Look-out for prime-chains with a prescribed number of mobility-degrees of freedom

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LOOK-OUT FOR PRIME-CHAINS WITH A PRESCRIBED NUMBER OF MOBILITY-DEGREES OF FREEDOM

TYPE-SYNTHESIS OF PRIME-CHAINS

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Abstract—The search for prime-chains has been triggered off by the query after the possible existence of closed kinematic chains not containing Assurgroups, but meeting a *given* mobility (f) in addition to a *given* girth (g). Those with a minimum number of links (n), have been entrusted with the name *prime-chains*. Prime-chains with more than $(g - 3)$ mobility-degrees of freedom are of no interest and have been left out of consideration. Spectacular has been the possibility to compose mobility- f chains without Assurgroups, with $(d - n + 1)$ independent linkage polygons, each with a given number of at least $(f + 4)$ sides. We found, for instance, a unique prime-chain in case $g = 4$ and $f = 0$ with $n = 5 - f$, also a unique one in case $g = 5$ and $f = 0$ or 1 with $n = 9 - f$, and again one if $g = 6$ and $f = 0, 1$ or 2 with $n = 13 - f$, and, lastly, a unique prime-chain in case $g = 8$ and $f = 0, 1$ or 2 with $n = 29 - f$. We found more than one of these chains when $g = 7$ with $n = 23 - f$. Namely, two for $f = 0$, three for $f = 1$, four for $f = 2$ and even 13 for $f = 3$. Finally, two prime-chains have been derived for $g = 8$ and $f = 3$ or 4 with $n = 29 - f$. Graph-theory has been used to find these results. Most of them have been derived from *cages*, representing particular chains having only ternary links, a minimum number of them for a given girth, and further having (-3) mobility-degrees of freedom.

1. INTRODUCTION

It is incredible how many different linkages may be built with only a few links.

Even if we restrict ourselves to turning-joints and *one* degree of mobility ($f = 1$), closed kinematic chains astoundingly increase in number with the number of links: with six links, for instance, we have only two possibilities, namely the Watt's form and the Stephenson's form; with eight links, we already have 16 different chains with constrained motion. Further, with 10 links, we find there are even 230 different possibilities [1]; and, if the closed chain contains 12 links, 6855 different structures arise to confound the designer [2]. (*Sub-chains* with a negative degree of mobility by the way, are not considered here.)

We may heavily decrease the number of these chains, by skipping the ones not meeting a prescribed *minimum looplength*, the so-called *girth* of the chains. (The length of a loop is the number of sides of a linkage-polygon, usually called a loop.)

Constrained 10-bar chains having girth 5, for instance, i.e. those not containing four-bar loops, number only 10.

However, four of them contain a *linkage-dyad*, i.e. an open sub-chain that may be removed without changing the mobility (f) nor the closeness of the chain. (See Figs 2(a)–(d) of Ref. [4].)

When, generally, a part of a closed chain can be removed without changing the mobility (f), nor the closeness of the chain, we will call that part an *Assur* group. Thus, in order to obtain fundamental insight in mobility- f chains, we shall have to confine ourselves to closed chains without Assur groups. Closed chains without Assur-groups will be named *fundamental main chains*. As Assur groups are sometimes difficult to recognize, we usually obtain them in two steps: we first skip the ones not meeting a prescribed girth in the chain, after which we delete the ones still containing an Assur group. The 230 mobility-1 10-bar chains, for example, are so reduced to only

Fig. 1. Derivation of a 10-bar fundamental main chain.

6 fundamental main-chains with girth 5. (See the Figs 1–6.) Observe that mobility-1 10-bar chains containing four-bar loops, automatically contain Assur groups. Whereas the four with girth 5 and a linkage-dyad are all to be reduced to a unique eight-bar linkage chain, composed with three *independent* pentagonal loops.

This mobility-1 eight-bar chain therefore, does not have Assur groups, but is the only one having girth 5 with the least number of links. (It is also the only one without Assur groups, see Fig. 7.)

The number C of *independent* loops in a mobility-1 chain has been calculated by Paul [3]. He found the simple equation:

$$2C = n - 2, \quad (1)$$

where n equals the number of links in the chain.

