Hecke operators and Euler products

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HECKE OPERATORS AND
EULER PRODUCTS

PROEFSCHRIFT TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE WIS- EN NATUURKUNDE AAN DE RIJKSUNIVERSITEIT TE UTRECHT, OP GEZAG VAN DE RECTOR MAGNIFICUS Dr. L. SEEKLES, HOOGLEERAAR IN DE FACULTEIT DER DIERGENEESKUNDE, VOLGENS BESLUIT VAN DE SENAAT DER UNIVERSITEIT, TEGEN DE BEDENKINGEN VAN DE FACULTEIT DER WIS- EN NATUURKUNDE TE VERDEDIGEN OP MAANDAG 28 OKTOBER 1957 TE 4 UUR PRECIES

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§ 0. Introduction.

The Fourier coefficients of modular forms have been studied by many mathematicians. Although many interesting results have been found, still enough problems remain to keep us interested in this subject. One of the powerful methods used in studying the coefficients is the theory of Hecke's operators $T_n$ (Hecke [1]). Hecke considered modular forms of integral dimension for congruence groups $\Gamma(N)$. The operators $T_n$ are linear combinations of transformed functions $f\left(\frac{a\tau+b}{d}\right)$, where \begin{pmatrix}a \\ b \\ 0 \\ d \end{pmatrix}$ runs through a complete system of in regard to $\Gamma(N)$ nonequivalent transformation matrices of order $n$. For instance, for modular functions of dimension $-r$ and step 1:

$$f \mid T_n = n^{r-1} \sum_{\substack{ad = n \\ b \equiv d \mod d, d > 0}} f\left(\frac{a\tau+b}{d}\right) d^{-r}.$$

The operators map the set of integral modular forms of dimension $-r$ and step 1 into itself. These operators were studied thoroughly (Hecke [1], Petersson [2]). It was proved that under certain conditions modular forms could be written as linear combinations of eigenfunctions of these operators.

If $f(\tau) = \sum_{n=0}^{\infty} a(n) e^{\frac{2\pi i n\tau}{N}}$ is eigenfunction of all $T_n$, the associated Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s}$ can be written as an Euler product $\prod_p (1 - a(p)p^{-s} + \varepsilon(p)p^{k-1-2s})^{-1}$.

To find analogous results for modular forms of non-integral dimension, operators of the type of $T_n$ would have to be defined. Hecke [2] considered the functions $x^a(\tau) \mathit{S}^b(\tau)$ where $a + b$ is an odd integer, and proved that operators $T_p$ could generally not be defined.
A general theory of "Hecke-operators" was given by Wohlfahrt [1]. The operators he defined, and some theorems concerning these operators, will be given in § 1. Now that we have these operators, we are of course interested in the results that can be found by applying them. Wohlfahrt considered applications to the functions $\vartheta(\tau, g, \mathfrak{A}, P_k)$, defined by Schoeneberg [1]. This gave formulae concerning the number of representations of an integer by quadratic forms in an odd number of variables.

For many years the coefficients of the powers of $\eta(\tau)$ have been studied, and special attention has been given to the finding of coefficients which are zero. The results concerning these coefficients, that have been found by Newman [1, 2, 3, 4] and Rademacher [1] suggest that they can be proved and generalized by applying Hecke-operators to the functions $\eta^4(\tau)$. This is done in § 4 of this thesis after the necessary operators have been defined in § 2 and § 3.

Wohlfahrt proved that all Hecke-operators could be built up out of the operators he defined except if the transformed functions occurring in these operators were linearly dependent; in which case more operators could be defined. Sometimes the operators can only be defined if there is linear dependence, as is the case for the functions $\eta^a(\tau) \vartheta^b(\tau)$, studied by Hecke. In § 5 we shall prove some theorems about this linear dependence, and find all integral modular forms for the groups $\Gamma(1)$ and $\Gamma_0$ for which this linear dependence occurs, under the condition that $2r$ is an integer, if $r$ represents the dimension of the modular forms.

§ 1. Definitions and notation.

1.1 $\Gamma(1)$ is the modular group, i.e. the group of unimodular two-rowed matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with rational integral elements.

$\Gamma(N)$ is the subgroup of matrices for which $a \equiv d \equiv 1 \pmod{N}$ and $b \equiv c \equiv 0 \pmod{N}$.

$\Gamma_0(N)$ is the subgroup of matrices for which $c \equiv 0 \pmod{N}$.

$\Gamma_0^0(N)$ is the subgroup of matrices for which $b \equiv 0 \pmod{N}$.

$\Gamma_0$ is the subgroup generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
We shall write \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = U \) and \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = T. \)

Let \( \Gamma \) be a subgroup of finite index of \( \Gamma(1), \) \( r \) a real number.

We shall call a function \( f(\tau) \), defined for \( \text{Im} \tau > 0 \), a modular form of dimension \(-r\) and with multiplier system \( v \) for the group \( \Gamma \) if it satisfies the following conditions (cf. Petersson [1]):

1. In every closed region in the halfplane \( \text{Im} \tau > 0 \), \( f(\tau) \) has as only singularities, at most a finite number of poles.

