, \infty] \). Moreover, \( H_p^{-1} \) is consistent with \( (\tilde{H}_p)^{-1} \) for all \( p \in [1, \infty] \).

It follows from (13) that

\[
0 \leq \tilde{K}_t(x; y) \leq c e^{(d-1)/2} \left( w(x) \right)^{-1/2} \left( w(y) \right)^{1/2} \left( 1 + t^{-d/2} \right)e^{-t^{-1}d(x,y)^2} t^{-1}
\]

uniformly for all \( t > 0 \) and \( x, y \in M \). Therefore for all \( x \neq y \) the Laplace transform \( \tilde{K}(x; y) = \int_0^\infty dt \tilde{K}_t(x; y) \) exists. But it follows from Lemma 4.3 that \( (x, y) \mapsto e^{\nu d(x; y)} \tilde{K}(x; y) \) satisfies the Conditions (3) and (4) with respect to the space \( M \), the metric \( d \) and measure \( \nu \). Moreover, if \( \varepsilon > 0 \) then by (11) and (10)

\[
\int_M d\nu(y) e^{-\varepsilon d(x; y)} \leq \sum_{n=1}^\infty \int_{B(x;n) \setminus B(x;n-1)} d\mu(y) w(y)^{-1} e^{-\varepsilon d(x; y)}
\]

\[
\leq \sum_{n=1}^\infty C_\varepsilon e^{\delta} \sum_{n=1}^\infty e^{-\varepsilon(n-1)} \mu(B(x; n))
\]

\[
\leq \sum_{n=1}^\infty C_\varepsilon^2 e^{\delta} e^{\varepsilon n} e^{-\varepsilon(n-1)} < \infty
\]

uniformly for all \( x \in M \), if one chooses \( \delta = \varepsilon/3 \). Therefore (2) is valid with respect to the space \( M \), the metric \( d \) and measure \( \nu \).

Hence we can apply Theorem 2.5 and deduce that \( \sigma(H_p^{-1}) = \sigma(H_2^{-1}) \) for all \( p \in [1, \infty] \) since \( H_p^{-1} = (\tilde{H}_p)^{-1} \). Hence \( \sigma(H_p) = \sigma(H_2) \) for all \( p \in [1, \infty] \).

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