Perspective on Weibull proportional-hazards models

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Perspective on Weibull Proportional-Hazards Models

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Key words — Proportional hazard, Weibull distribution, accelerated failure-time model.

Reader Aids —
General purpose: Tutorial, clarification
Special math needed for explanations: Basic probability & statistics
Special math needed to use results: Same
Results useful to: Reliability analysts

Summary & Conclusions — This note uses a paper of Elsayed & Chan (1990) to illustrate some of the advantages and some of the limitations of the proportional hazards approach. The role of proportional hazards as one of several tools for exploratory data analysis is described. The emphasis is on exploratory techniques as a way of 1) measuring the importance of factors influencing system behavior, and 2) determining the form of the model. The semi-parametric version of proportional hazards shows the relative importance of explanatory factors in determining the failure behavior regardless of whether the model is strictly correct. Thus the relative chance of failure can be assessed, but not the absolute chance. The advantage of proportional hazards is that it always yields a quantitative measure of importance for each influence factor. Although E&C clearly establish the importance of temperature as the most critical factor in thin-oxide breakdown, the other analysis technique indicates that more needs to be done to validate a particular model of system behavior. In this case, the failure mechanism remains open, and the use of accelerated test data to predict performance under usual conditions needs further investigation.

1. INTRODUCTION

This paper —
* uses an example from Elsayed & Chan [1] to illustrate the advantages & disadvantages of proportional hazards models,
* indicates when it is appropriate to use proportional hazards models,
* shows what the models can tell us.

The message is that while proportional hazards is a valuable exploratory data-analysis technique, caution must be exercised in interpreting the results. Ref [1] derives a physical model for the failure of thin oxide layers in a semi-conductor device. The physical model does not lead to a proportional hazards model. However, this does not prevent the use of proportional hazards to identify the order of importance of the explanatory variables; indeed it is not really necessary to hypothesize a model. In other words, the problem of model construction & confirmation can be carried out separately from an analysis which is simply meant to indicate which are the most influential variables in the system.

In this paper a fully parametric analysis supplements the semi-parametric proportional hazards analysis in [1].

Acronyms
AFTM accelerated failure-time model
K-S Kolmogorov-Smirnov (test)
PHM proportional-hazards model
Q-Q quantile plots.

Notation (same as [1])
λ hazard rate
Λ cumulative hazard function: Λ = \frac{1}{2} \lambda(u) du
Z covariate, describes the operating conditions
ψ, ψ(Z) relative risk function in proportional hazards model
b, h(Z) scale parameter in accelerated failure-time model
k shape parameter
N mean number of defects
κ0, κd constants for [intrinsic, 1-defect] breakdown.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

2. THE PHYSICAL MODEL

Ref [1: appendix] reports the failure rate for an n-defect capacitor as:

\[ \lambda_n = \kappa_0 + n \cdot \kappa_d \]  

By assuming a Poisson distribution for the defects, they derive a survival function for capacitors:

\[ S_n(t) = \exp(\kappa_0 \cdot t + n \cdot \exp(\kappa_d \cdot t)) \]  

When N is replaced by N(T), the cumulative hazard function and hazard rate are:

\[ \Lambda_n(t) = \kappa_0 \cdot t + N(T) \cdot \exp(\kappa_d \cdot t) \]  
\[ \lambda_N(t) = \kappa_0 + N(T) + t \left[ \frac{dN}{dt} + N(T) \cdot \frac{dk_d}{dt} \exp(-\kappa_d t) \right] \]  

allowing κ0 & κd to be time dependent. Now (3) is a cumulative hazard function only if \( \lambda_N(0) = 0 \) and \( \lambda_N(\infty) = \infty \). The first condition is always true for this function, but the second requires \( \kappa_0 > 0 \), whereas \( \kappa_0 = 0 \) leads to \( \lambda_N(\infty) = N \). This holds in both the time-independent and the time-dependent case. Thus, [1: assumption 6] cannot be used (as it stands) to obtain a hazard