For chains having f mobility-degrees of freedom, the formula would read:

$$2C = n - 1 - f \quad (2a)$$

or

$$C = d - n + 1, \quad (2b)$$

by replacing f by its expression, known as Grübler's formula:

$$f = 3(n - 1) - 2d, \quad (3a)$$

in which d represents the total number of turning-joints in the chain. Indeed, the unique mobility-1 eight-bar chain without Assur groups, does have $(d - n + 1) = 10 - 8 + 1 = 3$ independent pentagons.

As it is the only one with the *least* number of links with girth 5, we will call it the (1,5)-*prime*; in other words, it is a mobility-1 prime-chain with girth 5.

Thus, we define *prime-chains* as fundamental main chains with the *least number* of links in the chain for a *given girth*.

Fig. 2. Derivation of a 10-bar fundamental main chain.

Fig. 3. Derivation of a 10-bar fundamental main chain.

The six above-mentioned 10-bar fundamental main-chains, do not contain a prime-chain since the eight-bar just mentioned represents a solution with a fewer number of links.

The (1,6)-prime-chain has been found by Kurt Hain and treated in one of his "VDI-Bildungswerke" [5]. This chain contains neither pentagons nor four-bar loops. In accordance with Paul it has only five independent hexagons.

A symmetrical arrangement is shown in Fig. 8.

The (1,6)-prime is a constrained linkage-chain with 12 links. It further has 16 turning-joints, according to Grübler's formula for a mobility- f chain:

$$d = (3n - 3 - f)/2. \quad (3b)$$

2. THE SEARCH FOR CONSTRAINED PRIME-CHAINS AND FUNDAMENTAL MAIN-CHAINS HAVING GIRTH 5 OR 6, USING GRAPH-THEORY

The easiest way to find kinematic chains having certain structural (i.e. topological) properties such as the prime-chains, will be by way of graph-theory. The first to apply this theory has been Crossley in his search for all the 230 10-bar chains [1].

He turned each linkage into a *graph* by way of graphization. Through this transformation each link (no matter how many turning-joints it is equipped with) changes into a single *vertex*, whereas each turning-joint becomes an *edge* connecting two vertices representing the two links being coupled by this turning-joint.

It is important that the transformation represents a one-to-one mapping. That is to say, each graph represents a unique linkage and vice versa. Nonetheless, it is sometimes difficult to recognize the identity between two differently drawn, but principally identical graphs, usually named *isomorphic* graphs. This however, is true also for the kinematic chains themselves, though their identity is often even more difficult to recognize than the identity of their corresponding graphs.

Another important feature is that a graphization, or its inverse operation, turns k -sided linkage-polygons (in mechanism-theory named *kinematic loops*) into k -sided graphical polygons (in graph-theory named *cycles*).

Fig. 4. Derivation of a 10-bar fundamental main chain.

Fig. 5.

In other words, graphization doesn't change the *polygon-length* k .

As further rigid ternary-, quaternary- and other polygonal-formed links are each turned into a singular *vertex*, the graph of a linkage looks simpler than the original chain it came from. In the graph, the sides of such rigid polygons are represented by an equal number of edges joining the vertex representing the polygonal-formed link.

Fig. 6. Derivation of a 10-bar fundamental main chain.

Fig. 7.

In order to find the earlier mentioned six 10-bar fundamental main chains, knowing they do not enclose four-bar loops, we may start from a *basic* graph consisting of *two* interconnected pentagons having a common edge and common vertices of this edge.

We then simply look for the six different possibilities that exist to adjoin two graphical dyads (each of them representing a binary bar with two turning-joints) without introducing four-bar loops or linkage-dyads. Note that the required number of edges follows from Grübler's formula (3b).

With the same basic graph, but $n = 8$ instead of 10, we so found the single solution by adjoining only one edge. The method represents a short way to obtain the unique (1,5)-prime, having eight links. The choice of the basic graph (or corresponding chain) is entirely arbitrary: for instance, we may possibly start from *one* pentagonal cycle to obtain the (1,5)-prime. In that case we have to attach *two* graphical dyads instead of the single edge.

In order to find the (1,6)-prime, we may follow a similar method: with $n = 10$ we obtain no solutions, but with $n = 12$ we are able to start from an encouraging *basic* graph with 10 points as demonstrated in Fig. 8.

