A note on the reduced creep function corresponding to the quasi-linear visco-elastic model proposed by Fung

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References


A Note on the Reduced Creep Function Corresponding to the Quasi-Linear Visco-Elastic Model Proposed by Fung

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For description of the visco-elastic behavior of soft biological tissues, Fung proposed a visco-elastic model formulated in terms of a relaxation function and corresponding relaxation spectrum. For the corresponding creep function, Fung proposed an expression which needs correction to obtain a consistent formulation. This creep function and the corresponding creep spectrum are derived in this note.

Introduction

In order to describe the nonlinear visco-elastic behavior of soft biological tissues under uniaxial deformation, Fung (1981) proposed a quasi-linear visco-elastic model in which stress \( \sigma \) and strain \( \epsilon \) are related as

\[
\sigma(t) = \int_0^t G(t - \tau) \frac{d\sigma}{d\tau} d\tau
\]

(1)

In this expression \( t \) represents time and \( \sigma'(\epsilon) \) is the elastic response which can be a nonlinear-function of strain. \( G(t) \) is the so-called relaxation function (R.R.F) which, in general, can be written in terms of a relaxation spectrum \( S(\tau) \) (Fung, 1981):

\[
G(t) = \frac{1}{1 + \int_0^\infty S(\tau) e^{-t/\tau} d\tau}
\]

(2)

with

\[
G(t = 0) = 1
\]

\[
G(t \rightarrow \infty) = G(\infty) = \frac{1}{1 + \int_0^\infty S(\tau) d\tau}
\]

To account for the weakly frequency dependent behavior of soft biological tissues Fung (1981) proposed a specific spectrum of the form

\[
S(\tau) = \begin{cases} K/\tau_1 \leq \tau \leq \tau_2 \\ 0 \text{ else} \end{cases}
\]

(3)

In this particular case \( G(t) \) can be written as

\[
G(t) = \frac{1 + K(\tau_2/t - \tau_1/t)}{1 + K \ln(\tau_2/\tau_1)}
\]

(4)

where \( E_1(x) = \int_x^\infty e^{-y/y} dy \) is the exponential integral function.

The stress response at time \( t \) for a known strain history can easily be evaluated by means of Eq. (1). However, the use of this expression for the determination of strain at time \( t \) for a given stress history is much more laborious because this involves solution of a nonlinear integral equation. To avoid this problem it is useful to transform Eq. (1) into an expression in which the elastic response \( \sigma'(\epsilon) \) is directly related to stress \( \sigma \).

This can easily be done when use is made of the so-called reduced creep function (R.C.F.) \( J(t) \), by means of which (1) transforms into

\[
\sigma'(\epsilon(t)) = \int_0^t J(t - \tau) \frac{d\sigma}{d\tau} d\tau
\]

(5)

This expression is easily derived by taking the Laplace transform of Eq. (1)

\[
\tilde{\sigma}(s) = G(s) \cdot \tilde{\sigma}'(s)
\]

(6)

with the Laplace transform \( \tilde{f}(s) \) of a function \( f(t) \) given by

\[
\tilde{f}(s) = \int_0^\infty f(t) e^{-st} dt
\]

(7)

From Eq. (6) it follows

\[
\tilde{\sigma}'(s) = \frac{1}{sG(s)} \tilde{\sigma}(s) = \tilde{J}(s) \tilde{\sigma}(s)
\]

(8)

From Eq. (8), Eq. (5) is easily derived. Also we see that the R.R.F. and the R.C.F. are not independent

\[
\tilde{J}(s) = G(s) = 1/s^2
\]

(9)

or

\[
\int_0^t G(t - \tau) J(\tau) d\tau = t
\]
From Eq. (9) it is easily found that

\[ J(t=0) = 1 \]

\[ J(t=\infty) = J(\infty) = 1/G(\infty) \quad (11) \]

Fung (1981) gives an expression for the R.C.F. corresponding to the R.R.F. given by Eq. (4), denoted by \( J_f(t) \):

\[ J_f(t) = \frac{1 - K \left( E_1 \left( \frac{t}{K + \tau_2} \right) - E_1 \left( \frac{t}{K + \tau_1} \right) \right)}{1 - K \ln \left( \frac{K + \tau_2}{K + \tau_1} \right)} \quad (12) \]

with \( J(\infty) \) as defined by Eq. (11)

\[ J(\infty) = 1/G(\infty) = \frac{1}{1 + K \ln(\tau_2/\tau_1)} \quad (14) \]

Now \( J(t) \) can be obtained from equation (13) by taking the inverse Laplace transform, formally written as:

\[ J(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} J(s) e^{st} ds \quad (15) \]

with \( j^2 = -1 \).

To evaluate the Bromwich contour in (15), Cauchy’s residue theorem can be applied (Kuhfittig, 1978). Evaluation of the complex function \( J(s) \) reveals that \( J(s) \) has branchpoints at \( s = -\tau_2 \) and \( s = -\tau_1 \) where \( \tau_0 \) can be found from

\[ 1 + K \ln \left( \frac{1 - S_0 \tau_2}{1 - S_0 \tau_1} \right) = 0 \]

or

\[ S_0 = \frac{e^{1/K} - 1}{\tau_2 e^{1/K} - \tau_1} \quad (17) \]

As \( \tau_2 > \tau_1 \) and \( K > 0 \), \( S_0 \) satisfies

\[ 0 < S_0 < 1/\tau_2 \quad (18) \]

Furthermore \( J(s) \) has branchpoints at \( s = -\tau_2 \) and \( s = -\tau_1 \) in order to evaluate the Bromwich contour integral (15) two branch cuts are introduced as shown in Fig. 1. Then with the aid of Cauchy’s residue theorem \( J(t) \) can be found from

\[ J(t) = \text{Res}[J(s) e^{st}]_{s=0} + \text{Res}[J(s) e^{st}]_{s=-S_0} \]

\[ + \frac{1}{2\pi j} \int_{\text{branches}} J(s) e^{st} ds \quad (19) \]

where \( \text{Res} \left[ J(s) \right]_{s=0} \) is the residue of a function \( J(s) \) at \( s=a \).