The singular & plural of an acronym are always spelled the same.
rate. More needs to be said about the nature of the approximations involved, although even very small values of \( \kappa_0 \) are enough to yield a hazard function. Thus (4) could behave like a hazard rate over a large range of \( t \). Moreover, if \( \kappa_0 \) or \( \kappa_d \) were temperature dependent, there would be a mixture of proportional hazards and accelerated failure-time effects. It is interesting to contrast the lack of convincing physical arguments here with the comment of Lancaster [6: chapter 7] who states that there appear to be no economic-theoretical foundations for PHM in econometrics. The important point is that the correct physical model is not necessary for the use of PHM.

3. THE STATISTICAL MODELS

In its most general form the PHM is written [2: chapter 5]:

\[
\lambda(t; Z) = \psi(Z) \cdot \lambda_0(t)
\]

\( \lambda_0 \) is the baseline hazard.

If \( \psi > 1 \) the risk is increased and if \( \psi < 1 \) the risk is reduced.

The cumulative hazard and reliability functions satisfy:

\[
\Lambda(t; Z) = \int_0^t \lambda(u; Z) \, du = \psi(Z) \cdot \int_0^t \lambda_0(u) \, du
\]

\( \Lambda(t; Z) \cdot \lambda_0(t) \)

\( S(t; Z) = \exp(-\Lambda(t; Z)) = [S_0(t)]^{\psi(Z)} \)  

(6)

The only restriction on \( \psi \) is \( \psi > 0 \). Thus it is possible to choose forms for \( \psi \) other than the log-linear one in [1]:

\[
\psi(Z) = \exp(\Sigma_{\beta_i} Z_i)
\]

(8)

This choice is largely pragmatic as the simplest functional form which is non-negative, and because the quasi-linearity of the function lends simplicity to the derivatives required in the estimation procedure. The important factor is that the semi-parametric version allows the analysis to be carried out entirely in terms of \( \psi \).

A commonly used alternative to the PHM is the AFTM wherein the effects of the covariates are manifested as changes in a scale parameter. Formally the model is specified in terms of a standardized S:

\[
S(t; k, b(Z_i)) = S_0(t/b(Z_i); k)
\]

(9)

**Notation**

\( k \) form parameter

\( S_0(u; k) \) standardized S.

The relation between the hazard rates is:

\[
h_t(t; k, b(Z_i)) = h_0(t/b(Z_i); k)/b(Z_i)
\]

(10)

It follows from (9) that the percentiles of \( \ln(t) \) are proportional for:

\[
S_t(t; p) = S_0(t; p)/b(Z_i) = 1 - p
\]

(11)

\[
t_i(p)/b(Z_i) = t_0(p) = S_0^{-1}(1 - p)
\]

(12)

\[
t_i(p) = b(Z_i) \cdot t_0(p)
\]

(13)

Thus plots of the quantiles (Q-Q plots) for each level of \( Z \) should yield straight lines through the origin.

It also follows from (9) that as functions of \( \ln(t) \), the S satisfy:

\[
S(t; k, b(Z_i)) = S_0[\ln(t) - \ln(b(Z_i)); k]
\]

(14)

so that plots of the S on semi-log paper should be parallel curves shifted by \( -\ln(b(Z_i)) \). Thus, in the particular case of an AFTM model with a lognormal distribution, plots of the empirical S against time on lognormal paper gives parallel straight lines. The lognormal S, written in the shape & scale parameter form, is:

\[
S(t; k, b) = \text{gaufc}[\ln(t/b)/k]
\]

(15)

This leads to the equation for lognormal paper (and confirms the remark):

\[
\text{gaufc}^{-1}(S) = [\ln(t) - \ln(b)]/k
\]

(16)

To investigate if the PHM is reasonable, several graphical techniques are available, the simplest based on (4) is to note that:

\[
\ln(\Lambda(t; Z)) = \ln(\psi(Z)) + \ln(\Lambda_0(t))
\]