This graph possesses two interconnected hexagons having a common edge and common vertices of this edge. It further contains an additional edge, which is not difficult to place while observing the requirement that no quadrilaterals nor pentagons may occur. The graph still lacks two vertices and (according to Grübler's formula) also four edges. Adjoining them gives two seemingly different solutions: namely, one graph being drawn as a *line*-symmetric figure, and a second graph that appears to be *point*-symmetric. Each graph leads to a kinematic chain with a seemingly different topological structure. The two graphs however, as well as the two corresponding chains, are *isomorphic*: they both can be rearranged into the same topological chain as demonstrated. The

Fig. 8. Not containing quadrangles nor pentagons, but assembled with hexagons. The prime linkage does not contain sub-chains of mobility-1.

Fig. 9. Symmetrized (non-Hamilton) graph of a constrained 14-bar with three quaternary links and girth 6.

singular chain so found represents the earlier mentioned (1,6)-prime, having five independent hexagons.

Nine 14-bar constrained fundamental main-chains have been found with girth 6. Their graphs are demonstrated in the Figures 9–17. They may be found through a basic graph, containing two interconnected hexagons, that are to be completed in accordance with Grübler's formula.

One may prove that 14-bar fundamental main-chains (with girth 6) do not contain *pentagonal* links (nor more-sided ones). Thus,

$$0 = n_p = n_h = \dots,$$

further

$$14 = n = n_b + n_t + n_q + (n_p + n_h + \dots) \quad (4)$$

(in which n_b, n_t, n_q, n_p, n_h , etc. represent the respective numbers of binary-, ternary-, quaternary-, pentagonal- and hexagonal links.), whereas for the number of turning-joints (d):

$$2d = 2n_b + 3n_t + 4n_q + (5n_p + 6n_h + \dots). \quad (5)$$

Substitution of these expressions in Grübler's formula (3b) leads to the equation, valid for any chain:

$$n_b = (f + 3) + n_q + (2n_p + 3n_h + \dots). \quad (6)$$

For constrained 14-bar fundamental main-chains we find the list:

n_q	n_b	n_t	Number of constrained fundamental main chains
0	4	10	3 (Figs 15–17)
1	5	8	2 (Figs 13 and 14)
2	6	6	3 (Figs 10–12)
3	7	4	1 (Fig. 9)

Figure 9 demonstrates the solution with three quaternary links through a highly symmetrical *non-Hamilton graph*. (A *Hamilton-graph* is defined as a graph having a *cycle* containing all the vertices of the graph [6]. In other words, a *Hamilton-cycle* is a sub-graph being the cycle passing

Fig. 10. Symmetrized (non-Hamilton) graph of a constrained 14-bar with two quaternary links ($g = 6$).

Fig. 11. *Symmetrized (non-Hamilton) graph of a constrained 14-bar linkage with two quaternary links ($g = 6$).*

Fig. 12. *Unsymmetrical Hamilton-graph of a constrained 14-bar linkage with two quaternary links ($g = 6$).*

Fig. 13. *Non-Hamilton-graph of a constrained 14-bar linkage with one quaternary link ($g = 6$).*

Fig. 14. *Unsymmetrical Hamilton-graph of a constrained 14-bar linkage with one quaternary link with girth 6.*

Fig. 15. Symmetrized Hamilton-graph of a constrained 14-bar linkage having girth 6.

Fig. 16. Symmetrized Hamilton-graph of a constrained 14-bar linkage having girth 6.

through each of the vertices of the graph exactly once.) We find three solutions with two quaternary links of which only one (Fig. 12) is represented by a Hamilton-graph; the remaining two (Figs 10 and 11) are non-Hamiltonian but symmetric. The two solutions (Figs 13 and 14) with only *one* quaternary link are *un*-symmetrical; of them only one (Fig. 14) is represented by a Hamilton-graph.

We further find *two* ordinary symmetrical Hamilton-solutions (Figs 15 and 16); and, finally, *one* double-symmetrical Hamilton-solution (Fig. 17). (Note that the corresponding chain of a Hamilton graph is a *Hamilton-linkage*, since the longest polygon in the chain then contains all the links whereas the length of this polygon is the same as the one of the corresponding Hamilton-cycle in the graph.)

Fig. 17. Doubly symmetrized Hamilton-graph of a constrained 14-bar linkage with girth 6.