2. For \( L = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \in \Gamma \) and \( \text{Im} \tau > 0 \), the following relation holds:

\[
 f(L\tau) = f\left(\frac{a\tau + b}{c\tau + d}\right) = v(L) (c\tau + d)^r f(\tau),
\]

where \( v(L) \) depends only on \( L \) and \( |v(L)| = 1 \). (For \( v(L) \) we shall also write \( v\left(\begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}\right) \). Here \( (c\tau + d)^r \) is defined as follows:

For \( (c, d) \neq (0, 0) \) and \( \text{Im} \tau > 0 \), \( (c\tau + d)^r = e^{r \log(c\tau + d)} \), where that branch of \( \log(c\tau + d) \) is taken, for which \( \log(c\tau + d) \) is real if \( c\tau + d > 0, c \neq 0 \). And for \( c = 0 \) we define:

\[
 (c\tau + d)^r = d^r = |d|^r e^{-ir \frac{\text{sgn} d - 1}{2}}.
\]

If \( M = \begin{pmatrix} m_0 & m_3 \\ m_1 & m_2 \end{pmatrix} \) and \( S = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix} \) are rational unimodular matrices and \( MS = M' = \begin{pmatrix} m_0' & m_3' \\ m_1' & m_2' \end{pmatrix} \), the number \( \sigma(M, S) = \sigma(\tau)(M, S) \) is defined in the following way:

\[
 (m_1 S\tau + m_2)^r = \sigma(M, S) \frac{(m_1' \tau + m_2')^r}{(c\tau + d)^r}.
\]

Then \( \sigma(M, S) = e^{2\pi iv\tau} \), where \( v \) is an integer which does not depend on \( \tau \). For the multipliers the following relation holds:

\[
 v(L_1 L_2) = \sigma(L_1, L_2) v(L_1) v(L_2).
\]

3. In the closure of a fundamental region of \( \Gamma \), \( f(\tau) \) has as only possible singularities, poles in the various uniformizing variables. These poles do not cluster about real parabolic limit points of the fundamental region. So we have, if \( UN \in \Gamma \) and we write \( v(U^N) = e^{2\pi iv} \):
\[ f(\tau) = e^{\frac{2\pi i \tau}{N}} \sum_{n=-k}^{\infty} a(n) e^{\frac{2\pi in\tau}{N}}. \]  

\(|\Gamma, -r, v|\) will denote the class of modular forms for the group \(\Gamma\), with dimension \(-r\) and multiplier system \(v\). We call the class trivial if \(f(\tau) \equiv 0\) is the only modular form in the class.

If \(f(\tau)\) is a modular form of dimension \(-r\) and \(S = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) a rational matrix with \(|S| > 0\), we shall write:

\[ f(\tau) | S = |S|^r (c\tau + d)^{-r} f(S\tau). \]  

We then have:

\[ f|S_1S_2 = \sigma(S_1, S_2) f|S_1 | S_2. \]  

If \(f(\tau) \in |\Gamma, -r, v|\), and \(Q\) is a transformation matrix, we have:

\[ f|Q \in |\Gamma | Q, -r, v | Q|, \]  

where \(\Gamma | Q = \Gamma(1) \cap Q^{-1}\Gamma Q\) and \(v| Q\) is the multiplier system determined by:

\[ v(L) | Q = \frac{\sigma(QLQ^{-1}, Q)}{\sigma(Q, L)} v(QLQ^{-1}) \text{ for } L \in \Gamma | Q. \]  

For further theorems on modular forms of non-integral dimension, see Petersson [1].

We shall call a matrix \(Q\) primitive if the greatest common divisor of its elements is 1. If \(|Q| = n\), we say \(Q\) is of order \(n\).

Some modular forms we shall consider are the following:

\[ \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau}) = \sum_{n=1}^{\infty} \left( \frac{12}{n} \right) e^{\frac{\pi in\tau}{12}} \text{ (Im } \tau > 0). \]  

\(\eta(\tau)\) is a modular form for \(\Gamma(1)\), of dimension \(-\frac{1}{2}\). There are several expressions for the multiplier system (cf. Rademacher [2]). In the interior of the upper \(\tau\)-halfplane \(\eta(\tau)\) is free from poles and zeros, and \(\eta(\tau)\) vanishes at \(\tau = 0\) and \(\tau = i\infty\).

\[ \mathcal{G}(\tau) = 1 + 2 \sum_{n=1}^{\infty} e^{\pi in\tau} = \frac{\eta^2(\tau + 1)}{\eta(\tau + 1)}. \]  

