General equilibrium in
infinite dimensional economies;

a truncation approach

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Abstract

Mostly infinite dimensional economies can be considered limits of finite dimensional economies, in particular when we think of time or product differentiation. We investigate conditions under which sequences of quasi-equilibria in finite dimensional economies converge to a quasi-equilibrium in the infinite dimensional economy.

It is shown that convergence indeed occurs if the usual continuity assumption concerning the preference relations for finite dimensional commodity spaces is slightly modified.
1 Introduction

Rather recently economic theory has been enriched by the competitive general equilibrium analysis for economies with an infinite dimensional commodity space. For a survey we refer to Mas-Colell and Zame (1991). The infinite dimensionality is brought about in a variety of ways. One can think of an infinite horizon, an infinite number of differentiated commodities and of uncertainty. The present paper might be best understood on the context of economies with an infinite time domain. These were also the main motivation for the seminal work of Peleg and Yaari (1970) and Bewley (1972). But alternative interpretations are possible as well.

In order to prove the existence of a general equilibrium in an economy with an infinite dimensional commodity space certain assumptions have to be made, some of which closely resemble assumptions made for models in a finite dimensional setting. For example, consumption sets are bounded from below, the set of feasible allocations is compact (in some topology), initial endowments are interior, preference relations are monotonic and the like. One could argue that such assumptions are made to guarantee the existence of an equilibrium in the finite time analogue of the infinite time economy. Indeed, it usually doesn’t require too much imagination to conceive of the economy under study as the limit of finite time economies. Some authors are quite explicit in this respect. Boyd and McKenzie (1993) for example put forward: “This is a limiting form of the futures economy of Hicks”. The next step in this experiment of thought would be to argue that investigating the existence of equilibria in infinite horizon economies makes sense only if the existence of each analogue finite horizon equilibrium is warranted.

This suggests to study the conditions one has to impose on finite horizon economies equilibria in order to deduce the existence of an equilibrium in the infinite horizon economy. The present paper provides an outline of such an approach. We shall consider quasi-equilibria rather than equilibria because, as Mas-Colell and Zame (1991, p. 1855) state ‘the conditions which guarantee that the two notions coincide are entirely parallel to the well-understood, finite dimensional case’. In particular, we assume the existence of quasi-equilibria in each finite horizon economy (not caring about the conditions that have to hold for existence). It is shown next that under a rather mild condition with respect to the continuity of preferences, in addition to monotonicity, there exists a quasi-equilibrium in the infinite horizon economy,
with a sublinear price system. This continuity condition closely resembles the usual continuity requirement. It also includes a continuity condition used in an important paper by Prescott and Lucas (1972).

This article is organized as follows. In Section 2 we present the model, and state the theorem. In Section 3 we prove the main result. Section 4 discusses the assumptions made and provides an example. The conclusions are given in Section 5.

2 The infinite dimensional economy: model and problem formulation

The infinite dimensional economy is denoted by \( E \). The commodity space in the economy \( E \) is the vector space \( \mathbb{R}^N \), consisting of all functions (sequences) from \( \mathbb{N} \) into \( \mathbb{R} \). We introduce some notation. In \( \mathbb{R}^N \) the natural ordering is defined by

\[
(2.1) \quad x \geq y : \iff x(s) \geq y(s) \text{ for all } s \in \mathbb{N} .
\]

By \( \mathbb{R}^N_+ \) we denote the positive cone related to \( \geq \), so

\[
\mathbb{R}^N_+ := \{ x \in \mathbb{R}^N | \forall s \in \mathbb{N} : x(s) \geq 0 \} .
\]

Moreover, we write \( x > y \) if \( x \geq y \) and \( x \neq y \), and \( x \gg y \) if \( x(s) > y(s) \) for all \( s \in \mathbb{N} \). For each \( t \in \mathbb{N} \) the projections \( Q_t \) are defined by

\[
(2.2) \quad (Q_t x)(s) = \begin{cases} x(s) & \text{for } 1 \leq s \leq t , \\ 0 & \text{for } s > t . \end{cases}
\]

Hence \( x \geq y \) implies \( Q_t x \geq Q_t y \) for all \( t \in \mathbb{N} \).

