A new sufficient condition for robust root clustering of linear state-space models with structured uncertainty

Citation for published version (APA):

DOI:
10.1049/ip-cta:19960298

Document status and date:
Published: 01/01/1996

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:
• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication
New sufficient condition for robust root clustering of linear state space models with structured uncertainty

J.S. Luo
P.P.J. van den Bosch
S. Weiland

Indexing terms: Robust root clustering, Linear state space models, D-stability, Structured uncertainty

Abstract: A new sufficient condition is proposed for robust root clustering in an arbitrary subregion of the complex plane for both linear continuous time and linear discrete time uncertain systems. The systems are described by state space models with structured parameter uncertainty. Explicit bounds on uncertain parameters are derived to guarantee root clustering inside any specified region which may be demanded by stability and performance robustness. Illustrative examples show that the results of the proposed method can be much less conservative compared with the results of existing methods.

1 Introduction

In recent years the problem of stability and performance robustness of uncertain systems has received considerable research interest due to its fundamental importance. In particular, there are many methods presented to analyse stability robustness by the condition that the closed-loop uncertain system has all its eigenvalues in the open left half of the complex plane for continuous time systems and the unit circle with centre at the origin for discrete time systems. However, the more general D-stability robustness problem where D is an arbitrary subregion of the complex plane has received much less attention. The choice of D region may be determined by the requirements of performance robustness, for example, shaping of dynamic response, acceptable worst case behaviour and pole-placement. In the literature, most of the available D-stability robustness analysis methods are based on state space model approaches. Juang et al. [1, 2] and Juang [3] proposed some sufficient conditions of root clustering for some special choices of D-regions using a Lyapunov equation approach. Yedavalli [4, 5] presented some explicit bounds on structured and unstructured uncertainty for root clustering of linear state space models based on generalised Lyapunov theory approach or Kroneker product approach. One of the most important advantages of the state space model approach is that many physical laws in engineering sciences are represented by linear differential equations for continuous time systems and linear difference equations for discrete time systems. Uncertain parameters which have explicit physical meaning are often more easily displayed using a state space model.

In this paper we propose a new sufficient condition for robust root clustering of linear state space models with structured parameter uncertainty. The open left-half of the complex plane and the unit circle with centre at the origin become special cases of arbitrary choices of the D-regions. By searching the optimal choice of similarity transformation matrices, the least bound on uncertainty to maintain root clustering in a given subregion of the complex plane may be approached. The illustrative examples are used to show the much less conservative results of our proposed method compared with the results of some existing methods [1–5].

2 Description of some special D-regions

The boundary points of any D-region in the complex plane are defined as follows:

2.1 Definition

Let Y be a metric space and let $H \subset Y$. A point $y \in Y$ is called a boundary point of $H$ if every neighbourhood of $y$ contains points of $H$ as well as points not belonging to $H$.

Fig. 1 Common stability regions

Let $C$ be the complex plane, $C_e \subset C$ a non-empty subset and $C_a := C - C_e$ its complement (the subscript
with the superscript $C^e$ is the closure, and $C^j_\xi$ represents the interior of $C^e$. It is assumed that there exists a parameterisation $\pi : X \to B$ with $X \subset \mathbb{R},$ that is for all $b \in B$ there exists a real number $\xi \in X$ such that $b = \pi(\xi) = a(\xi) + jg(\xi)$ where $j = -1$ and $f(\xi) = \Re(\xi)$ and $g(\xi) = \Im(\xi)$ for all $\xi \in X$.

Some examples of borders for special D-regions are described as follows and shown in Fig. 1.