3. THE THREE CONSTRAINED PRIME-CHAINS WITH GIRTH 7

In order to find these chains, we could start from a *basic graph* containing *four rings* (in this case *heptagons*) instead of two that we used earlier to find the (1,6)-prime. (With only three rings in the basic chain, we are obviously too far off from the end-result meeting Grübler's formula.) After many try-outs, we obtain the best results if we start from a basic chain having four heptagonal rings complete with two edges as shown in Fig. 18.

According to Grübler's formula, we then still have to adjoin *two* edges as well as *two* graphical dyads to bring the graph into its final form. We find there are only three possible ways to do this. Each possibility leads to a different (1,7)-prime-chain.

Fig. 18. The three different prime-graphs of constrained 22-bar linkages with girth 7, with 31 turning-joints and 10 independent heptagons.

No other solutions with less than 22 links are possible. Furthermore, because no quaternary links appear, all three graphs representing a solution, contain $f + 3 = 4$ binary links in accordance with equation (6). In addition, they all show to have $(22 - 4) = 18$ ternary links as well as $(22/2) - 1 = 10$ independent heptagons.

Another way to find prime-chains would be by starting from a *Hamilton-cycle* as a basic graph. This cycle will then be completed in accordance with the given girth. The mobility (f) of the linkage-chain corresponding to the resulting graph, may then be calculated through Grübler's formula and compared with the required degree of mobility.

If they are not the same, the Hamilton-cycle will be enlarged until it does. Using this procedure, it is first recommended to restrict ourselves to symmetrical graphs with one or more axes of reflection. Later, when the required mobility is attained, we may try to diminish the length of the Hamilton-cycle in order to obtain the prime-chain(s) with a minimum number of vertices. This may be done by then giving up one or more of the assumed axes of reflection.

This method of obtaining prime-chains, though a general one, does not necessarily lead to all prime-chains, as only the Hamilton linkages are attained this way. In case there are more prime-chains and also in case none of the prime-chains for given g and f contain a Hamilton-cycle, we have to start with a shorter cycle in addition to one or more extraneous vertices.

The main feature of the three constrained (1,7)-prime-chains, however, appears to be the fact that they are all derivable from a singular so-called *cage* with—in this case—24 vertices (See Fig. A.4.) A *cage* is defined as a graph of a linkage containing only *ternary* links, but with the *least* number of links for a given girth. See the Appendix and Ref. [7].

Fig. 19. Derivation of the constrained (1,8)-prime chain from a basic graph using Grübler's formula for constrained motion.

Fig. 20. *Hamilton-graph* of constrained (1,8)-prime chain with 28 links, 40 turning joints and 13 independent octagons ($f = 1$).

Fig. 21. Tutte's (3,8)-Cage of 30-bar with only *ternary* links and girth 8 ($f = -3$). Erasing *any* unit of the form $\bigcirc - \bigcirc - \bigcirc$ regains the *Hamilton-graph* of the constrained (1,8)-prime chain again.

By deleting two directly connected ternary links in this cage, we regain the three (1,7)-prime-graphs again, as there are three non-isomorphic ways to carry this out. Indeed, each time we do this, four binary links appear, namely the ones earlier connected with the two deleted ternary links. If we erase, for instance, a *diameter-edge* together with its two vertices and edges, the cage changes into a graph still being double-symmetric. It is then isomorphic with graph AA of Figs 18 or 28.

When we omit two connected non-diameter vertices with their edges, it is possible to prove that the cage turns into a *point-symmetric* (as well as *line-symmetric*) *Hamilton-graph*. (See graph BB of Fig. 18.) And, finally, when a diameter and a non-diameter vertex, with their edges are left out, the cage transforms into the singular symmetric *Hamilton-graph* AB of Fig. 18.

4. DERIVATION AND PROPERTIES OF THE CONSTRAINED (1,8)-PRIME-CHAIN

A graph having girth 8 and containing two concentric cycles of length 12 is taken as the basic graph to start the formation of the (1,8)-prime chain. As the kinematic chain, represented by this graph, has $3(24 - 1) - 2 * 28 = 13$ d.f. in motion, we have to complete the chain with four turning-joints and four binary links to turn it into a constrained motion linkage, whence the basic graph has to be completed with four edges and four graphical dyads. There is only one way of doing this if the girth of the graph has to be 8. The result is demonstrated in Fig. 19. A redesign turns it into a graph being a double-symmetric cycle with 26 vertices and two separate components which are two graphical dyads symmetrically adjoined to the cycle.