For \( p \in \mathbb{R}^N_+ \) and \( z \in \mathbb{R}^N \) we introduce

\[
(2.3) \quad p[z] := \limsup_{t \to \infty} \sum_{s=1}^{t} p(s) z(s) .
\]

We observe that \( p[z] \) can be \( \pm \infty \). Also, for \( z \in \mathbb{R}^N_+ \)

\[
p[z] = \sup_{t \in \mathbb{N}} \sum_{s=1}^{t} p(s) z(s) = \sum_{s=1}^{\infty} p(s) z(s) ,
\]
and for \( z \in IR^N \) and \( t \in IN \)

\[
p_t(Q_tz) = \sum_{s=1}^{t} p(s)z(s) .
\]

In \( E \) there are \( H \) consumers labelled by \( h \), each having an initial endowment \( \omega_h \in IR^N_+ \) and a consumption set \( X_h = K \). We assume that \( K \) is a subset of \( IR^N_+ \), satisfying

\[
\begin{align*}
0 & \in K \\
Q_t(K) & \subset K \text{ for all } t \in IN \\
x \in K, \  \hat{x} \in K, \ x \leq \hat{x} & \Rightarrow ax + (1 - a)\hat{x} \in K \text{ for } 0 \leq a \leq 1
\end{align*}
\]

Each consumer is endowed with a preference relation, denoted by \( \succ_h \), on \( K \). Here \( x \succ_h y \) means that \( x \) is strictly preferred to \( y \). The preference relation is assumed to satisfy the following conditions.

**Assumption 1.**

(2.5.i) monotonicity:

\[
\forall x \in K \ \forall \hat{x} \in K : \hat{x} > x \Rightarrow \hat{x} \succ_h x .
\]

(2.5.ii) continuity:

Let \( (x^n)_{n \in N} \) be a sequence in \( K \), \( x \in K, \ b \in K \) and \( \hat{x} \in K \) such that

\[
\forall n \in N : x^n \leq b
\]

\[
\forall s \in N : \lim_{n \to \infty} x^n(s) = x(s) ,
\]

and \( \hat{x} \succ_h x \). Then

\[
\exists n_0 \in N \ \exists n_0 \in N \ \forall t \geq t_0 \ \forall n \geq n_0 : Q_t \hat{x} \succ_h x^n .
\]

So the restriction of the preference relation \( \succ_h \) to \( Q_t(K) \) is monotone and upper-hemi-continuous.
In $E$ there are $F$ producers, labeled by $j$, each with a production set $Y_j \subset \mathbb{R}^N$. Each $Y_j$ is assumed to satisfy $0 \in Y_j$ and $Q_t(Y_j) \subset Y_j$ for all $t \in IN$.

By $A$ we denote the subset of $K$ consisting of the feasible aggregate consumption bundles,

$$(2.6) \quad A = \{ x \in K \mid \exists y_j \in Y_j, j = 1, \ldots, F : x \leq \sum_j y_j + \sum_h \omega_h \}.$$ 

The conditions imposed on the consumption sets $X_h$ and the production sets $Y_j$ guarantee that $A$ satisfies (2.4).

Additionally, we postulate

Assumption 2.

$$(2.7) \quad \exists a \in A : a \gg 0.$$ 

Now consider the so called truncated economies $E^T$, $T \in IN$, related to $E$, where each consumer $h$ has the consumption set $X_h^T = Q_T(K)$, initial endowment $\omega_h^T = Q_T \omega_h$ and preference relation $\succ_h$, and where each producer has the production set $Y_j^T = Q_T(Y_j)$. The economies $E^T$ are finite in the sense that the corresponding commodity spaces $Q_T(\mathbb{R}^N)$ are finite dimensional.

We assume that all these truncated economies $E^T$ admit a quasi-equilibrium, i.e.

Assumption 3.

For all $T \in IN$ there exist

$$\xi_h^T \in X_h^T, \quad \eta_j^T \in Y_j^T \quad \text{and} \quad p^T \in Q_T(\mathbb{R}^N_+) \quad \text{with} \quad p^T(1) > 0,$$

such that

$$(2.8.i) \quad \sum_h \xi_h^T = \sum_h \omega_h^T + \sum_j \eta_j^T \quad \text{(feasibility)},$$

$$(2.8.ii) \quad \forall h \forall x \in X_h^T : x \succ_h \xi_h^T \Rightarrow p^T[x] \geq p^T[\xi_h^T] \quad \text{(expenditure minimization)},$$

$$(2.8.iii) \quad \forall_j \forall y \in Y_j^T : p^T[y] \leq p^T[\eta_j^T] \quad \text{(profit maximization)}.$$
Note that (2.8.iii) can be replaced by

\( (2.8.\text{iii}') \quad \forall f \forall y \in V_f : \ p^T[y] \leq p^T[\tilde{y}_f] \).