\(\mathcal{G}(\tau)\) is a modular form of dimension \(-\frac{1}{2}\) for the group \(\Gamma_0\).
In the fundamental region of $\Gamma_0$, $\vartheta(\tau)$ has no poles. The behaviour in $\tau = -1$ can be found from (1.09) to be as follows:

$$\left(\tau + 1\right)\vartheta(\tau) = e^{\frac{2\pi i}{8} \left(\frac{\tau}{\tau + 1}\right)} \sum_{n=0}^{\infty} c_n e^{2\pi i \left(\frac{\tau}{\tau + 1}\right)^n}. \quad (1.10)$$

C. The Eisenstein series:

$$G_k(\tau) = \frac{1}{2\xi(k)} \sum' \left(m_1 \tau + m_2 \right)^{-k} = 1 + \frac{(2\pi i)^k}{\xi(k). (k - 1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i m n}, \quad (1.11)$$

where $\sigma_r(n) = \sum_{d|n, d > 0} d^r$, and $k$ is an even integer $\geq 4$, are modular forms in $\{\Gamma(1), -k, 1\}$. Every modular form $\in \{\Gamma(1), -k, 1\}$ can be written uniquely as $\sum_n c_n \Delta^n(\tau) G_{k-12n}(\tau)$, $(0 \leq n \leq \frac{k}{12}, k - 12n \neq 2)$, where $G_0 = 1$, and $\Delta(\tau) = \eta^{24}(\tau)$ (cf. Hecke [1]).

1.2 We shall now give some definitions and theorems from Wohlfahrt's general theory of Hecke-operators (Wohlfahrt [1]). If $\Gamma_2$ is a subgroup of finite index of $\Gamma_1$, we write $\{\Gamma_1 : \Gamma_2\}$ for a complete system of representatives of a decomposition of $\Gamma_1$ into left cosets of $\Gamma_2$.

**Theorem 1.** If $K = \{\Gamma, -r, v\}$ and $\Lambda = \{\theta, -r, v^*\}$ are two classes of modular forms, $Q$ is a transformation matrix with $v \mid Q = v^*$ on $\theta \cap \Gamma \mid Q$, and $\theta \cap \Gamma \mid Q$ is of finite index in $\Gamma(1)$, the operator $T^K_{\Lambda}(Q)$ defined by

$$f \mid T^K_{\Lambda}(Q) = \sum_{V \in \{\theta \cdot \theta \cap \Gamma \mid Q\}} v^*(V)^{-1} f \mid Q \mid V \quad (1.12)$$

maps $K$ into $\Lambda$.

Remark: If $Q^* = n Q^{-1}$ where $n = |Q|$, and $T^K_{\Lambda}(Q)$ is defined, then $T^K_{\Lambda}(Q^*)$ can also be defined.

We refer to this as

$$\quad (1.13)$$
Theorem 2. We define a Hecke-operator to be an operator which transforms the modular forms \( f \) of a class \( [\Gamma, -r, v] \) into forms of \( [\varnothing, -r, v^*] \), the operator being a linear combination (with coefficients independent of \( f \)) of transformed functions \( f|Q \) with primitive \( Q \) of a fixed order. Then all Hecke-operators defined for the whole class \( [\Gamma, -r, v] \) can be built up out of the operators defined in theorem 1 with the following exception: Let \( Q \) run through a complete system of primitive transformation matrices of fixed order, which are left-inequivalent with respect to \( \Gamma \). If for all \( f \) of \( [\Gamma, -r, v] \) the transformed functions \( f|Q \) are linearly dependent, other Hecke-operators than those mentioned above can be defined.

Remark: It is also of interest to consider those functions of \( [\Gamma, -r, v] \) for which linear relations for the \( f|Q \) exist. For these functions other Hecke-operators can be defined. Of course these operators are not necessarily Hecke-operators for the other functions of the class. Functions for which this holds are the functions \( \eta(\tau), \eta^3(\tau), \eta(\tau) \) and \( \eta^4(\tau) \eta^{-1}(\tau) \), studied by Hecke [2]. We shall give more examples in § 5.

Sometimes the operators defined in theorem 1 can be written in a form which is easier to handle. Consider \( K = [\Gamma, -r, v] \), \( \Lambda = [\varnothing, -r, v^*] \). Let \( \Gamma(N) \subseteq \Gamma \cap \varnothing \) and let \( T^\Lambda_K(Q) \) be defined and \( (n, N) = 1 \), where \( n = |Q| \). Then the system:

\[
\mathcal{C} = \mathcal{C}^{(n)}_N = [R : R = \begin{pmatrix} a & bN \\ 0 & d \end{pmatrix}, ad = n, \]
\[
a > 0, b \text{ mod } d, (a, b, d) = 1 \}
\]

is a complete system in regard to \( \Gamma(1) \) left-inequivalent, primitive transformation matrices of order \( n \).

