Solution to problem 95-16 : A function arising in one-dimensional percolation

Citation for published version (APA):

Boersma, J. (1996). Solution to problem 95-16 : A function arising in one-dimensional percolation. *SIAM Review*, 38(4), 671-675. <https://doi.org/10.1137/1038127>

DOI:

[10.1137/1038127](https://doi.org/10.1137/1038127)

Document status and date:

Published: 01/01/1996

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

$$\begin{aligned}
S^* &= \frac{1}{4} \sum_{i=1}^k \sum_{j=1}^k \left(\frac{3}{2} + \epsilon_i \sqrt{3} - \epsilon_i^2 + \dots \right) \left(\frac{3}{2} + \epsilon_j \sqrt{3} - \epsilon_j^2 + \dots \right) [(\epsilon_i - \epsilon_j)^2 + \dots] \\
&+ \frac{1}{8} \sum_{i=1}^k \sum_{j=k+1}^n \left(\frac{3}{2} + \epsilon_i \sqrt{3} - \epsilon_i^2 + \dots \right) \left(\frac{3}{2} - \epsilon_j \sqrt{3} - \epsilon_j^2 + \dots \right) \left[\frac{3}{2} + (\epsilon_j - \epsilon_i) \sqrt{3} \right. \\
&\quad \left. - (\epsilon_j - \epsilon_i)^2 + \dots \right] \\
&+ \frac{1}{8} \sum_{i=k+1}^n \sum_{j=1}^k \left(\frac{3}{2} - \epsilon_i \sqrt{3} - \epsilon_i^2 + \dots \right) \left(\frac{3}{2} + \epsilon_j \sqrt{3} - \epsilon_j^2 + \dots \right) \left[\frac{3}{2} + (\epsilon_i - \epsilon_j) \sqrt{3} \right. \\
&\quad \left. - (\epsilon_i - \epsilon_j)^2 + \dots \right] \\
&+ \frac{1}{4} \sum_{i=k+1}^n \sum_{j=k+1}^n \left(\frac{3}{2} - \epsilon_i \sqrt{3} - \epsilon_i^2 + \dots \right) \left(\frac{3}{2} - \epsilon_j \sqrt{3} - \epsilon_j^2 + \dots \right) [(\epsilon_i - \epsilon_j)^2 + \dots] \\
&= \frac{27}{32} k(n-k) - \frac{9}{8} (2n-3k) \sum_{j=1}^k \epsilon_j^2 - \frac{9}{8} (3k-n) \sum_{j=k+1}^n \epsilon_j^2 - \frac{9}{8} \left(\sum_{j=1}^k \epsilon_j - \sum_{j=k+1}^n \epsilon_j \right)^2 + \dots,
\end{aligned}$$

in which $27k(n-k)/32 = 27\lfloor n/2 \rfloor (n - \lfloor n/2 \rfloor)/32$ is the value of the given S when solution (2) is inserted into it and the result neglects only fourth and higher degree terms in the ϵ 's. Note that

$$2n - 3 \left\lfloor \frac{n}{2} \right\rfloor > 0 \quad \text{for } n > 2 \quad \text{and} \quad 3 \left\lfloor \frac{n}{2} \right\rfloor - n > 0 \quad \text{for } n \geq 2, \quad n \neq 3,$$

both expressions being equal to $n/2$ when n is even. For n odd,

$$2n - 3 \left\lfloor \frac{n}{2} \right\rfloor = \frac{n+3}{2} > 0 \quad \text{and} \quad 3 \left\lfloor \frac{n}{2} \right\rfloor - n = \frac{n-3}{2} \begin{cases} > 0, & n \geq 5, \\ = 0, & n = 3. \end{cases}$$

The obtained formula proves that $S = 27k(n-k)/32$ is a local maximum, since there exists a neighborhood around (2) in which S can take on only smaller values than at (2).

A Function Arising in One-Dimensional Percolation

Problem 95-16, by M. L. GLASSER (Clarkson University).

There has been controversy over the function

$$G(t) = \int_0^\infty e^{-t/x} e^{-\frac{1}{2}x^2} dx,$$

which arises in studies of hopping transport for one-dimensional percolation, particularly concerning its behavior for large and small t [1]. Find computationally effective expansions for this function.

REFERENCE

- [1] J. BERNASCONI, *Hopping transport in a one-dimensional percolation model: A comment*, Phys. Rev. B, 25 (1982), pp. 1394–1395.