(17)

\[
\ln[-\ln(S(t; Z))] = \ln(\psi(Z)) + \ln[-\ln(S_0(t))]
\]

(18)

Thus a plot of \( \ln[-\ln(S(t; Z))] \) against \( t \) for various values of \( Z \) gives parallel curves. For the Weibull distribution:

\[
S(t) = \text{weifc}(t/b; k);
\]

(6) becomes:

\[
\ln[-\ln(S(t; Z))] = \ln(\psi(Z)) - k \cdot \ln(b) + k \cdot \ln(t)
\]

\[
= -k \cdot \ln[b/\psi(Z)]^{1/k} + k \cdot \ln(t)
\]

(19)

Eq (19) shows that for the Weibull we cannot distinguish between PHM and AFTM, the factor \( \psi \) can also appear as part of the scale parameter \( b/\psi(Z)^{1/k} \). Moreover (18) shows that parallel straight lines on Weibull paper indicate a family of Weibull distribution.

The use of the Weibull distribution obscures the distinction between PHM and AFTM. Indeed [1] reports that Mil-Hdbk-217 models for electronic devices are a class of Weibull PHM, this means that they are also Weibull AFTM.
4. WHY USE PROPORTIONAL HAZARDS?

The great advantage of the PHM over the AFTM is that a partial likelihood [2: chapter 7] can be constructed which estimates the parameters of $\psi(Z)$ without needing to specify a baseline hazard rate. This is valuable if we are interested only in relative magnitudes of the effects of the explanatory variables $Z$ on the experimental subjects. However, in physics-of-failure models it is also usual to require a parametric form for the whole model, ie, both $\psi$ & $\lambda_0$ must be specified. This means we can often use a PHM to analyze a system as a black-box where we seek an indication of the strength of the relation between inputs, the $Z$, and the output, the survival probability. In short, PHM almost always gives a reasonable measure of the importance of factors, but tells nothing about the mechanism itself [5]. Proportional hazards analysis is therefore an excellent exploratory data-analysis technique. It is also possible to carry out a fully parametric analysis of a PHM.

In [1] the simple graphical approach does not distinguish between proportional hazards and accelerated failure time. However, little is lost. The essential misspecifications are [5] proportional hazards when accelerated life holds and vice versa. There is no direct duality between the models if the non-parametric form of the proportional hazards argument is used. Solomon [5] has shown that, in the absence of censoring, the relative importance of the factors is identical in both models. Thus the relative magnitudes of the parameters $\beta_i$ are the same, not necessarily their absolute values. In the face of censoring, there is no clear-cut answer, and the specification of the models needs to be carefully addressed. Although if we are reasonably certain that a Weibull model is appropriate it makes no difference since the models are identical.

5. EXPLORATORY DATA ANALYSIS

Before the more formal analysis, it is worth exploring the data in appendix A.1 [1]. The conclusions are contradictory. There is one striking pattern in the data from the 3 experiments:

- a bunch of early failures,
- a relatively long period with no failures,
- a bunch of later failures.

The gap becomes shorter as the temperature increases, this can be clearly seen with the aid of a stem & leaf plot (table 1). The pattern is not typical of any single distribution, and might even suggest a mixture of failure modes. In the stem & leaf plot, it is clear that the life is reduced by about a factor of 10 at each temperature step.

These data are handled later as grouped data, since in the first reported experiment (170°C), 8 failures took place at precisely 2495.0 hours. The ties in the other data sets also suggest grouped data; thus it is reasonable to assume that the failure times are actually only known to fall in an interval.

<table>
<thead>
<tr>
<th>170°C unit = 100</th>
<th>200°C unit = 10</th>
<th>250°C unit = 1</th>
</tr>
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</table>

Table 1: Stem & Leaf Plot

Figure 1. Weibull Plots — Least-Squares Fit

Figure 2. Lognormal Plots — Least-Squares Fit
200°C are roughly linear, at least over a reasonable range of values, but the plot for the 200°C quantiles against the 170°C quantiles is highly non-linear. This, together with the non-linear behavior near the origin, also raises the possibility of a mixture of failure modes: burn-in failures and a wearout mode.