Theorem 3. If \( \mathcal{C} \subseteq \Gamma Q \varnothing \) and \( [\varnothing \cap \Gamma(1) Q : \varnothing \cap \Gamma | Q] = 1 \) we have

\[
f| T^\Lambda_K(Q) = \sum_{R \in \mathcal{C}} \lambda(R)^{-1} f| R. \tag{1.15}
\]

Here \( \lambda(R) = \sigma(M, Q) \sigma(MQ, V) \nu(M) \nu^*(V) \) if \( R = MQV \) with \( M \in \Gamma, V \in \varnothing \). We call this the normal form of \( T^\Lambda_K(Q) \).
In defining the operators $T_K^\Lambda(Q)$ the main difficulty is proving that $v^* = v|Q$. In fact if $K$ is given and $\theta$ is a given group it is possible that for $\theta$ no multiplier system $v^*$ exists so that $v^* = v|Q$ and $\{\theta, -r, v^*\}$ is non-trivial. For certain groups we shall study the multiplier systems in § 2, and then define some special operators of the type $T_K^\Lambda(Q)$ in § 3.

§ 2. Some properties of multiplier systems.

We shall consider multiplier systems for the groups $\Gamma(1)$, $\Gamma_0(2)$ and $\Gamma_\theta$, establishing relations between $v_{(\begin{array}{cc} a & b \\ cp & d \end{array})}$ and $v_{(\begin{array}{cc} a & bp \\ c & d \end{array})}$.

First we remark that for all groups $\Gamma$ with $-I \in \Gamma$, we have:

$$v(-I) = e^{-nir}.$$  

(2.01)

2.1 The case $\Gamma = \Gamma(1)$. $\Gamma(1)$ is generated by $U$ and $T$ for which the following relations hold: $T^2 = -I$ and $(UT)^3 = -I$. From these we find for $\lambda = v(U)$ and $\mu = v(T)$ the relations:

$$\lambda = \zeta_6^r e^{\frac{nir}{6}} \quad \text{and} \quad \mu = \lambda^{-3} = \pm e^{\frac{nir}{2}}.$$  

(2.02)

Here $\zeta_6$ is a 6-th root of unity. We see that for $\Gamma(1)$ only 6 different multiplier systems are possible (for every $r$).

In the following we shall use the fact that for every $M \in \Gamma(1)$ there is an integer $w(M)$, independent of $r$, so that:

$$v(M) = \lambda^{w(M)}.$$  

(2.03)

Proof: Expressing $M$ in the generators $U$ and $T$, we find for $v(M)$ a product of factors $\sigma(M_i, M_j)$ and powers of $\lambda$ and $\mu$. All $\sigma(M_i, M_j)$ are powers of $e^{2nir}$, therefore powers of $\lambda$. Using $\mu = \lambda^{-3}$ we find the result (2.03).

If $f(\tau) \in \{\Gamma(1), -r, v^*\}$, application of (1.06) gives:

$$f(p\tau) f^{-p}(\tau) \in \{\Gamma_0(p), r(p-1), v^*\},$$  

(2.04)

where $v^*_{(\begin{array}{cc} a & b \\ cp & d \end{array})} = v_{(\begin{array}{cc} a & bp \\ c & d \end{array})} v^{-p}_{(\begin{array}{cc} a & b \\ cp & d \end{array})} = \lambda^{w_{(\begin{array}{cc} a & bp \\ c & d \end{array})} - p w_{(\begin{array}{cc} a & b \\ cp & d \end{array})}}$. 


For $f(\tau) = \eta(\tau)$ and $p > 3$ a prime, $\nu^* \left( \frac{a}{c} \frac{b}{d} \right) = \varepsilon(d) = \left( \frac{d}{p} \right)$, where $\left( \frac{d}{p} \right)$ is the quadratic rest symbol (Hecke [2]).

This gives us the relation:

$$e^{\frac{2\pi i}{12} \left\{ w \left( \frac{a}{c} \frac{b}{d} \right) - n \cdot w \left( \frac{a}{c} \frac{b}{d} \right) \right\}} = \left( \frac{d}{p} \right).$$  \hspace{1cm} (2.05)

Using $f(\tau) = \eta^3(\tau)$ we find:

$$e^{\frac{3\pi i}{4} \left\{ w \left( \frac{a}{c} \frac{b}{d} \right) - 3w \left( \frac{a}{c} \frac{b}{d} \right) \right\}} = \left( \frac{d}{3} \right).$$  \hspace{1cm} (2.06)

We now prove:

**Theorem 4.** If $\left\{ \Gamma(1), -r, v \right\}$ is non-trivial, $r$ is an integer and $p > 3$ is a prime, then $v \left( \frac{a}{c} \frac{b}{d} \right) = v \left( \frac{a}{c} \frac{b}{d} \right)$.

Proof: (2.05) implies $w \left( \frac{a}{c} \frac{b}{d} \right) - p \cdot w \left( \frac{a}{c} \frac{b}{d} \right) \equiv 0 \pmod{12}$ and as in this case $\lambda^{12} = e^{2\pi i r} = 1$, the first equation is proved. Taking the $p$-th powers of both sides we find $v^p \left( \frac{a}{c} \frac{b}{d} \right) = v^p \left( \frac{a}{c} \frac{b}{d} \right)$. From this the second equation follows because $p^2 \equiv 1 \pmod{24}$ and all multipliers are $12$-th roots of unity. If $3/r$ and $\lambda = \pm e^{\frac{i}{6}}$, these results are also true for $p = 3$, which can be proved using (2.06).

