\[
S^* = \frac{1}{4} \sum_{i=1}^{k} \sum_{j=1}^{k} \left( \frac{3}{2} + \epsilon_i \sqrt{3} - \epsilon_i^2 + \cdots \right) \left( \frac{3}{2} + \epsilon_j \sqrt{3} - \epsilon_j^2 + \cdots \right) \left[ (\epsilon_i - \epsilon_j)^2 + \cdots \right]
\]
\[
+ \frac{1}{8} \sum_{i=1}^{k} \sum_{j=k+1}^{n} \left( \frac{3}{2} + \epsilon_i \sqrt{3} - \epsilon_i^2 + \cdots \right) \left( \frac{3}{2} - \epsilon_j \sqrt{3} - \epsilon_j^2 + \cdots \right) \left[ \frac{3}{2} + (\epsilon_j - \epsilon_i) \sqrt{3} \right]
\]
\[
- (\epsilon_j - \epsilon_i)^2 + \cdots \]
\[
+ \frac{1}{8} \sum_{i=k+1}^{n} \sum_{j=1}^{k} \left( \frac{3}{2} - \epsilon_i \sqrt{3} - \epsilon_i^2 + \cdots \right) \left( \frac{3}{2} + \epsilon_j \sqrt{3} - \epsilon_j^2 + \cdots \right) \left[ \frac{3}{2} + (\epsilon_i - \epsilon_j) \sqrt{3} \right]
\]
\[
- (\epsilon_i - \epsilon_j)^2 + \cdots \]
\[
+ \frac{1}{4} \sum_{i=k+1}^{n} \sum_{j=k+1}^{n} \left( \frac{3}{2} - \epsilon_i \sqrt{3} - \epsilon_i^2 + \cdots \right) \left( \frac{3}{2} - \epsilon_j \sqrt{3} - \epsilon_j^2 + \cdots \right) \left[ (\epsilon_i - \epsilon_j)^2 + \cdots \right]
\]
\[
= \frac{27}{32} k(n - k) \frac{9}{8} (2n - 3k) \sum_{j=1}^{k} \epsilon_j^2 - \frac{9}{8} (3k - n) \sum_{j=k+1}^{n} \epsilon_j^2 - \frac{9}{8} \left( \sum_{j=1}^{k} \epsilon_j - \sum_{j=k+1}^{n} \epsilon_j \right)^2 + \cdots,
\]
in which \(27k(n - k)/32 = 27[n/2](n - [n/2])/32\) is the value of the given \(S\) when solution (2) is inserted into it and the result neglects only fourth and higher degree terms in the \(\epsilon\)'s. Note that
\[
2n - 3 \left\lfloor \frac{n}{2} \right\rfloor > 0 \quad \text{for} \quad n > 2 \quad \text{and} \quad 3 \left\lfloor \frac{n}{2} \right\rfloor - n > 0 \quad \text{for} \quad n \geq 2, \quad n \neq 3,
\]
both expressions being equal to \(n/2\) when \(n\) is even. For \(n\) odd,
\[
2n - 3 \left\lfloor \frac{n}{2} \right\rfloor = \frac{n + 3}{2} > 0 \quad \text{and} \quad 3 \left\lfloor \frac{n}{2} \right\rfloor - n = \frac{n - 3}{2} \left\{ \begin{array}{ll}
> 0, & n \geq 5, \\
= 0, & n = 3.
\end{array} \right.
\]
The obtained formula proves that \(S = 27k(n - k)/32\) is a local maximum, since there exists a neighborhood around (2) in which \(S\) can take on only smaller values than at (2).

**A Function Arising in One-Dimensional Percolation**

*Problem 95-16, by M. L. Glasser (Clarkson University).*

There has been controversy over the function

\[
G(t) = \int_{0}^{\infty} e^{-t/x} e^{-\frac{1}{2}x^2} \, dx,
\]

which arises in studies of hopping transport for one-dimensional percolation, particularly concerning its behavior for large and small \(t\) [1]. Find computationally effective expansions for this function.

**REFERENCE**

We shall derive two expansions for the function $G(t)$, namely, a series expansion in powers of $t$ and $\log t$ that is convergent for $t \geq 0$, and an asymptotic expansion valid for $t \to \infty$.

First we determine the Mellin transform of $G(t)$:

$$\mathcal{M}\{G(t)\} = \int_0^\infty G(t)t^{z-1}dt = \int_0^\infty e^{-x^2/2}dxe^{-t/x}t^{z-1}dt$$

$$= \Gamma(z) \int_0^\infty e^{-x^2/2}x^2dx = 2^{(z-1)/2}\Gamma(z)\Gamma\left(\frac{1}{2}z + \frac{1}{2}\right),$$

valid for $\Re z > 0$. Then, inversely, $G(t)$ can be represented by the Mellin–Barnes integral

$$G(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z)\Gamma\left(\frac{1}{2}z + \frac{1}{2}\right)2^{(z-1)/2}t^{-z}dz, \quad c > 0.$$