We now assume that the equilibrium allocations \( \tilde{x}_h^T \) and \( \tilde{y}_f^T \) have a pointwise limit in \( X_h \) and \( Y_f \), respectively. The central result of this paper can now be stated as follows.

**Theorem.** Let there exist \( \tilde{x}_h \in X_h, h = 1, \ldots, H \) and \( \tilde{y}_f \in Y_f, f = 1, \ldots, F \) such that

\[
\begin{align}
(2.9.\text{i}) & \quad \lim_{T \to \infty} \tilde{x}_h^T(s) = x_h(s), \quad s \in IN, \\
(2.9.\text{ii}) & \quad \lim_{T \to \infty} \tilde{y}_f^T(s) = y_f(s), \quad s \in IN.
\end{align}
\]

Moreover, assume

\[
\begin{align}
(2.10.\text{i}) & \quad \exists h \in \mathcal{K} \forall h \forall \tau \in IN : \tilde{x}_h^T \leq b, \\
(2.10.\text{ii}) & \quad \forall h \exists h \in \mathcal{A} : \tilde{x}_h \leq c_h.
\end{align}
\]

Then there is \( p \in \mathbb{M}_+^{NV} \), \( p \neq 0 \), such that

\[
\begin{align}
(2.11.\text{i}) & \quad \lim_{T \to \infty} p^T[\tilde{x}_h^T] = p[x_h] < \infty, \quad h = 1, \ldots, H, \\
(2.11.\text{ii}) & \quad \lim_{T \to \infty} p^T[\omega_h] = p[\omega_h] < \infty, \quad h = 1, \ldots, H, \\
(2.11.\text{iii}) & \quad \lim_{T \to \infty} p^T[\tilde{y}_f^T] = p[\tilde{y}_f] = \sum_{s=1}^{\infty} p(s)\tilde{y}_f(s), \quad f = 1, \ldots, F, \\
(2.11.\text{iv}) & \quad \sum_h \tilde{x}_h = \sum_h \omega_h + \sum_f \tilde{y}_f \quad \text{(feasibility)}, \\
(2.11.\text{v}) & \quad \forall x \in \mathcal{K} : x \succ_h x_h \Rightarrow p[x] \geq p[\tilde{x}_h] \quad \text{(expenditure minimization)}, \\
(2.11.\text{vi}) & \quad \forall f \forall y \in V_f : p[y] \leq p[\tilde{y}_f] \quad \text{(profit maximization)}.
\end{align}
\]

Before we present the proof of this theorem we shall normalize the truncation prices \( p^T \), \( T \in IN \).
By assumption (2.10.ii) and monotonicity (2.5.i) we have \( c_h \succ h \bar{x}_h \) for all \( h \). Then according to (2.10.i), continuity (2.5.ii) and (2.9.i) there are \( t_0 \) and \( T_0 \) such that for all \( t \geq t_0 \) and \( T \geq T_0 \),

\[
(2.12) \quad Q_t c_h \succ t \bar{x}_h
\]

Take \( \hat{t} = t_0 + T_0 \). We impose the normalization assumption

\[
(2.13) \quad \sum_{s=1}^{i} p^T(s) = 1
\]

which makes sense because of \( p^T(1) > 0 \) for all \( T \).

Since \( Q_{\hat{t}} c_h \succ h \bar{x}_h \) for all \( T \geq \hat{t} \), expenditure minimization (2.8.ii) yields

\[
(2.14) \quad p^T[\bar{x}_h] \leq p^T[Q_{\hat{t}} c_h] \leq \gamma , \quad T \geq \hat{t}
\]

where

\[
\gamma := \max\{c_h(s) \mid h = 1, \ldots, H, s \leq \hat{t}\}
\]

3 Proof of the Theorem

We proceed in steps.

Step 1. The construction of \( p \in \mathbb{N}_+^{N} \).