- The vertical line (with $a$ a constant) 
  \begin{align}
  f(\xi) &= a \\
  g(\xi) &= \xi \\
  X &= \mathbb{R}
  \end{align}

- A one sided horizontal strip (with $b > 0$ constant) 
  \begin{align}
  f(\xi) &= \min\{0, b - |\xi|\} \\
  g(\xi) &= \sign(\xi) \min\{b, |\xi|\} \\
  X &= \mathbb{R}
  \end{align}

- The sector area ($\theta \in (0, \pi)$ is a constant angle) 
  \begin{align}
  f(\xi) &= -|\xi| \cos(\theta) - a \\
  g(\xi) &= |\xi| \sin(\theta) \\
  X &= \mathbb{R}
  \end{align}

- The circle (centred at $a + jb$ with radius $r > 0$) 
  \begin{align}
  f(\xi) &= r \cos(\xi) + a \\
  g(\xi) &= r \sin(\xi) + b \\
  X &= [0, 2\pi]
  \end{align}

- The ellipse (centred at $(a_0, b_0)$ with length of major axis $(2a, 2b)$) 
  \begin{align}
  f(\xi) &= a \cos(\xi) - x_0 \\
  g(\xi) &= b \sin(\xi) - y_0 \\
  X &= [0, 2\pi]
  \end{align}

- The parabola (vertex at $(a_0, b_0)$) 
  \begin{align}
  f(\xi) &= x_0 - \xi^2/a \\
  g(\xi) &= \xi \\
  X &= \mathbb{R}
  \end{align}

The systems considered in this paper are described by state space models, i.e. either 
\begin{align}
\frac{d}{dt} x(t) &= (A + E) x(t) \\
\text{for continuous time systems, or} \\
x(k + 1) &= (A + E) x(k)
\end{align}

for discrete time systems. Here, $x(t) \in \mathbb{R}^n$ and $A, E \in \mathbb{R}^{n \times n}$. The $A$ matrix is the nominal system matrix with all eigenvalues lying in the specified $D$ region. The $E$ matrix represents the uncertainty in the system model. It is assumed that the system is subject to structured parametric uncertainties which can be described by:

\begin{align}
E &= \sum_{i=1}^{m} e_i E_i, \\
e_i &\in [-\varepsilon, +\varepsilon]
\end{align}

where $e_i, (i = 1, 2, ..., m)$ are uncertain parameters, $E_i (i = 1, ..., m)$ are constant real matrices and $\varepsilon \geq 0$. Our interest is to derive an upper bound $\bar{\varepsilon}$ of $\varepsilon$ such that for all $0 \leq \varepsilon < \bar{\varepsilon}$ the eigenvalues of the uncertain system (eqn. 7 or eqn. 8) with $E$ satisfying eqn. 9, are lying inside the given $D$-region of the complex plane.

Due to the fact that the eigenvalues of a matrix depend continuously on the elements of the matrix [6], we have the following lemma.

### 2.2 Lemma
Suppose that the system is described by eqn. 7 or eqn. 8, and that all eigenvalues of $A$ are located inside a $D$-region of the complex plane. A necessary and sufficient condition to maintain root clustering in the $D$-region for the perturbed matrices $A + E$ with $E$ satisfying eqn. 9 is that:

\begin{align}
\det \{\pi(\xi)I - A - \sum_{i=1}^{m} \varepsilon_i E_i\} &\neq 0 \\
\text{for all } \xi \in X \text{ and all } \varepsilon_i &\in [-\varepsilon, +\varepsilon].
\end{align}

Here, $\det(\cdot)$ is the determinant of a square matrix and $\pi(\xi)$, $\xi \in X$ is the function which parametrises the boundary of the $D$-region. $I$ is a unit matrix of appropriate dimensions.

### 2.3 Proof
Since all the eigenvalues of $A$ lie inside the specified $D$-region of the complex plane, and the eigenvalues of $A + E$ depend continuously on $e_i$, $(i = 1, ..., m)$, an eigenvalue of $A + E$ lies outside the $D$-region only if the root locus of $A + E$ intersects the border of the $D$-region. Then, there must exist at least one set of $e_i$, $i = 1, ..., m$, which we denote $e^*_i$, which makes one of the eigenvalues of $A + E$ for some $\xi^* \in X$ to satisfy:

\begin{align}
\det \{\pi(\xi^*)I - A - \sum_{i=1}^{m} \varepsilon^*_i E_i\} &= 0
\end{align}

Therefore, if eqn. 10 is satisfied for all $\xi \in X$, we know that the eigenvalues of $A + E$ matrix remain inside the specified $D$-region for the given class of uncertainties.