Thus, the (1,8)-prime chain is to be drawn as a double symmetric chain containing 28 links. It is a unique linkage chain. The applied numbering additionally shows the corresponding graph to be a *Hamiltonian graph*, clearly demonstrated in the redrawn configuration of Fig. 20.

As before with the (1,7)-prime, it is similarly possible to obtain the (1,8)-prime from a *cage* in which each vertex of the *Hamilton-cycle* has the *degree* 3. (The degree or valency of a vertex is the number of edges meeting in that vertex.) The cage here used to form the (1,8)-prime clearly has 30 vertices and girth 8. The solution is shown in Fig. 21.

Fig. 22.

Fig. 23.

Deleting a *random* pair of two interconnected vertices together with their edges then obviously turns the cage into the (1,8)-prime graph of the corresponding (1,8)-prime linkage of mobility 1. Indeed, as the linkage corresponding to the cage has $3(30 - 1) - 2 * 45 = -3$ d.f. in motion, the deletion of two coupled ternary links with their five turning-joints increases the mobility by $3 * (-2) - 2 * (-5) = 4$ degrees. This process turns the chain into a mobility-1 linkage.

5. STATICALLY DETERMINATE PRIME-TRUSSES ($f = 0$), CONSTRAINED PRIME-CHAINS ($f = 1$), AND PRIME-CHAINS WITH MOBILITY 2 OR MORE, DERIVED FROM $(3, g)$ -CAGES

Regular graphs of degree 3 and girth g are denoted as $(3, g)$ -graphs. They are called regular as long as all vertices of these graphs have the same degree, which means that each vertex of the graph meets the same number of edges. A regular $(3, g)$ -graph with the *least number of vertices* is called a *cage*. (See the Appendix for the derivation of $(3, g)$ -cages.)

Lemma 1

All linkages, that are the graphization-equivalent of $(3, g)$ -cages have (-3) d.f. in motion.

Indeed, if we calculate the mobility f through Grübler's formula, we obtain $f = 3(n - 1) - 2d = -3$ when also $3n = 2d$. This is true, since the "handshaking lemma" says that the sum of the vertex-degrees will be twice the number of edges (d).

Lemma 2

Erasing one vertex and its three edges from a $(3, g)$ -cage, results into a $(0, g)$ -prime graph of a statically determinate prime-truss of girth g .

Indeed, $f_{\text{truss}} = f_{\text{cage}} + (\Delta f) = -3 + (3\Delta n - 2\Delta d) = -3 + 3 * (-1) - 2 * (-3) = 0$.

Further, as the cage contained only ternary links, the truss derived from it is left with exactly three binary links, in accordance with equation (7). When we erase, for instance, only *one* vertex with its edges from the $(3, 4)$ -cage, a distinct $(0, 4)$ -prime graph results, representing an overconstrained five-bar chain for which $f = 0$. (See Fig. 22.)

Fig. 24.

Fig. 25.

Similarly, when we take the *Peterson-cage* as a basic graph, a deletion of one vertex equally leads a statically determinate structure being a prime-truss for which $f = 0$. Both, the prime-truss as well as the (0,5)-prime graph, representing the structure, are demonstrated in Fig. 23.

In the same manner, we may use the *Heawood-(3,6)-cage* as a basic graph for the derivation of the (0,6)-truss. The statically determinate structure obtained in this way, then encloses 10 ternary and three binary links. Observe, that since the vertices in the Heawood-cage all have the same topological position within the cage, it does not matter which one is deleted; for this reason only *one* (0,6)-prime truss with zero degrees of mobility results. (See Fig. 24.)

The deletion of one vertex and its edges from the (3,7)-cage of *McGee* and *Tutte* leads to *two* different graphs for which $f = 0$. The reason is, that the cage contains two different kinds of vertices: (A) those located at the end of a *diagonal-edge*, and (B) those at the end of the other edges forming a regular octagon with an inscribed circle. The *two* statically determinate (0,7)-prime trusses, represented by these graphs, are in fact Hamilton-linkages with 20 ternary and three binary links. (See the Figs 26 and 27.)