Since \( p^T[\omega_h] \geq 0, \ p^T[\bar{y}_j] \geq 0 \), because \( 0 \in Y_f \) and \( \sum_h \omega_h^T + \sum_f \bar{y}_j^T = \sum_h \bar{x}_h^T \) (feasibility), we obtain

\[
\sum_h p^T[\omega_h] + \sum_f p^T[\bar{y}_j] = \sum_h p^T[\bar{x}_h]
\]

and so by (2.14)

\[
\sum_h p^T[\omega_h] \leq \gamma H , \quad \sum_f p^T[\bar{y}_j] \leq \gamma H .
\]
Let $x \in \mathcal{A}$. Then $x \leq \sum_{f} y_{f} + \sum_{k} \omega_{k}$ for some $y_{f} \in Y_{f}$, $f = 1, \ldots, F$ and

$$0 \leq P^{T}[x] \leq \sum_{f} P^{T}[y_{f}] + \sum_{k} P^{T}[\omega_{k}] \leq \sum_{f} P^{T}[\mathcal{F}_{f}] + \sum_{k} P^{T}[\omega_{k}] \leq \gamma H .$$

We see that there is $M > 0$ such that

(3.1.i) $\forall x \in \mathcal{A} \forall T \in \mathbb{N} : 0 \leq P^{T}[x] \leq M ,$

(3.1.ii) $\forall k \forall T \in \mathbb{N} : 0 \leq P^{T}[\omega_{k}] \leq M ,$

(3.1.iii) $\forall f \forall y \in Y_{f} \forall T \in \mathbb{N} : P^{T}[y] \leq M .$

In (2.7) we assumed existence of $a \in \mathcal{A}$ with $a \gg 0$; so by (3.1.i)

$$P^{T}(s) \leq \frac{M}{a(s)} , \quad T \in \mathbb{N} .$$

This means that the sequence $(P^{T})_{T \in \mathbb{N}}$ is pointwise bounded in $\mathbb{R}_{+}^{N}$. A diagonal argument yields a subsequence $(P^{T_{k}})_{k \in \mathbb{N}}$ with a pointwise limit $P \in \mathbb{R}_{+}^{N}$, i.e. $\lim_{k \to \infty} P^{T_{k}}(s) = p(s), s \in \mathbb{N}$. For convenience and without damaging generality we suppose that $\lim_{T \to \infty} P^{T}(s) = p(s), s \in \mathbb{N}$, in the sequel. Due to the normalization (2.13), $\sum_{s=1}^{i} p(s) = 1$ and so $p > 0$. Before we come to the proof of (2.11.i-vi) we present some auxiliary results first.

**Step 2.**

- For all $x \in \mathcal{A}$ and all $t \in \mathbb{N}$, $Q_{t}x \in \mathcal{A}$ so that

$$p[Q_{t}x] = \lim_{T \to \infty} P^{T}[Q_{t}x] \leq M , \quad t \in \mathbb{N} .$$

We conclude that $p[x] = \sum_{s=1}^{\infty} p(s)x(s) \leq M$, i.e. the latter series is convergent for all $x \in \mathcal{A}$ and in particular for $x = \mathcal{F}_{k}$.

- The argumentation that $p[\omega_{k}] = \sum_{s=1}^{\infty} p(s)\omega_{k}(s)$ is the same as for $p[x_{k}]$. Indeed, for all $t \in \mathbb{N}$

$$p[Q_{t}\omega_{k}] = \lim_{T \to \infty} P^{T}[Q_{t}\omega_{k}] \leq M$$

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and so the result mentioned.

Let \( y \in Y_f, f = 1, \ldots, F \). Then for all \( t \in \mathbb{N} \)
\[
\sum_{s=1}^{t} p(s)y(s) = p(Q_t y) = \lim_{T \to \infty} p^T[Q_t y] \leq M
\]
so that
\[
p[y] = \lim_{t \to \infty} \sup_t p[Q_t y] \leq M.
\]

**Step 3.**

For all \( \varepsilon > 0 \) there are \( t_1 \in \mathbb{N} \) and \( T_1 \in \mathbb{N} \) such that for all \( h \) and all \( T > T_1 \)
\[
\sum_{s > t_1} p^T(s)\overline{y}_h^T(s) < \varepsilon.
\]