Thus the evidence appears to be contradictory: the Weibull & lognormal plots indicate an AFTM relation between the two experiments at the lower temperatures, while the Q-Q plots contradict this and suggest an AFTM relationship except for the two lower temperatures.

While the choice between PHM & AFTM remains difficult, the proportional hazards analysis [1] shows the relative importance of the explanatory variables. In both model-group I and model-group-II the reciprocal of absolute temperature is the most important factor and the electric field has much less
effect; moreover the interaction appears to be weak. For the levels of temperature reported, and with the electric field held constant, the relative risks for each temperature are (170°C is normalized to unity) given in table 2; the last rows give the relative values of \( \psi \) for \( k=1 \) & \( k=0.4 \), these are the relative values of the mean life for a Weibull model with shape parameter \( k \). The last row shows that the mean life under usual conditions is about 4 times that at 170°C and a reduction by a factor of 10 for each of the temperatures above 170°C.

6. PARAMETRIC ANALYSIS

On the basis of the discussion in sections 1 - 5, fully parametric analyses of the Weibull PHM/AFTM, lognormal AFTM, and inverse Gaussian AFTM models were used (see appendix 2). All the distributions are written in shape & scale parameter form. Again, assume that the data are not recorded exactly; likelihood estimators for the fully parametric models are readily calculated. A set of intervals:

\[ [0, a_1 + \Delta_1], [a_2 - \Delta_2, a_2 + \Delta_2], \ldots, [a_n - \Delta_n, a_n + \Delta_n] \]

were constructed with \( d_i \) failure times in \( [a_j - \Delta_j, a_j + \Delta_j] \) to give a likelihood [7, 8] for experiment \( i \):

\[
\mathcal{L}_i(k_i, b_i) = \prod_{j=1}^{n} \left[ S(a_j - \Delta_j) - S(a_j + \Delta_j) \right]^{d_i}
\]

The data are in appendix A.1. Since only 3 data sets are available, the AFTM can be examined using a constant \( k \), and allowing \( b_i \) for each set \( i \), to differ. Then the likelihood for the experiment is the product of the likelihoods for each data set.

\[
\mathcal{L}(k, b_1, b_2, b_3) = \mathcal{L}_1(k_1, b_1) \cdot \mathcal{L}_2(k_2, b_2) \cdot \mathcal{L}_3(k_3, b_3)
\]

The likelihood for the 170°C experiment contains one extra term, \( S(2948.0; k, b_1) \) to account for the 1 censored observation.

| TABLE 2 |
| Relative Performance Based On Proportional Hazards |

<table>
<thead>
<tr>
<th>Temperature</th>
<th>150°C</th>
<th>170°C</th>
<th>200°C</th>
<th>250°C</th>
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</thead>
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<tr>
<td>( \psi )</td>
<td>0.6</td>
<td>1.0</td>
<td>2.1</td>
<td>5.7</td>
</tr>
<tr>
<td>( \psi^{-1} )</td>
<td>1.7</td>
<td>1.0</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>( \psi^{-1/k} ) (k=0.4)</td>
<td>3.83</td>
<td>1.00</td>
<td>0.17</td>
<td>0.01</td>
</tr>
</tbody>
</table>

| TABLE 3 |
| Estimators & Test-Of-Fit for the Weibull Analysis |

| Maximum Likelihood |
| shape parameter | scale parameter | K-S distance | s-significance level (%) | log-likelihood |
| 170°C | 0.59 | 1191.00 | 0.288 | 7.3 | -156.86 |
| 200°C | 0.46 | 110.14 | 0.159 | 69.4 | -127.10 |
| 250°C | 0.58 | 13.14 | 0.169 | 64.7 | -88.67 |