**Corollary:** if $f(\tau) \in \left\{ \Gamma(1), -r, v \right\}$, r an integer, then

$$f(\tau) \in \left\{ \Gamma(12), -r, 1 \right\}.$$

Proof: If $\left( \frac{a}{c} \frac{b}{d} \right) \in \Gamma(12)$ we have $\mu \cdot v \left( \frac{a}{c} \frac{b}{d} \right) = v \left( \frac{b-a}{d-c} \right)$. Now $(d, 6) = 1$ so we can apply theorem 4.
We find, using $\lambda^{12} = \mu^4 = 1$:

$$
\mu \cdot \nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} 
\nu^d \begin{pmatrix} b - ad & \\
1 & -c \\
-1 & 1 \\
\end{pmatrix} = \mu^d = \mu, & (d > 0) \\
\nu^{-d} \begin{pmatrix} b & ad \\
-1 & -c \\
1 & 1 \\
\end{pmatrix} = \mu^{-3d} = \mu, & (d < 0).
\end{cases}
$$

From this follows $\nu \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$ if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(12)$.

**Theorem 5.** If, in $\{\Gamma(1), -r, \nu\}$, $2r = 1$ is an odd integer, and $p > 3$ is a prime, then $\nu \begin{pmatrix} a & bp \\ c & d \end{pmatrix} = \left(\frac{d}{p}\right)^{\nu_p(a,b)} \begin{pmatrix} a & b \\ cp & d \end{pmatrix}$.

**Proof:** In this case $\lambda = \zeta_6 \cdot e^{\frac{12}{12}} = \zeta_{12} \cdot e^{\frac{12}{12}}$ where $\zeta_6 = e^{\frac{2\pi i}{6}} = e^{\frac{6\pi i}{12}} = 1$.

The first equation now follows from (2.05) and the second can be proved in the same way as in theorem 4, if we remark that now all multipliers are 24-th roots of unity. As in theorem 4, the result is also true for $p = 3$ if $3/l$ and $\lambda = \pm e^{\frac{6\pi i}{6}}$.

Applying theorem 4 or theorem 5 twice, we find:

**Theorem 6.** If, in $\{\Gamma(1), -r, \nu\}$, $2r$ is an integer, and $p > 3$ is a prime, then $\nu \begin{pmatrix} a & bp^2 \\ c & d \end{pmatrix} = \nu \begin{pmatrix} a & b \\ cp^2 & d \end{pmatrix}$.

Again, this is also true for $p = 3$ if $3/2r$ and $\lambda = \pm e^{\frac{6\pi i}{6}}$.

From these equations, others, e.g. $\nu \begin{pmatrix} ap^2 & b \\ c & d \end{pmatrix} = \nu \begin{pmatrix} a & b \\ cp & dp^2 \end{pmatrix}$ can be proved easily ($2r$ an integer). With these theorems the multipliers can easily be determined in those cases where $(c, 6) = 1$ or $(d, 6) = 1$.

We shall now prove analogous results for the groups $\Gamma_6(2)$ and $\Gamma_9$. First we remark that, by (1.06), if $f(r) \in \{\Gamma_6(2), -r, \nu\}$, then:

$$
f\left(\frac{r + 1}{2}\right) \in \{\Gamma_9, -r, \nu \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\}. \quad (2.07)$$
2.2 The case $\Gamma = \Gamma_0(2)$. $\Gamma_0(2)$ is generated by $U$ and $T^* = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, for which the relation $(U^{-1} T^* )^2 = -I$ holds. Writing $v(U) = \lambda^*$ and $v(T^*) = \mu^*$, we have:

$$\mu^* = \pm \lambda^* e^{\frac{\pi i r}{2}}. \quad (2.08)$$

Using (2.08) we find that for every $M \in \Gamma_0(2)$ there are numbers $w_1(M)$ and $w_2(M)$, independent of $r$, so that:

$$v(M) = \lambda^{w_1(M)} \mu^{w_2(M)}. \quad (2.09)$$

Considering the multiplier systems of $\eta(\tau)$ and $\eta(2\tau)$, we now find two relations:

$$e^{\frac{\pi i}{12}} \left[ w_1(a \ b_p) - 2w_2(a \ b_p) \right] - p \left[ w_1(a \ c_p \ d) - 2w_2(a \ c_p \ d) \right] = \left( \frac{d}{p} \right), \quad (2.10)$$

$$e^{\frac{\pi i}{12}} \left[ 2w_1(a \ b_p) - w_2(a \ b_p) \right] - p \left[ 2w_1(a \ c_p \ d) - w_2(a \ c_p \ d) \right] = \left( \frac{d}{p} \right). \quad (2.11)$$

Applying these formulae in the same way as was done for $\Gamma(1)$, we find:

**Theorem 7.** If we consider only multiplier systems in which $\lambda^*$ is a 24-th root of unity (and therefore all multipliers are 24-th roots of unity), the theorems 4, 5 and 6 hold if $\Gamma(1)$ is replaced by $\Gamma_0(2)$. 