**Proof.** By (2.10.iii) there is \( c_h \in A \) with \( c_h > \overline{y}_h \) and so for \( 0 < \alpha < 1 \) we have
\[
\alpha c_h + (1 - \alpha)\overline{y}_h \in \mathcal{K} \quad \text{and}
\]
\[
c_h > \alpha c_h + (1 - \alpha)\overline{y}_h = \overline{y}_h + \alpha(c_h - \overline{y}_h) > \overline{y}_h.
\]

It follows from monotonicity (2.5.i) that \( \alpha c_h + (1 - \alpha)\overline{y}_h \succ_h \overline{y}_h \). Continuity of \( \succ_h \) (2.5.ii) yields \( t_0(\alpha) \) and \( T_0(\alpha) \) such that for all \( t \geq t_0(\alpha) \) and \( T \geq T_0(\alpha) \)
\[
Q_t(\alpha c_h + (1 - \alpha)\overline{y}_h) \succ_h \overline{y}_h^T.
\]

From expenditure minimization (2.8.iii) we conclude that for all \( t \geq t_0(\alpha) \) and \( T \geq T_0(\alpha) \)
\[
p^T[Q_t(\alpha c_h + (1 - \alpha)\overline{y}_h)] \geq p^T[\overline{y}_h^T]
\]
and so
\[
\sum_{s > t_1} p^T(s)\overline{y}_h^T(s) \leq \alpha p^T[c_h] + p^T[Q_t(x_h - \overline{y}_h^T)].
\]

From Step 1 we have that for all \( T \in \mathbb{N} \), \( p^T[c_h] \leq M \).

Now let \( \varepsilon > 0 \) and take \( \alpha = \frac{1}{2}\varepsilon/M \). Then for all \( T \in \mathbb{N} \), \( \alpha p^T[c_h] \leq \frac{1}{2}\varepsilon \). Since
$$\lim_{T \to -\infty} p^T (s)(\overline{x}_h(s) - \overline{x}_h^T(s)) = 0$$

for all $s \in \mathbb{N}$, there is $T_1 \geq T_0(\alpha)$ such that for all $T \geq T_1$

$$|p^T [Q_{t_0(\alpha)}(\overline{x}_h - \overline{x}_h^T)|] < \frac{1}{2}\varepsilon.$$

Thus the result is established. $\square$

Step 4.

Subsequently we prove (2.11.i-vi). So first we prove

(2.11.i) $\forall_h : \lim_{T \to -\infty} p^T [\overline{x}_h^T] = p[\overline{x}_h]$.

Proof. Write

$$p[\overline{x}_h] - p^T [\overline{x}_h^T] = \sum_{s=1}^{t_1} (p(s)\overline{x}_h(s) - p^T(s)\overline{x}_h^T(s))$$

$$+ \sum_{s > t_1} p(s)\overline{x}_h(s) - \sum_{s > t_1} p^T(s)\overline{x}_h^T(s).$$

Let $\varepsilon > 0$. According to the results of Step 2 and Step 3 there are $T_1 \in \mathbb{N}$ and $t_1 \in \mathbb{N}$ such that

$$\sum_{s > t_1} p(s)\overline{x}_h(s) < \frac{1}{3}\varepsilon$$

and for all $T > T_1$

$$\sum_{s > t_1} p^T(s)\overline{x}_h^T(s) < \frac{1}{3}\varepsilon.$$

Since $\lim_{T \to -\infty} |p(s)\overline{x}_h(s) - p^T(s)\overline{x}_h^T(s)| = 0$ for all $s \in \mathbb{N}$ there is $T_2 \geq T_1$ such that

$$|\sum_{s=1}^{t_1} p(s)\overline{x}_h(s) - p^T(s)\overline{x}_h^T(s)| < \frac{1}{3}\varepsilon$$

for all $T > T_2$. $\square$

We next show
(2.11.ii) \( \forall h : \lim_{T \to \infty} p^T[\omega_h] = p[\omega_h] \)

and

(2.11.iii) \( \forall f : \lim_{T \to \infty} p^T[\varphi_f] = p[\varphi_f] \).