### 3 Main result
It is of practical importance for stability and performance robustness analysis to get an upper bound $\bar{\varepsilon}$ (eqn. 9) to guarantee root clustering in a given subregion of a complex plane for an uncertain linear system (eqn. 7 or eqn. 8). Such a $D$-stability measure should be as accurate as possible to avoid conservative judgments in a specific application.

#### 3.1 Theorem
Let the system be described by eqn. 7 or eqn. 8 with structured parameter uncertainty described by eqn. 9. An upperbound $\bar{\varepsilon}$ of $\varepsilon$ in eqn. 9 which guarantees root clustering in a given $D$-region bordered by the set $B = \pi(X)$ for all possible values of $\varepsilon_i \in [-\varepsilon, +\varepsilon], (i = 1, ..., m)$ with $\varepsilon < \bar{\varepsilon}$ is given by:

\begin{align}
\bar{\varepsilon} = \left(\min_{M \in \mathbb{R}^{n \times n}} \max_{\xi \in X} \|M^{-1} P(\xi) G_i^T M^{-1} P(\xi) G_i\|\right)^{-1}
\end{align}

where $M$ is the set of non-singular matrices, $P(\xi) = (\pi(\xi) I - A)^{-1}, G_i, i = 1, ..., 2^m$ are vertex matrices of eqn. 9, $\|\cdot\|$ represents the spectral norm of the matrix, and superscript $T$ denotes matrix transpose.

#### 3.2 Proof
From lemma 2.2, it is known that root clustering in any given subregion bordered by $\pi(\xi)$ with $\xi \in X$ in the
complex plane implies that:

$$\det \{ \pi(\xi)I - A - E \} \neq 0$$  \hspace{1cm} (13)

It follows that

$$\det \{ \pi(\xi)I - A \} \det \{ I - [\pi(\xi)I - A]^{-1}E \} \neq 0$$  \hspace{1cm} (14)

for all $\xi \in X$, as the eigenvalues of $A$ are all inside $D$. From lemma 2.2, we infer $\det \{ \pi(\xi)I - A \} \neq 0$, so eqn. 14 implies that

$$\det \{ I - [\pi(\xi)I - A]^{-1}E \} \neq 0$$  \hspace{1cm} (15)

for all $\xi \in X$. Since the determinant of a matrix is equal to the product of all eigenvalues of the matrix, eqn. 15 yields

$$\lambda \{ \pi(\xi)I - A \}^{-1}E \neq 1$$  \hspace{1cm} (16)

for all $\xi \in X$, where $\lambda(\cdot)$ denotes the set of eigenvalues. It follows that for all $\xi \in X$

$$\lambda \{ \pi(\xi)I - A \}^{-1}E \neq 1$$  \hspace{1cm} (17)

The eigenvalues do not change under similarity transformation. Hence, for all $\xi \in X$

$$\lambda \{ M^{-1}[\pi(\xi)I - A]^{-1}EM \} \neq 1$$  \hspace{1cm} (18)

which is the case if

$$\Re \lambda \{ M^{-1}[\pi(\xi)I - A]^{-1}EM \} \neq 1$$  \hspace{1cm} (19)

for all $\xi \in X$. One sufficient condition for this to hold is:

$$\begin{align*}
\lambda \left\{ \frac{1}{2} \left[ M^{-1}[\pi(\xi)I - A]^{-1}EM \right] \\
+ \frac{1}{2} \left[ M^{-1}[\pi(\xi)I - A]^{-1}EM \right]^T \right\} \neq 1
\end{align*}$$  \hspace{1cm} (20)

Denote $F(\xi) = [\pi(\xi)I - A]^{-1}$ and $G_p$, ($p = 1, 2, ..., 2^{m}$) as vertex matrices of $E$. The vertex matrices are formed by taking extreme point values of uncertain parameters in $E$. Then, eqn. 20 immediately yields [7] an upper bound $\bar{\varepsilon}$

$$\bar{\varepsilon} = \left( \min_{M, \varepsilon \in \mathbb{C}^n} \max_{\xi \in X} \left\{ \frac{1}{2} \left( M^{-1}F\{\xi\}G_pM^{-1}F\{\xi\}M^T \right) \right\} \right)^{-1}$$  \hspace{1cm} (21)

which guarantees root clustering in any given subregion $D$ of the complex plane.