2.3 The case $\Gamma = \Gamma_\theta$. $\Gamma_\theta$ is generated by $U^2$ and $T$. We write $v(U^2) = \lambda^*$ and $v(T) = \mu = \pm e^{\frac{\pi i r}{2}}$. The calculations in this case are rather tedious. We use the functions $\eta \left( \frac{r - 1}{2} \right)$ and $\theta(\tau)$.

We shall only state the result:

**Theorem 8.** If we consider only multiplier systems in which $\lambda^*$ is a 24-th root of unity, the theorems 4, 5 and 6 hold if $\Gamma(1)$ is replaced by $\Gamma_\theta$. 

§ 3. Some special operators \( T^\Lambda_K(Q) \).

In this paragraph we shall consider operators \( T^\Lambda_K(Q) \) with \( Q = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \) or \( \begin{pmatrix} p^2 & 0 \\ 0 & p \end{pmatrix} \). If possible, we shall write these operators in the normal form (1.15). In the following we have \( K = \{ \Gamma_1, -r, v \}, \Delta = \{ \Gamma_2, -r, v^* \}, l = 2r \). Furthermore \( p \) will always represent a prime number larger than 3.

3.1 \( \Gamma_1 = \Gamma_2 = \Gamma(1), \) \( r \) an integer, \( Q = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \).

By (2.03) the multiplier systems are determined by \( \lambda = v(U) \) and \( \lambda^* = v^*(U) \). If we have \( \lambda = e^{2\pi i k_1 \frac{2ir}{6}} e^\frac{\pi ir}{6} \), we choose \( k_2 \equiv pk_1 + \frac{r(p-1)}{2} \) (mod 6). Then \( \lambda^* = e^{2\pi i k_2 \frac{2ir}{6}} e^\frac{\pi ir}{6} \) satisfies \( \lambda^* = \lambda^p \), and so by (2.03) and theorem 4 we have:

\[
\lambda^* \begin{pmatrix} a \\ b \end{pmatrix} = \lambda^p \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix}.
\]

Therefore \( \lambda^* = v \mid Q \) on \( \Gamma_0(p) = \Gamma_2 \cap \Gamma_1 \mid Q \).

Both conditions in theorem 1 are satisfied. Hence we can define \( T^\Lambda_K(Q) \). (Remark: \( T^K_\Lambda(Q) \) can also be defined for these classes).

In view of later applications we choose \( N = 24 \) to write \( T^\Lambda_K(Q) \) in the normal form (see (1.15)). The conditions of theorem 3 are satisfied. \( \mathcal{R} \) consists of the \( p+1 \) matrices

\[
\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 24b \\ 0 & p \end{pmatrix}, \quad (b = 0, \ldots, p-1).
\]

(3.01)

We find:

\[
\lambda(R) = \begin{cases} 1 & \text{for } R = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \\ \frac{r(p-1)}{2} & \text{for } R = \begin{pmatrix} 1 & 24b \\ 0 & p \end{pmatrix}. \end{cases}
\]

(3.02)

We shall write \( T(p) \) for the operator we have defined. We found: When \( f(r) \in \{ \Gamma(1), -r, v \}, r \) an integer, then

\[
f(r) \mid T(p) \in \{ \Gamma(1), -r, v^* \}.
\]
where

\[ f(\tau) | T(p) = p^r f(p\tau) + (-1)^{\frac{r(p-1)}{2}} \sum_{b=0}^{p-1} f\left(\frac{\tau + 24b}{p}\right). \]  

(3.03)

If \(3/r\) and \(\lambda = \pm e^{\frac{i\pi}{6}}\) we can define \(T(3)\):

\[ f(\tau) | T(3) = 3^r f(3\tau) + (-1)^{\frac{r}{2}} \sum_{b=0}^{2} f\left(\frac{\tau + 8b}{3}\right). \]  

(3.04)

If \(l = 2r\) is an odd integer, we cannot define an operator \(T(p)\) as was done above for \(\Gamma_1 = \Gamma_2 = \Gamma(1)\).

Proof: A necessary condition is \(v^* = v|Q\), therefore \(\lambda^* = \lambda^p\).

Then \(v^* = v^p\) for all \(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)\). We then have for \(\begin{pmatrix} a & b \\ cp & d \end{pmatrix} \in \Gamma_0(p)\):

\[ v^* \begin{pmatrix} a & b \\ cp & d \end{pmatrix} = v \begin{pmatrix} a & bp \\ c & d \end{pmatrix} = \begin{pmatrix} d & -bp \\ p & a \end{pmatrix} v^p \begin{pmatrix} a & b \\ cp & d \end{pmatrix}, \]  

and this is equal to \(v^p \begin{pmatrix} a & b \\ cp & d \end{pmatrix}\) only if \(\left(\frac{d}{p}\right) = 1\), therefore not for all matrices in \(\Gamma_0(p)\).