**Proof.** For all \( T \in \mathbb{N} \),

\[
\sum_h x^T_h = \sum_h \omega^T_h + \sum_f \varphi^T_f
\]

and therefore

\[
\sum_h \sum_{s > t} p^T(s)x^T_h(s) = \sum_h \sum_{s > t} p^T(s)\omega^T_h(s) + \sum_f \sum_{s > t} p^T(s)\varphi^T_f(s) .
\]

If for some \( T \in \mathbb{N} \) and \( t \in \mathbb{N} \)

\[
\sum_{s > t} p^T(s)\varphi^T_f(s) < 0 ,
\]

then we would have

\[
p^T(Q_f\varphi^T_f) > p^T(\varphi^T_f) \text{ and } Q_f\varphi^T_f \in Y_f
\]

contradicting profit maximization in \( \mathcal{E}^T \) (2.8.iii). So, for all \( T \in \mathbb{N} \) and \( t \in \mathbb{N} \)

\[
\sum_{s > t} p^T(s)\varphi^T_f(s) \geq 0 .
\]

Let \( \varepsilon > 0 \). Then due to the results of Step 3 there are \( t_1 \in \mathbb{N} \) and \( T_1 \in \mathbb{N} \) such that for all \( t > t_1 \) and \( T > T_1 \)

\[
0 \leq \sum_{s > t_1} p^T(s)\omega_h(s) < \varepsilon
\]

and

\[
0 \leq \sum_{s > t_1} p^T(s)\varphi_f(s) < \varepsilon .
\]

Consequently,
\begin{align*}
\lim_{T \to \infty} p^T[\omega_h] &= p[\omega_h] = \sum_{s=1}^{\infty} p(s)\omega_h(s) \\
\lim_{T \to \infty} p^T[\overline{y}_f] &= p[\overline{y}_f] = \sum_{s=1}^{\infty} p(s)\overline{y}_f(s).
\end{align*}

and

\begin{align*}
\lim_{T \to \infty} p^T[\overline{y}_f] &= p[\overline{y}_f] = \sum_{s=1}^{\infty} p(s)\overline{y}_f(s). \quad \square
\end{align*}

Feasibility, i.e. (2.11.iv), is an immediate consequence of the assumed pointwise convergence of \((\overline{x}_h)\) and \((\overline{y}_f)\).

We also prove

\begin{align*}
(2.11.v) \forall_h \forall x \in K : x \succ_h x_h \Rightarrow p[x] \geq p[\overline{x}_h].
\end{align*}

**Proof.** Let \(x \in K\) with \(x \succ_h x_h\). Then continuity (2.5.ii) yields \(t_0 \in \mathbb{N}\) and \(T_0 \in \mathbb{N}\) such that for all \(t \geq t_0\) and \(T \geq T_0\)

\begin{align*}
\overline{Q}_t x \succ_h \overline{x}_h
\end{align*}

and by expenditure minimization (2.8.ii)

\begin{align*}
p^T[\overline{Q}_t x] \geq p^T[\overline{x}_h]
\end{align*}

So for \(T \to \infty\) we get

\begin{align*}
p[x] \geq p[\overline{Q}_t x] \geq p[\overline{x}_h].
\end{align*}

\( \square \)

Finally we show

\begin{align*}
(2.11.vi) \forall_f \forall y \in \mathbb{Y}_f : p[y] \leq p[\overline{y}_f].
\end{align*}

**Proof.** Let \(y \in \mathbb{Y}_f\). Since \(Q_t(\mathbb{Y}_f) \subseteq \mathbb{Y}_f\) for all \(t \in \mathbb{N}\), profit maximization (2.8.iii) yields

\begin{align*}
\forall T \in \mathbb{N} \forall t \in \mathbb{N} : p^T[Q_t y] \leq p^T[\overline{y}_f].
\end{align*}
Letting $T \to \infty$, we see that

$$\forall i \in \mathbb{N} \quad p(Q_i, y) \leq p(y)$$

which means

$$p[y] \leq p(y)$$.

\[ \square \]

## 4 Discussion

In this section we comment on the assumptions made in Section 2 and provide some ideas on the modification of our theorem and its proof under alternative assumptions.

It was already remarked in the Introduction that the model is most easily interpreted in a dynamic setting. Here we shall stick to this interpretation. The definition of the commodity space might then seem a little bit odd, because it allows for only one marketed commodity per period of time. However it is easily seen that the analysis is not affected at all if we assume that there are markets for some arbitrary number of commodities in each period of time. This would just require a simple rearrangement. If this number is equal for each period of time then we could also easily introduce per period consumption and production sets. The total consumption and production sets would then just be the Cartesian products of the per period consumption and production sets.