Here the integrand has simple poles at $z = -2n$ and double poles at $z = -2n - 1$, where $n = 0, 1, 2, \ldots$. By closing the contour to the left, the integral is evaluated by standard residue calculus with the result

$$G(t) = 2^{-1/2}\pi \sum_{n=0}^{\infty} \frac{(-1)^n2^{-n}}{(2n)!\Gamma(n + \frac{1}{2})}t^{2n}$$

$$- \sum_{n=0}^{\infty} \frac{(-1)^n2^{-n}}{(2n+1)!n!} \left[ \psi(2n + 2) + \frac{1}{2}\psi(n + 1) - \log(2^{-1/2}t) \right]t^{2n+1}$$

$$= \left(\frac{\pi}{2}\right)^{1/2} {}_0F_2\left(\frac{1}{2}, \frac{1}{2}; -t^2/8\right) - \left[1 - \frac{3}{2}\gamma - \log(2^{-1/2}t)\right]t_0F_2\left(1, \frac{3}{2}; -t^2/8\right)$$

$$- t \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(\frac{1}{2}n)!} \left(\sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{2k+1}\right)\left(\frac{t^2}{8}\right)^n.$$

Here, $\psi(x) = \Gamma'(x)/\Gamma(x)$, while $\gamma$ denotes Euler’s constant and the notation $0F_2$ stands for the generalized hypergeometric function. Obviously, the series expansion for $G(t)$ is convergent for $t \geq 0$.

Next we turn to the asymptotics of $G(t)$ as $t \to \infty$. Since the exponent $t/x + x^2/2$ is stationary at $x = t^{1/3}$, we substitute $x = t^{1/3}u$, whereupon the integral passes into

$$G(t) = t^{1/3} \int_0^\infty \exp\left[-t^{2/3}\left(u^{-1} + \frac{1}{2}u^2\right)\right]du.$$
Around \( x = 0 \) the inverse function \( u = u(x) \) has the Taylor expansion

\[
  u = u(x) = 1 + \sum_{n=1}^{\infty} a_n x^n,
\]

in which the coefficients \( a_n \) are readily determined by means of a computer algebra program package. The asymptotic expansion of \( G(t) \) now follows by the use of Watson’s lemma [1, §4.1]. Thus we replace \( du(x)/dx \) by its Taylor expansion and integrate term by term, with the result

\[
  G(t) \sim t^{1/3} \exp \left[ -\frac{3}{2} t^{2/3} \right] \sum_{n=1}^{\infty} n a_n \int_{-\infty}^{\infty} \exp \left[ -\frac{3}{2} t^{2/3} x^2 \right] x^{n-1} dx
\]

\[
  = \exp \left[ -\frac{3}{2} t^{2/3} \right] \sum_{n=0}^{\infty} (2n + 1) \Gamma \left( n + \frac{1}{2} \right) a_{2n+1} \left( \frac{2}{3} \right)^{n+\frac{1}{2}} t^{-2n/3} \quad (t \to \infty).
\]

By the use of MAPLE we have evaluated the first seven coefficients \( a_n \). The corresponding four-term asymptotic expansion is found to be

\[
  G(t) = \left( \frac{2\pi}{3} \right)^{1/2} \exp \left[ -\frac{3}{2} t^{2/3} \right] \left\{ 1 - \frac{1}{18} t^{-2/3} + \frac{25}{648} t^{-4/3} - \frac{1225}{34992} t^{-2} + O(t^{-8/3}) \right\} \quad (t \to \infty).
\]

It has been verified that for \( t \geq 3 \) the present expansion reproduces \( G(t) \) with a relative error less than 0.0013. Over the range \( 0 \leq t \leq 3 \), the function \( G(t) \) may be calculated from the power series expansion presented above.

**Additional remarks.** (1) The Mellin–Barnes integral representation of \( G(t) \) can be rewritten in terms of the definition [2, Form. 5.3(1)] of Meijer’s \( G \)-function, thus leading to

\[
  G(t) = (2\pi)^{-1/2} G_{03}^{30} \left( t^2/8 \mid 0, \frac{1}{2}, \frac{1}{2} \right).
\]

(2) It is easily verified that \( y = G(t) \) is a solution of the differential equation

\[
  ty''' + y'' + y = 0,
\]

which has a regular singularity at \( t = 0 \). Using the method of Frobenius [3, Chap. XVI], one may construct three independent series solutions viz.

\[
  y_1(t) = {}_0F_2 \left( \frac{1}{2}, \frac{1}{2}; -t^2/8 \right), \quad y_2(t) = t {}_0F_2 \left( 1, \frac{3}{2}; -t^2/8 \right),
\]

\[
  y_3(t) = \frac{\partial}{\partial \sigma} \left[ t^\sigma {}_1F_3 \left( 1; \frac{1}{2} + \frac{1}{2} \sigma, \frac{1}{2} + \frac{1}{2} \sigma, 1 + \frac{1}{2} \sigma; -t^2/8 \right) \right]_{\sigma=1}
\]

\[
  = t \log t {}_0F_2 \left( 1, \frac{3}{2}; -t^2/8 \right) - t \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{1}{2k+1} \right) \left( \frac{t^2}{8} \right)^n.
\]