We see that operators \(T(p)\) can only be defined for these classes in the exceptional case mentioned in theorem 2 (see § 5).

3.2 \(\Gamma_1 = \Gamma_2 = \Gamma(1), r\ an\ integer, \ Q = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}\).

We now take \(v^* = v\). Then by theorem 6: \(v^* = v|Q\) on \(\Gamma_0(p^2) = \Gamma_2 \cap \Gamma_1 |Q\). The conditions of theorem 1 are satisfied.

To write \(T_K^\Lambda(Q)\), which operator we shall write as \(T(p^2)\), in the normal form, we take:

\[ \mathcal{C}_R = \left\{ \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} p & 24b \\ 0 & p \end{pmatrix} \text{ with } (b = 1, \ldots, p - 1); \begin{pmatrix} 1 & 24b \\ 0 & p^2 \end{pmatrix} \text{ with } (b = 0, \ldots, p^2 - 1) \right\}. \]  

(3.05)

We find:

\[ \lambda(R) = \begin{cases} 1 & \text{for } \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 24b \\ 0 & p^2 \end{pmatrix} \\ (-1)^{\frac{r(p-1)}{2}} & \text{for } \begin{pmatrix} p & 24b \\ 0 & p \end{pmatrix} \end{cases}. \]  

(3.06)
So we have: if \( f(\tau) \in \Gamma(1), -r, v \), \( r \) an integer, then 
\[
f(\tau) \mid T(p^2) \in \Gamma(1), -r, v
\]
where 
\[
f(\tau) \mid T(p^2) = \\
p^r f(p^{2r}) + (-1)^{\frac{r(p-1)}{4}} \sum_{b=0}^{p^2-1} f\left(\frac{p\tau + 24b}{p}\right) + p^{2r} \sum_{b=1}^{p-1} f\left(\frac{\tau + 24b}{p^2}\right).
\]
(3.07)

If we denote \( f(\tau) \mid T(p) \) by \( f(\tau) \mid T(p)^2 \) the following relation holds:
\[
f(\tau) \mid T(p)^2 = f(\tau) \mid T(p^2) + (-1)^{\frac{r(p-1)}{2}} p^r (p + 1) f(\tau).
\]
(3.08)

If \( 3/r \) and \( \lambda = \pm \frac{e^{it}}{6} \) we can define \( T(9) \) by replacing \( 24 \) by \( 8 \) in (3.07). Then (3.08) holds for \( p = 3 \).

For \( \Gamma_1 = \Gamma_2 = \Gamma(1), r \) an integer, \( Q = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \) where \((n, 6) = 1\), we can define an operator \( T(n) \) in the same way \( T(p) \) was defined in 3.1. In the case \( v^* = v = 1 \), these operators can be compared with Hecke’s operators \( T_n \) (Hecke [1]). Wohlfart [1] found the relation \( T_n = n^{-1} \sum_{h > 0, h^2/n} h^r T\left(\frac{n}{h^2}\right) \).

For the operators \( T(n) \) we can find a multiplication theorem. Wohlfahrt [1] proved that:
\[
T(m) T(n) = T(mn) \text{ if } (m, n) = 1. \quad (3.09)
\]

In the same way as for \( T(p) \) and \( T(p^2) \) we find the normal form:
\[
f(\tau) \mid T(p^k) = \\
p^{kr} f(p^{kr}) + \sum_{v=1}^{k-1} \sum_{\begin{subarray}{c} b=1 \\ (b, p) = 1 \end{subarray}}^{p^v-1} (-1)^{\frac{r(p^v-1)}{2}} p^{(k-v)r} f\left(\frac{p^{k-v}\tau + 24b}{p^v}\right) + \\
+ \sum_{b=0}^{p^k-1} (-1)^{\frac{r(p^k-1)}{2}} f\left(\frac{\tau + 24b}{p^k}\right).
\]
(3.10)

This gives us for \( k > 2 \):
\[
T(p^k) = T(p) T(p^{k-1}) - (-1)^{\frac{r(p-1)}{2}} p^{r+1} T(p^{k-2}). \quad (3.11)
\]
Combining (3.08), (3.09) and (3.11) we find:

$$\sum_{(n, 6) = 1} \frac{T(n)}{n^s} = \prod_{p > 3} \left( \frac{1 - (-1)^{\frac{p-1}{2}} p^{r-2s}}{1 - T(p) \cdot p^{-s} + (-1)^{\frac{p-1}{2}} p^{r+1-2s}} \right). \quad (3.12)$$

Here the series and the product are to be considered formally only.