The assumptions with respect to the consumption sets are rather innocuous. Note that we don’t require solidness which would imply free disposal in consumption. With respect to the production sets we require the possibility of zero production and the possibility of truncation. This latter assumption guarantees that the producer can always decide to do nothing from some future instant of time on. What is excluded here is the occurrence of negative external effects over time. For example, present production does not cause future pollution which may harm consumers. Note that convexity is not imposed on consumption and production sets.

Preferences are assumed to display monotonicity and continuity. Monotonicity is used in the strong sense: a larger bundle with at least one strictly larger item is strictly preferred. This should be seen in relation to the assumption that there exists for each consumer a
feasible consumption bundle which is larger in at least one item than the bundle that is the candidate equilibrium bundle in the infinite dimensional economy (Assumption 2.10.ii). If that assumption would be strengthened by requiring the existence of a feasible consumption bundle that is greater in each item, then the monotonicity assumption could be relaxed to weak monotonicity. A second alternative would be to maintain Assumption 2.10.ii, to impose weak monotonicity and to add strict quasi-concavity of the preference relation. With respect to the continuity assumption we wish to remark that it is indeed necessary to require that the sequence of approaching bundles is bounded by some $b$ in $K$. We consider the following example, where a preference relation on $(\mathbb{R}_+^m)^N$ is induced by a function $u : \mathbb{N} \times \mathbb{R}_+^m \rightarrow \mathbb{R}_+$ in the following sense:

$$U(x) := \sum_{t=1}^{\infty} u(t, x(t)),$$

where $u(t, 0) = 0$ and $u(t, \cdot)$ is increasing for all $t$. We define the subset $K \subset (\mathbb{R}_+^m)^N$ by

$$x \in K : \Leftrightarrow U(x) < \infty$$

and $x \succ y : \Leftrightarrow U(x) > U(y)$; $x \in K, y \in K$.

Moreover, $x \succ y$ implies $U(x) > U(y)$. This preference relation on $K$ satisfies the continuity (Assumption 2.5.ii) just due to the fact that only approaching bundles $x^n$, dominated by some $b \in K$, are to be considered. Note that the sequence $b$ need not be bounded. Note also the similarity of our continuity assumption and the one made by Prescott and Lucas (1972). They require that if $x \in K, z \in K$ and $x \succ_h z$ then $Q_t x \succ_h z$ for all $t$ sufficiently large. We require somewhat more because their consumption sets are $L^+_\infty$.

Assumption 2 states that there is a feasible bundle with positive consumption in all coordinates. Arrow and Hahn (1971, p.65) call this a "surely innocuous proposition". For the paper at hand it is important in the construction of the normalized prices.

In Assumption 3 we postulate the existence of a quasi-equilibrium in every finite horizon economy. Note that what we actually need is that such equilibria exist for economies with sufficiently large horizons. The boundedness of the quasi-equilibrium, consumption bundles has already been discussed. The finite horizon quasi-equilibria naturally entail a non-zero price vector by their definition. In Assumption 3 we postulate that the price of the first
commodity is positive. We have to make such an assumption explicitly because it cannot be excluded that in a quasi-equilibrium with horizon $T$ only the price at $T$ is positive. If we would depart from an equilibrium, not a quasi-equilibrium, then this assumption can be dropped. The remaining assumption to be discussed is the existence of limit allocations (Assumptions 2.9.i and 2.9.ii). It is easy to give conditions which guarantee point-wise convergence. For example, if the equilibrium allocations in the finite dimensional economies are uniformly bounded or if the production sets are uniformly bounded, we get the desired result. We have refrained from making such assumptions in order to be as general as possible: limits may exist even if the assumptions mentioned above are not satisfied.

5 Conclusions

We have derived conditions guaranteeing that the sequences of general quasi-equilibria in finite horizon economies converge to a general quasi-equilibrium in the corresponding infinite horizon economy. Basically all that is required is the existence of limits of the finite horizon equilibrium allocations and a rather straightforward extension of the usual continuity assumption with respect to the preference relation.

Our approach is a generalization of earlier work by Van Geldrop et al. (1991) who use specific production and utility functions. The advantage of our approach seems to be that it is analytically rather straightforward and allows for a nice economic interpretation.

References