The problem is formulated into a minimax framework. For each given choice of similarity transformation $M$ matrix, a corresponding $\varepsilon$ can be obtained from eqn. 21. It is a very hard problem to get an explicit analytical solution of $M$ for a maximal upper bound $\bar{\varepsilon}$. The numerical solution of the maximal $\varepsilon$ can be easily obtained by using a minimax optimisation program (for example, 'Minimax' of MATLAB).

4 Illustrative examples

Example 1

Consider a simple example used in [5]. The linear continuous-time system is described by state space model eqn. 7 with

$$A = \begin{pmatrix} -4.3 & -0.4 \\ 0.2 & -3.4 \end{pmatrix}; E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$  \hspace{1cm} (22)

The eigenvalues of the $A$ matrix are $\lambda_1 = -4.2, \lambda_2 = -3.5$. The specified root clustering region is a unit circle centered at $(a, b) = (-4, 0)$ as described in eqn. 4. The result of [5] is $\bar{\varepsilon} = 0.0317$.

The result of our new condition (eqn. 21), taking $M = I$, results in $\bar{\varepsilon} = 0.4$. This has drastically improved on the previous result. In fact, because $\lambda [A + 0.4(E_1 + E_2)] = -3$ which intersects the border of the specified unit circle, we know that $\bar{\varepsilon} < 0.4$ is actually the necessary and sufficient condition to guarantee root clustering in the given circle region. This also implies that searching a better similarity transformation $M$ matrix is not necessary.

Example 2

The continuous linear time system is described by a state space model (eqn. 7) with

$$A = \begin{pmatrix} -6 & -2 \\ 5 & 9 \end{pmatrix}; E_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}; E_2 = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix};$$  \hspace{1cm} (23)

The eigenvalues of $A$ are $\lambda_1, \lambda_2 = -3 \pm j$. The specified root clustering region is the common area bordered by a vertical line described by eqn. 1 with $a = -1$ and a sector area described by eqn. 3 with $a = 0$ and $\theta = \pi/4$. The eigenvalues of $A$ are inside the region.

The result of the sufficient condition of [1, 2] is $\bar{\varepsilon} = 0.1566$. The result of our new sufficient condition, taking $M = I$, yields $\bar{\varepsilon} = 1.0899$. The conservatism of the previous result is significantly reduced. Employing the optimisation program 'Minimax' of MATLAB, a better choice of $M$ is found to be

$$M = \begin{pmatrix} 0.9755 & 0.2713 \\ 0.2407 & 1.0167 \end{pmatrix};$$  \hspace{1cm} (24)

which results in $\bar{\varepsilon} = 1.1006$, which is approaching the necessary and sufficient condition $\varepsilon < 1.2192$ with $\lambda [A + 1.2192(E_1 + E_2)] = -1.7808 \pm j1.7808$ intersecting the border of the given region.

5 Conclusion

This paper presents a unified sufficient condition of root clustering for linear state space models (either continuous-time or discrete-time systems) subject to structured parameter uncertainty. The explicit upper bound for tolerable perturbation ranges is naturally a D-stability robustness measure. It is illustrated by examples that much less conservative results can be obtained by simply taking the similarity transformation $M$ matrix as a unit matrix in our proposed method compared with some existing methods. The involvement of the $M$ matrix provides a possibility to get more accurate results at the cost of more computation effort.

6 References

6. SHI, Z.C., and GAO, W.B.: 'A necessary and sufficient condition for tolerable perturbation ranges is naturally a D-stability robustness measure. It is illustrated by examples that much less conservative results can be obtained by simply taking the similarity transformation $M$ matrix as a unit matrix in our proposed method compared with some existing methods. The involvement of the $M$ matrix provides a possibility to get more accurate results at the cost of more computation effort.

6 References