Lemma 3

Erasing $(f + 1) \leq (g - 3)$ directly connected vertices with their $(2f + 3)$ edges from a $(3, g)$ -cage, results in a fundamental main-chain with girth g and mobility f . When $g \leq 8$ the obtained fundamental main-chains are prime-chains.

Indeed, since $f = f_{\text{cage}} + \Delta f = -3(3\Delta n - 2\Delta d) = -3 + 3\{-(f + 1)\} - 2\{-(2f + 3)\} = f$, the linkage must have the mobility f . Further, Assur groups do not appear as long as $f \leq g - 4$. (See also the Lemma's 4 and 5.)

(0,7)-prime graph (A)

Fig. 26.

(0,7)-prime graph (B)

Fig. 27.

Fig. 28.

Note that the cage does have a least number of vertices in comparison with other $(3,g)$ -graphs, just like the (f,g) -prime chain has the least number of links in comparison with (f,g) -fundamental main-chains. Note further, that not all (f,g) -prime graphs are derivable from cages.

A first graph to apply Lemma 3 on may be the $(3,4)$ -cage of Fig. A.1 with six vertices. This $(3,4)$ -cage leads to the quadrilateral as the $(1,4)$ -prime graph when two vertices are removed; and

Fig. 29. Nine *probable* $(1,9)$ -prime graphs each representing a *constrained* prime-chain having girth 9 and 56 links.

thus to the four-bar linkage as the first *constrained* prime linkage. Similarly removing two vertices from *Peterson's* (3,5)-cage leads to the well-known constrained 8-bar (1,5)-prime chain as indicated in Fig. 7. Then, the *Heawood*-graph of Fig. A.3 identifies the unique (3,6)-cage. Application of Lemma 3 for $f = 1$ on this cage so leads to the constrained 12-bar (1,6)-prime linkage of Kurt Hain. see Fig. 8. Further, the *Mc.Gee-Tutte* graph of Fig. A.4 represents the unique (3,7)-cage [8, 9]. Application of Lemma 3 similarly gives *three* different (1,7)-prime graphs and corresponding 22-bar (1,7)-prime linkages. (See Figures 18 and 28.)

Also, the *Tutte*-graph of Fig. A.5 or 21 is a (3,8)-cage on which it is similarly possible to apply Lemma 3. This way, we obtain the unique 28-bar (1,8)-prime linkage of Fig. 20. Finally, the (3,9)-graph of *Biggs* and *Hoare* with 58 vertices *may* be a (3,9)-cage. Application of Lemma 3 on this graph does lead to possible 56-verticed (1,9)-prime graphs and so to 56-bar (1,9)-prime

Fig. 31. Nine (3,7)-prime graphs as sub-graphs of the (3,7)-cage.

Fig. 32. Four particular (3,7)-prime graphs, having mobility-3 and girth 7. With these, a graphical dyad is not to be adjoined ($n = 20$).

linkages. However, using the method of basic graphs, we also found *nine* double-symmetric solutions with 56 links. Their graphs are all demonstrated in Fig. 29.

Mobility-2 prime linkages may be obtained by erasing three vertices from a cage. If we do this with Peterson's cage, the resulting graph will contain a pentagon in addition to an Assur group. As prime-linkages may not contain Assur groups, the result will be this pentagon-linkage indeed having the mobility 2. Thus, the pentagon-linkage actually represents the (2,5)-prime chain. In fact, the mobility (f) of prime-linkages is limited by the girth (g) of these linkages, especially so when we obtain them by subtraction from (3, g)-cages. If we remove, for instance, ($g - 2$) vertices from a (3, g)-graph or from a (3, g)-cage, we obtain ($g - 3$) mobility degrees of freedom of the remaining linkage in accordance with Lemma 3. However, this is the same mobility as of a singular g -polygon, with the result that the resulting linkage has to be an Assur-group if we subtract this g -polygon. Whence, the g -polygon actually represents the ($g - 3, g$)-prime chain. This leads to the following lemmas.

Lemma 4

A fundamental main-chain or a prime-chain with girth g has at most ($g - 3$) mobility-degrees of freedom.