Clearly,

\[
  G(t) = \left( \frac{\pi}{2} \right)^{1/2} y_1(t) - \left( 1 - \frac{3}{2} y + \frac{1}{2} \log 2 \right) y_2(t) + y_3(t).
\]
(3) The differential equation (*) may also be solved by the method of Laplace transformation [3, Chap. XVIII], leading to contour-integral solutions of the form

\[ y(t) = \int_C e^{-t/z} W(z) \, dz. \]

This function \( y(t) \) satisfies the differential equation provided that

\[ \frac{d}{dz} (z^{-1} W) + (z^{-2} + 1) W = 0, \quad \text{hence} \quad W(z) = e^{-z^2/2}, \]

and

\[ [z^{-1} e^{-t/z} W(z)]_C = [z^{-1} e^{-t/z} - z^2/2]_C = 0. \]

The latter condition is satisfied if the contour \( C \) is taken along the positive real axis, whereupon the original integral representation for \( G(t) \) is recovered. Other possible choices for \( C \) are a simple closed contour encircling \( z = 0 \) in the positive direction, or a contour along the real axis with an indentation above or below \( z = 0 \). The corresponding contour-integral solutions can be shown to be related to the series solutions \( y_1(t) \) and \( y_2(t) \) by

\[
\int_{(0+)} e^{-t/z} e^{-z^2/2} \, dz = -2\pi i y_2(t),
\]

\[
\int_{-\infty}^{\infty} e^{-t/z} e^{-z^2/2} \, dz = (2\pi)^{1/2} y_1(t) + \pi i y_2(t),
\]

\[
\int_{-\infty}^{\infty} e^{-t/z} e^{-z^2/2} \, dz = (2\pi)^{1/2} y_1(t) - \pi i y_2(t).
\]

(4) The transformation of variables \( x = x(u) \) is given by

\[ x^2 = \frac{2}{3} \left( u^{-1} + \frac{1}{2} u^2 \right) - 1 = \frac{(u - 1)^2(u + 2)}{3u}, \quad \text{sgn } x = \text{sgn}(u - 1); \]

hence

\[ x = x(u) = (u - 1) \left( \frac{u + 2}{3u} \right)^{1/2}. \]

The Taylor coefficients \( a_n \) of the inverse function \( u = u(x) \) can also be determined by means of Lagrange's inversion formula [4, §7.32] viz.

\[ a_n = \frac{1}{n!} \frac{d^{n-1}}{du^{n-1}} \left[ \left( \frac{3u}{u + 2} \right)^{n/2} \right] = \frac{1}{n!} \frac{d^{n-1}}{du^{n-1}} \left[ \left( \frac{1 + u}{1 + u/3} \right)^{n/2} \right]_{u=0}. \]

As for the coefficients \( a_{2n+1} \) of odd index appearing in the asymptotic expansion of \( G(t) \), the 2nth derivative may be evaluated via the expansion of \((1 + u)^{n+1/2}\) and \((1 + u/3)^{-n-1/2}\) in binomial series. Thus it is found that \( a_{2n+1} \) is expressible in terms of a hypergeometric function \( F \) of argument 1/3 viz.

\[ a_{2n+1} = \frac{(-1)^n (-\frac{1}{2})^n}{2^n n!} F\left(-2n, n + \frac{1}{2}; \frac{3}{2} - n; \frac{1}{3}\right). \]
By an application of Petkovsek's algorithm, Koepf [5] has proven that there is no closed-form expression for the present $F\left(\frac{1}{3}\right)$. An application of the computer algebra procedure "sumrecursion," due to Koepf [6], leads to the recurrence relation

$$6(n + 1)(2n + 3)a_{2n+1} + (12n^2 + 12n + 1)a_{2n+1} + 2(n - 1)(2n + 1)a_{2n-1} = 0,$$

valid for $n = 0, 1, 2, \ldots$, with initial values $a_{-1} = 0$ and $a_1 = 1$. This relation is most useful for the calculation of the coefficients $a_{2n+1}$.

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Expansions for Elementary Functions

Problem 95-19, by JOE KEANE (Union City, CA).

Consider this series:

$$S = \sum_{j=0}^{\infty} (A + jB) \left\{ M \prod_{k=1}^{j} \frac{18k(2k - 1)}{(6k + 1)(6k + 5)} X \right\}.$$

Show that we obtain $S = \arctan t$ with the following values:

$$A = 15 + 25t^2 + 8t^4,$$
$$B = 18 + 30t^2 + 8t^4,$$
$$M = t/(15(1 + t^2)^2),$$
$$X = -4t^6/[27(1 + t^2)^2].$$

Also show that we obtain $S = \log z$ with the following values:

$$A = 10z(z^2 + 10z + 1) - (z - 1)^4,$$
$$B = 12z(z^2 + 10z + 1) - 2(z - 1)^4,$$
$$M = (z - 1)/(60z^2(z + 1)),$$
$$X = (z - 1)^6/[108z^2(z + 1)^2].$$