Solution by J. BOERSMA (Eindhoven University of Technology, Eindhoven, The Netherlands).

We shall derive two expansions for the function $G(t)$, namely, a series expansion in powers of t and $\log t$ that is convergent for $t \geq 0$, and an asymptotic expansion valid for $t \rightarrow \infty$.

First we determine the Mellin transform of $G(t)$:

$$\begin{aligned}\mathcal{M}\{G(t)\} &= \int_0^\infty G(t)t^{z-1}dt = \int_0^\infty e^{-x^2/2}dx \int_0^\infty e^{-t/x}t^{z-1}dt \\ &= \Gamma(z) \int_0^\infty e^{-x^2/2}x^zdx = 2^{(z-1)/2}\Gamma(z)\Gamma\left(\frac{1}{2}z + \frac{1}{2}\right),\end{aligned}$$

valid for $\operatorname{Re} z > 0$. Then, inversely, $G(t)$ can be represented by the Mellin–Barnes integral

$$G(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(z)\Gamma\left(\frac{1}{2}z + \frac{1}{2}\right) 2^{(z-1)/2}t^{-z}dz, \quad c > 0.$$

Here the integrand has simple poles at $z = -2n$ and double poles at $z = -2n - 1$, where $n = 0, 1, 2, \dots$. By closing the contour to the left, the integral is evaluated by standard residue calculus with the result

$$\begin{aligned}G(t) &= 2^{-1/2}\pi \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-n}}{(2n)! \Gamma(n + \frac{1}{2})} t^{2n} \\ &\quad - \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-n}}{(2n+1)! n!} \left[\psi(2n+2) + \frac{1}{2}\psi(n+1) - \log(2^{-1/2}t) \right] t^{2n+1} \\ &= \left(\frac{\pi}{2}\right)^{1/2} {}_0F_2\left(\frac{1}{2}, \frac{1}{2}; -t^2/8\right) - \left[1 - \frac{3}{2}\gamma - \log(2^{-1/2}t)\right] t {}_0F_2\left(1, \frac{3}{2}; -t^2/8\right) \\ &\quad - t \sum_{n=1}^{\infty} \frac{(-1)^n}{n! (\frac{3}{2})_n n!} \left(\sum_{k=1}^n \frac{1}{k} + \sum_{k=1}^n \frac{1}{2k+1} \right) \left(\frac{t^2}{8}\right)^n.\end{aligned}$$

Here, $\psi(x) = \Gamma'(x)/\Gamma(x)$, while γ denotes Euler's constant and the notation ${}_0F_2$ stands for the generalized hypergeometric function. Obviously, the series expansion for $G(t)$ is convergent for $t \geq 0$.

Next we turn to the asymptotics of $G(t)$ as $t \rightarrow \infty$. Since the exponent $t/x + x^2/2$ is stationary at $x = t^{1/3}$, we substitute $x = t^{1/3}u$, whereupon the integral passes into

$$G(t) = t^{1/3} \int_0^\infty \exp\left[-t^{2/3}\left(u^{-1} + \frac{1}{2}u^2\right)\right] du.$$

The exponent $u^{-1} + \frac{1}{2}u^2$ attains its absolute minimum $3/2$ at $u = 1$. Following Laplace's method [1, §5.1], we apply the transformation of integration variables $x = x(u)$ or, inversely, $u = u(x)$, specified by

$$u^{-1} + \frac{1}{2}u^2 = \frac{3}{2}(1+x^2), \quad \operatorname{sgn}(u-1) = \operatorname{sgn} x.$$

Then the integral for $G(t)$ transforms into

$$G(t) = t^{1/3} \exp\left[-\frac{3}{2}t^{2/3}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{3}{2}t^{2/3}x^2\right] \frac{du(x)}{dx} dx.$$

Around $x = 0$ the inverse function $u = u(x)$ has the Taylor expansion

$$u = u(x) = 1 + \sum_{n=1}^{\infty} a_n x^n,$$

in which the coefficients a_n are readily determined by means of a computer algebra program package. The asymptotic expansion of $G(t)$ now follows by the use of Watson's lemma [1, §4.1]. Thus we replace $du(x)/dx$ by its Taylor expansion and integrate term by term, with the result

$$\begin{aligned} G(t) &\sim t^{1/3} \exp \left[-\frac{3}{2} t^{2/3} \right] \sum_{n=1}^{\infty} n a_n \int_{-\infty}^{\infty} \exp \left[-\frac{3}{2} t^{2/3} x^2 \right] x^{n-1} dx \\ &= \exp \left[-\frac{3}{2} t^{2/3} \right] \sum_{n=0}^{\infty} (2n+1) \Gamma \left(n + \frac{1}{2} \right) a_{2n+1} \left(\frac{2}{3} \right)^{n+\frac{1}{2}} t^{-2n/3} \quad (t \rightarrow \infty). \end{aligned}$$