Lemma 5

The ($(g - 3), g$)-prime-chain is the polygon with length g .

Thus, in practice, at most ($g - 3$) vertices are to be removed from a cage, otherwise Assur groups arise. From this, we derive the list:

f	-3	0	1	2	3	4	f	$f(\max) = g - 4$
Δf	0	3	4	5	6	7	$f + 3$	$g - 1$
$-\Delta n$	0	1	2	3	4	5	$f + 1$	$g - 3$
$-\Delta d$	0	3	5	7	9	11	$2f + 3$	$2g - 5$

So leads, as an example of mobility-2 chains, the *Heawood*-cage to the (2,6)-prime chain. To obtain this result, we may start from the (1,6)-prime graph already obtained from this cage. Deletion of *one* out of the four degree-2 vertices with its edges then leads to the singular (2,6)-prime graph, representing the mentioned (2,6)-prime chain with mobility-2. (See Fig. 25.) As observed in the figure, the chain contains six ternary links as well as five bars. *Four* different (2,7)-prime graphs are to be obtained by deletion of three connected vertices from the (3,7)-cage of *McGee* and *Tutte*. This can be done in four ways, to be indicated by the erased formations (BAB), (AAB), (ABB), and (BBB). (See Fig. 30.) The vertices *A* in this notation, are to be found at the end of a *diagonal* edge of the cage, whereas the vertices *B* are the remaining ones of the cage.

Nine different (3,7)-prime graphs are obtainable by deletion of $g - 3 = 4$ connected vertices from the same (3,7)-cage. They are indicated by the erased formations like (AABB), (BAAB) (2x), (ABBA), (BABB), (ABBB) and (BBBB), and further by the two erased star-formed groups in which either vertex *A* or *B* is in the center, in both cases being connected with three vertices, namely with two *B*'s and one *A*. These nine (3,7)-prime-graphs of which the corresponding chains have the mobility-3, are demonstrated in Fig. 31. *Four* entirely different (3,7)-prime graphs have been found not to be extended to or derivable from the (3,7)-cage. (See Fig. 32.) Note that one of them contains a vertex of degree four. Thus, prime-chains may contain quaternary links. (The proof that these

four graphs are prime graphs has been left out for reasons of brevity.) It is further not to be excluded that more (3,7)-prime graphs can be found.

The prime-linkage-number is clearly dependent on the girth (g) and on the number (f) of mobility-degrees of freedom. This number has been listed below:

	$g = 4$ $n = 5 - f$	$g = 5$ $n = 9 - f$	$g = 6$ $n = 13 - f$	$g = 7$ $n = 23 - f$	$g = 8$ $n = 29 - f$
$f = 0$	1	1	1	2	1
$f = 1$		1	1	3	1
$f = 2$			1	4	1
$f = 3$				(9 + 4)	2
$f = 4$					2

7. CONCLUSIONS

A deep penetration of closed linkage systems has been endeavoured in order to explore the ultimate borders of the possibilities that compose planar linkages. This investigation into the type-synthesis of closed kinematic chains lead to the discovery of new types such as, for example, the fundamental main chains, being those not containing Assur groups.

In this context, the term prime-chain has been coined for a special type of these chains with the least number of links for a given girth. The investigation didn't stop at constrained prime-chains, but allowed them to have less or more than one mobility-degree of freedom. This lead, for example, to the spectacular discovery of a (3,7)-prime chain containing a quaternary link, proving that prime-chains may contain such links. We so obtained (statically determinate) prime-trusses and prime-chains with up to $(g - 4)$ mobility-degrees of freedom.

Though not directly treated in the manuscript, the text opens a way to the investigation of *prime Assur*-groups, being open chains with zero mobility-degrees of freedom, but restricted to those not to be split up into other Assur groups with a lesser number of links.

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APPENDIX

Derivation of (3,g)-Cages

For $g \leq 8$, (3,g)-cages may be obtained from *trees*. (A tree is a *connected* graph not containing cycles.) (3,g)-cages with an *even* girth ($g = 2k$) are built with *bicentral trees*; whereas (3,g)-cages having an *odd* girth ($g = 2k + 1$) may be set up with *central trees* as basic graphs. The (3,4)-cage for instance, starts from a bicentral tree, each half having *depth* 1. We then close the loose branches of the tree with connecting edges, but in such a way that the emerging cycles are never shorter than the given girth. (See Fig. A.1.)