The residue at \( s=0 \) can easily be found from

\[ \text{Res}[J(s)], s=0 = \lim_{s \to 0} (s - 0) J(s) e^{st} \]

whereas the residue at \( s = -S_0 \) is found from

\[ \text{Res}[J(s) e^{st}]_{s=-S_0} = \lim_{s \to -S_0} (s + S_0) J(s) e^{st} \]

\[ = -J(\infty) \left( \frac{(1 - S_0 \tau_2) (1 - S_0 \tau_1)}{K S_0 (\tau_2 - \tau_1)} \right) e^{-S_0 t} \quad (20) \]

In order to work out the contour integrals in (19) first the integral along the lower branch (LB) is considered (see also Fig. 1). As on the lower branch \( s = x e^{-IT} \) (with real \( x \) it can be written:

\[ \int_{\text{LB}} J(s) e^{st} ds \]

\[ = J(\infty) \left( \frac{e^{-S_0 t}}{1 + K \ln \left( \frac{\tau_2 - x}{1 - \tau x} \right)} - K j \pi \right) \]

On the upper branch (UB) \( s = x e^{IT} \) which results in

\[ \int_{\text{UB}} J(s) e^{st} ds =
\]

\[ J(\infty) \left( \frac{e^{-S_0 t}}{1 + K \ln \left( \frac{\tau_2 - x}{1 - \tau x} \right)} + K j \pi \right) \]

From Eqs. (22) and (23) it can be found

\[ \frac{1}{2\pi j} \int_{\text{branches}} J(s) e^{st} ds =
\]

\[ J(\infty) K \left( \frac{1}{1 - \tau_1} \right) e^{-\tau_1 t} \frac{1}{\tau_1} \left( \frac{\tau_2 - \tau}{\tau - \tau_1} \right)^2 + \left( K j \pi \right)^2 \frac{d\tau}{\tau} \]

From Eqs. (19), (20), (21), and (24) the final expression for \( J(t) \) is found to be

\[ J(t) = J(\infty) \left( \frac{1 - (S_0 \tau_2) (1 - S_0 \tau_1)}{K S_0 (\tau_2 - \tau_1)} \right) e^{-S_0 t} \]

\[ - K \left( \frac{1}{1 - \tau_1} \right) e^{-\tau_1 t} \frac{1}{\tau_1} \left( \frac{\tau_2 - \tau}{\tau - \tau_1} \right)^2 + \left( K j \pi \right)^2 \frac{d\tau}{\tau} \quad (25) \]

In the next section this expression for \( J(t) \) will be discussed further.

**Interpretation of the Reduced Creep Function**

As depicted in the introduction, the R.R.F. can be formul-
lated in terms of a relaxation spectrum $S(\tau)$. A similar formulation can be given for the reduced creep function, as it can be written

$$J(t) = J(\infty) \left[ 1 - \int_0^\infty \psi(\tau) e^{-t/\tau} d\tau \right]$$

(26)

where $\psi(\tau)$ is the so-called creep spectrum. By introducing

$$\tau_o = 1/S_o = \frac{\tau_0 e^{\tau_0/K} - \tau_1}{e^{\tau_0/K} - 1}$$

(27)

with $\tau_0 > \tau_2$, according to Eq. (18), eq. (25) can be rewritten to obtain $\psi(\tau)$ as

$$\psi(\tau) = \frac{K}{\tau} \left\{ 1 + K \ln \left( \frac{\tau_2 - \tau}{\tau - \tau_1} \right) \right\} + (K\pi)^2$$

for $\tau_1 \leq \tau \leq \tau_2$

$$\psi(\tau) = \frac{(\tau_o - \tau_2)(\tau_o - \tau_1)}{K \tau_o (\tau_2 - \tau_1)} \delta(\tau - \tau_o) \text{ at } \tau = \tau_o$$

$$\psi(\tau) = 0 \text{ else}$$

(28)

Here $\delta(\tau - \tau_o)$ is Dirac's delta function at $\tau = \tau_o$ with the fundamental property

$$\int_{-\infty}^\infty g(\tau) \delta(\tau - a) \, d\tau = g(a)$$

(29)

for arbitrary functions $g(\tau)$.

$S(\tau)$ and $\psi(\tau)$ are illustrated in Fig. 2 for a set of values of $K$, $\tau_1$, and $\tau_2$ of 0.05, 0.01 s and 100 s, respectively. It is observed that the creep spectrum $\psi(\tau)$ is not a smooth function as the relaxation spectrum $S(\tau)$. Apart from the Dirac function at $\tau = \tau_o$ for $\tau_1 \leq \tau \leq \tau_2$, $\psi(\tau)$ has a local maximum near $\tau = \tau_1$ and $\tau = \tau_2$ and in between a local minimum, with $\psi(\tau_1) = \psi(\tau_2) = 0$.

This specific shape clearly is a result of the selection made for the relaxation spectrum $S(\tau)$ and the interdependence of the relaxation and creep spectra as given by Eq. (10).

References