By the use of MAPLE we have evaluated the first seven coefficients a_n . The corresponding four-term asymptotic expansion is found to be

$$G(t) = \left(\frac{2\pi}{3} \right)^{1/2} \exp \left[-\frac{3}{2} t^{2/3} \right] \left\{ 1 - \frac{1}{18} t^{-2/3} + \frac{25}{648} t^{-4/3} - \frac{1225}{34992} t^{-2} + O(t^{-8/3}) \right\} \quad (t \rightarrow \infty).$$

It has been verified that for $t \geq 3$ the present expansion reproduces $G(t)$ with a relative error less than 0.0013. Over the range $0 \leq t \leq 3$, the function $G(t)$ may be calculated from the power series expansion presented above.

Additional remarks. (1) The Mellin–Barnes integral representation of $G(t)$ can be rewritten in terms of the definition [2, Form. 5.3(1)] of Meijer's G -function, thus leading to

$$G(t) = (2\pi)^{-1/2} G_{03}^{30} \left(t^2/8 \mid 0, \frac{1}{2}, \frac{1}{2} \right).$$

(2) It is easily verified that $y = G(t)$ is a solution of the differential equation

$$(*) \quad t y''' + y'' + y = 0,$$

which has a regular singularity at $t = 0$. Using the method of Frobenius [3, Chap. XVI], one may construct three independent series solutions viz.

$$y_1(t) = {}_0F_2 \left(\frac{1}{2}, \frac{1}{2}; -t^2/8 \right), \quad y_2(t) = t {}_0F_2 \left(1, \frac{3}{2}; -t^2/8 \right),$$

$$y_3(t) = \frac{\partial}{\partial \sigma} \left[t^\sigma {}_1F_3 \left(1; \frac{1}{2} + \frac{1}{2}\sigma, \frac{1}{2} + \frac{1}{2}\sigma, 1 + \frac{1}{2}\sigma; -t^2/8 \right) \right]_{\sigma=1}$$

$$= t \log t {}_0F_2 \left(1, \frac{3}{2}; -t^2/8 \right) - t \sum_{n=1}^{\infty} \frac{(-1)^n}{n! (\frac{3}{2})_n n!} \left(\sum_{k=1}^n \frac{1}{k} + \sum_{k=1}^n \frac{1}{2k+1} \right) \left(\frac{t^2}{8} \right)^n.$$

Clearly,

$$G(t) = \left(\frac{\pi}{2} \right)^{1/2} y_1(t) - \left(1 - \frac{3}{2} \gamma + \frac{1}{2} \log 2 \right) y_2(t) + y_3(t).$$

(3) The differential equation (*) may also be solved by the method of Laplace transformation [3, Chap. XVIII], leading to contour-integral solutions of the form

$$y(t) = \int_C e^{-t/z} W(z) dz.$$

This function $y(t)$ satisfies the differential equation provided that

$$\frac{d}{dz}(z^{-1}W) + (z^{-2} + 1)W = 0, \quad \text{hence} \quad W(z) = e^{-z^2/2},$$

and

$$[z^{-1}e^{-t/z}W(z)]_C = [z^{-1}e^{-t/z-z^2/2}]_C = 0.$$

The latter condition is satisfied if the contour C is taken along the positive real axis, whereupon the original integral representation for $G(t)$ is recovered. Other possible choices for C are a simple closed contour encircling $z = 0$ in the positive direction, or a contour along the real axis with an indentation above or below $z = 0$. The corresponding contour-integral solutions can be shown to be related to the series solutions $y_1(t)$ and $y_2(t)$ by

$$\begin{aligned} \int_{(0+)} e^{-t/z} e^{-z^2/2} dz &= -2\pi i y_2(t), \\ \oint_{-\infty}^{\infty} e^{-t/z} e^{-z^2/2} dz &= (2\pi)^{1/2} y_1(t) + \pi i y_2(t), \\ \oint_{-\infty}^{\infty} e^{-t/z} e^{-z^2/2} dz &= (2\pi)^{1/2} y_1(t) - \pi i y_2(t). \end{aligned}$$