Fig. A.1

Fig. A.2

Fig. A.3

The (3,5)-cage starts from a central tree with depth 2, as indicated. If we then close the loose branches such that the appearing cycles aren't shorter than length 5, the famous *Peterson*-graph will be regained. (See Fig. A.2.) The basic graph of the (3,6)-cage has to be the bicentral tree with depth 2. Closing the loose branches in this case will lead to the so-called *Heawood*-cage. (See Fig. A.3.)

The procedure applied to the central tree with depth 3, however, does not immediately succeed in creating the (3,7)-cage. Nonetheless, if we additionally adjoin two extraneous vertices, it is still possible to obtain this cage, nowadays named after *McGee* and *Tutte*. (See Fig. A.4.)

The (3,8)-cage can be found in the normal way again. That is to say, we then start from a bicentral tree with depth 3. Closing the ultimate branches then leads to the *Tutte*-cage as demonstrated. (See Fig. A.5.)

Fig. A.4

Fig. A.5

The procedure allows a first guess in the number of vertices for these cages. For *even* girth (i.e. $g = 2k$), we find that:

$$n \geq 2(2^k - 1),$$

whereas for (3, g)-cages having an *odd* girth (i.e. $g = 2k + 1$) we get:

$$n \geq 1 + 3(2^k - 1).$$

When $g = 4, 5, 6, 8$ and 12 the number equals the right-hand side of these inequalities. For other girths the cages possess more vertices than the number of these lower bounds. For the (3,9)-cage, for instance, it is known that $54 \leq n \leq 58$. It is further possible to find *three* (3,10)-cages with 70 vertices. They were found respectively by Balaban [10] and Harries and Wong [12]. And, for the (3,11)-cage, finally, the literature [11] merely indicates that $96 \leq n \leq 112$, as a (3,11)-graph has been derived from the (3,12)-cage. Unknown cages may still be found using Hamilton-cycles and an extremely fast computer.

ÜBER DIE TYP-SYNTHESE VON PRIM-KETTEN

Zusammenfassung—Es ist offenbar möglich Polygonketten gleicher *Gürtel*-(länge) (g) als Baustein für die Zusammenstellung von geschlossenen, zwangsläufigen Drehgelenkketten zu verwenden. So genügt es, beim Aufbau von zwangsläufigen, n -gliedrigen, sogenannten *Prim*-Ketten, nur $((n/2) - 1)$ Polygonketten von vorgegebener Gürtel(länge) zu benutzen. Ein bekanntes Beispiel ist das achtegliedrige Getriebe ohne Gelenkviereck, das mit drei *Fünfecken* aufgebaut werden kann. Ein weiteres Beispiel findet man in dem zwölfgliedrigen Getriebe von Kurt Hain, das mit nur fünf *Sechsecken* zusammengesetzt werden kann. Weiterhin fanden wir sogar drei zwangsläufige Prim-Ketten mit 22 Gliedern und 10 unabhängigen *Siebenecken*. Mit *Achtecken* braucht man 13 derer zum Aufbau einer 28-gliedrigen Prim-Kette, von der nur eine existiert. Grüblersche Drehgelenkketten von Laufgrad (f) ungleich eins, möchten ebenfalls mit $(n - 1 - f)/2$ dieser g -eckigen Unterketten aufgebaut werden. So gab es z.B. vier 21-gliedrige Prim-Ketten von Laufgrad 2 und Gürtel 7, während *dreizehn* 20-gliedrige Prim-Ketten von Laufgrad 3 und Gürtel 7 gefunden sind. Alle diese Primgetriebe sind im Grunde genommen Drehgelenkgetriebe ohne Assur-gruppen aber mit Laufgrad, Gürtel und Mindestanzahl der Glieder die vorgegeben sind. In der Praxis möchten solche Getriebe für sogenannte Identifizierungsprogrammen, d.h. für Computerprogrammen benutzt werden, wobei es sich um die Wiedererkennung oder Einteilung von gleichfalls komplizierten Drehgelenkgetrieben mit Einfachgelenken handelt. Im Allgemeinen sind diese Grundketten auch sehr geeignet um universelle Analyse-Programme zu testen.