AFTM - Likelihood Estimators

| shape parameter | = 0.53, log-likelihood = -373.15 |

| TABLE 4 |
| Estimators & Test-Of-Fit for the LogNormal Analysis |

| Maximum Likelihood |
| shape parameter | scale parameter | K-S distance | s-significance level (%) | log-likelihood |
| 170°C | 2.72 | 422.30 | 0.243 | 18.7 | -161.27 |
| 200°C | 2.90 | 30.32 | 0.225 | 26.3 | -129.59 |
| 250°C | 2.22 | 4.71 | 0.192 | 48.3 | -90.42 |

AFTM, Likelihood Estimators

| shape parameter | = 2.63, log-likelihood = -381.93 |

The number of significant figures is not intended to imply any accuracy in the estimates, but to illustrate the arithmetic.
While the inverse Gaussian likelihood estimators exist for the experiments treated individually, the estimators in the AFTM model yield unacceptable values, therefore it can be ruled out. The comparison between the lognormal and Weibull models is more interesting. The graphical evidence is discussed in section 3. The K-S test rates the fit of the Weibull models to the 170°C as only marginal, although the fits to the other experiments are good. The lognormal model appears to fit slightly less well at 200°C and 250°C but also fits satisfactorily for the 170°C data. The lognormal AFTM model seems to fit slightly better than the Weibull PHM/AFTM model.

In terms of representing the relative risks ensuing from various operating conditions, the situation is not too bad, but the choice of failure model remains difficult. However, the analyses combine to suggest that the experiment at 170°C shows a different failure pattern than the other two. If this is true, then the desired extrapolation from higher stress to the nominal stress levels is unlikely to yield useful predictions.

7. DISCUSSION

The emphasis in [1] has been changed to the exploratory techniques as a way of measuring the importance of factors influencing system behavior (exploratory data analysis) and as a way of determining the form of the model. However, the whole picture given by the range of techniques indicates that much more needs to be done if we wish to validate a particular model of system behavior. In comparison with the other techniques, PHM always yields a quantitative measure of importance for each factor in the \( \beta_i \). Although the importance of temperature as the most critical factor in thin-oxide breakdown is clearly established, the rest of the models indicate that more needs to be done if we wish to validate a particular model of system behavior. In this case, the failure mechanism remains open, and further, the use of accelerated test data to predict performance under usual conditions needs further investigation.

APPENDIX

A.1 Experimental Data [1]

<table>
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<tr>
<th>Recorded failure time ( a_i )</th>
<th>( \Delta_i )</th>
<th>( a_i - \Delta_i )</th>
<th>( a_i + \Delta_i )</th>
<th>number of failures ( d_i )</th>
<th>censored observations</th>
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<td>50.50</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>120.0</td>
<td>.50</td>
<td>119.50</td>
<td>120.50</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
A.2 Distributions

The pdf's & Cdf's used here are all written with \( k & b \); thus with \( u = t/b \), they are:

**Weibull**

\[
 f(t;k,b) = \frac{k}{b} u^{k-1} \exp(-u^k)
\]

\[
 S(t;k,b) = \expfc(u^k)
\]

**Lognormal**

\[
 f(t;k,b) = \frac{1}{b u \sqrt{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{\ln(u)}{k}\right)^2\right)
\]

\[
 S(t;k,b) = \text{gaufc}\left(\frac{\ln(u)}{k}\right)
\]

**Inverse Gaussian**

\[
 f(t;k,b) = \frac{1}{b u^{3/2}} \sqrt{\frac{k}{2\pi}} \exp\left(-\frac{1}{2} \left(\frac{k(u-1)}{u}\right)^2\right)
\]

\[
 S(t;k,b) = \text{gauf}\left(1-u\right) \sqrt{\frac{k}{u}} \exp(2k) \text{gauf}\left(-1+u\right) \sqrt{\frac{k}{u}}
\]

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**REFERENCES**


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