(4) The transformation of variables $x = x(u)$ is given by

$$x^2 = \frac{2}{3} \left(u^{-1} + \frac{1}{2} u^2 \right) - 1 = \frac{(u-1)^2(u+2)}{3u}, \quad \text{sgn } x = \text{sgn}(u-1);$$

hence

$$x = x(u) = (u-1) \left(\frac{u+2}{3u} \right)^{1/2}.$$

The Taylor coefficients a_n of the inverse function $u = u(x)$ can also be determined by means of Lagrange's inversion formula [4, §7.32] viz.

$$a_n = \frac{1}{n!} \frac{d^{n-1}}{du^{n-1}} \left[\left(\frac{3u}{u+2} \right)^{n/2} \right]_{u=1} = \frac{1}{n!} \frac{d^{n-1}}{du^{n-1}} \left[\left(\frac{1+u}{1+u/3} \right)^{n/2} \right]_{u=0}.$$

As for the coefficients a_{2n+1} of odd index appearing in the asymptotic expansion of $G(t)$, the $2n$ th derivative may be evaluated via the expansion of $(1+u)^{n+1/2}$ and $(1+u/3)^{-n-1/2}$ in binomial series. Thus it is found that a_{2n+1} is expressible in terms of a hypergeometric function F of argument $1/3$ viz.

$$a_{2n+1} = \frac{(-1)^n \left(-\frac{1}{2}\right)_n}{2^{2n} n!} F \left(-2n, n + \frac{1}{2}; \frac{3}{2} - n; \frac{1}{3} \right).$$

By an application of Petkovsek's algorithm, Koepf [5] has proven that there is no closed-form expression for the present $F(\frac{1}{3})$. An application of the computer algebra procedure "sumrecursion," due to Koepf [6], leads to the recurrence relation

$$6(n+1)(2n+3)a_{2n+3} + (12n^2 + 12n + 1)a_{2n+1} + 2(n-1)(2n+1)a_{2n-1} = 0,$$

valid for $n = 0, 1, 2, \dots$, with initial values $a_{-1} = 0$ and $a_1 = 1$. This relation is most useful for the calculation of the coefficients a_{2n+1} .

REFERENCES

- [1] N. BLEISTEIN AND R. A. HANDELSMAN, *Asymptotic Expansions of Integrals*, Dover, New York, 1986.
- [2] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, 1953.
- [3] E. L. INCE, *Ordinary Differential Equations*, Dover, New York, 1956.
- [4] E. T. WHITTAKER AND G. N. WATSON, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge, UK, 1958.
- [5] W. KOEPF, private communication.
- [6] ———, *Algorithms for m-fold hypergeometric summation*, J. Symbolic Comput., 20 (1995), pp. 399–417.

Also solved by PAUL BRACKEN (University of Florida), BRIAN BRADIE (Christopher Newport University), HONGWEI CHEN (Christopher Newport University), S. L. COLE and J. J. LEWANDOWSKI (Rensselaer Polytechnic Institute), CARL C. GROSJEAN (University of Ghent, Belgium), W. B. JORDAN (Scotia, NY), KEE-WAI LAU (Hong Kong), ALLEN R. MILLER (Washington, DC), and the proposer.

Expansions for Elementary Functions

Problem 95-19, by JOE KEANE (Union City, CA).

Consider this series:

$$S = \sum_{j=0}^{\infty} (A + jB) \left\{ M \prod_{k=1}^j \frac{18k(2k-1)}{(6k+1)(6k+5)} X \right\}.$$

Show that we obtain $S = \arctan t$ with the following values:

$$A = 15 + 25t^2 + 8t^4,$$

$$B = 18 + 30t^2 + 8t^4,$$

$$M = t / \{15(1 + t^2)^2\},$$

$$X = -4t^6 / \{27(1 + t^2)^2\}.$$

Also show that we obtain $S = \log z$ with the following values:

$$A = 10z(z^2 + 10z + 1) - (z - 1)^4,$$

$$B = 12z(z^2 + 10z + 1) - 2(z - 1)^4,$$

$$M = (z - 1) / \{60z^2(z + 1)\},$$

$$X = (z - 1)^6 / \{108z^2(z + 1)^2\}.$$