3.3 \( \Gamma_1 = \Gamma_2 = \Gamma(1), l = 2r \) an odd integer, \( Q = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \).

As in 3.2 we take \( v^* = v \) and by the same reasoning as is used there, we find that \( T_K(Q) \) can be defined. We shall also call this operator \( T(p^2) \). For \( \mathcal{R} \) we choose (3.05) and find:

$$\lambda(R) = \begin{cases} 
1 & \text{for } R = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 24b \\ 0 & p^2 \end{pmatrix}, \\
\frac{\pi ir(p-1)}{2} \left( \frac{24b}{p} \right) = \begin{cases} 
\left( \frac{3b}{p} \right) \text{ when } p \equiv 1 \pmod{4} \\
\left( \frac{3b}{p} \right) \text{ when } p \equiv 3 \pmod{4} 
\end{cases} & \text{for } R = \begin{pmatrix} p & 24b \\ 0 & p \end{pmatrix}.
\end{cases} \quad (3.13)$$

So we have: If \( f(\tau) \in \{ \Gamma(1), -r, v_1 \}, l = 2r \) an odd integer, then \( f(\tau) \mid T(p^2) \in \{ \Gamma(1), -r, v_1 \} \), where:

$$f(\tau) \mid T(p^2) = \quad (3.14)$$

with \( \gamma_p = 1 \) when \( p \equiv 1 \pmod{4} \) and \( \gamma_p = i^{-1} \) when \( p \equiv 3 \pmod{4} \).

For \( 3/r, \lambda = \pm e^{\pi ir} \), we can define \( T(9) \):

$$f(\tau) \mid T(9) = \quad (3.15)$$
We can define \( T(n^2) \) for all \( n \) with \((n, 6) = 1\) in the same way. We find:

\[
f(r) \mid T(p^{2k}) = p^{2k} f(p^{2k}r) + \sum_{p-1}^{2k-1} \sum_{b=0}^p \frac{(24b)^v}{p^2} \cdot p^{2k-v} f\left(p^{2k-v} + 24b\right) + \sum_{b=0}^{p^2-1} f\left(\frac{r + 24b}{p^{2k}}\right). \tag{3.16}
\]

This gives us:

\[
T(p^2) T(p^2) = T(p^4) + p^{2r+1}(p + 1), \tag{3.17}
\]

and for \( k > 1 \):

\[
T(p^{2k}) T(p^2) = T(p^{2k+2}) + p^{2r+2} T(p^{2k-2}). \tag{3.18}
\]

From (3.16), (3.17) and \( T(n^2) T(m^2) = T(n^2m^2) \) if \((n, m) = 1\) we find:

\[
\sum_{(n, 6) = 1} T(n^2) \frac{1}{n^s} = \prod_{p > 3} \left(1 - \frac{1 - p^{2r+1-s}}{1 - T(p^2) p^{-s} + p^{2r+2-2s}}\right). \tag{3.19}
\]

(Series and product are to be considered formally).

3.4 \( \Gamma_1 = \Gamma_2 = \Gamma_0(2) \) or \( \Gamma_6, 2r = l \) an odd integer, \( Q = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \).

The reasoning is the same as in 3.3. We now use theorem 7 and theorem 8. In the case \( \Gamma_0(2) \), we take (3.05) for \( \mathcal{R} \), finding \( \lambda(R) = 1 \) for \( R = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \) and \( R = \begin{pmatrix} 1 & 24b \\ 0 & p^2 \end{pmatrix} \), while for \( R = \begin{pmatrix} p & 24b \\ 0 & p \end{pmatrix} \) we find:

\[
\lambda(R) = \begin{cases} \left(\frac{3b}{p}\right) \text{ when } p \equiv 1 \pmod{4} \\ i^l \left(\frac{3b}{p}\right) \text{ when } p \equiv 3 \pmod{4} \end{cases} \quad \text{if } \lambda^{*12} = -1, \tag{3.20}
\]

\[
\lambda(R) = \begin{cases} \left(\frac{6b}{p}\right) \text{ when } p \equiv 1 \pmod{4} \\ i^l \left(\frac{6b}{p}\right) \text{ when } p \equiv 3 \pmod{4} \end{cases} \quad \text{if } \lambda^{*12} = 1.
\]

In the case \( \Gamma_6 \) we take \( \mathcal{R} \) as in (3.05) but replace 24 by 48.
We find:

\[ \lambda(R) = 1 \text{ for } R = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } R = \begin{pmatrix} 1 & 48b \\ 0 & p^2 \end{pmatrix}, \text{ while for } R = \begin{pmatrix} p & 48b \\ 0 & p \end{pmatrix} \text{ we have} \]

\[ \lambda(R) = \begin{cases} \left( \frac{6b}{p} \right) & \text{when } p \equiv 1 \pmod{4}, \\ \left( \frac{b}{p} \right) & \text{when } p \equiv 3 \pmod{4}. \end{cases} \]  

(3.21)

For the \( T_K^\lambda(Q) \) we find formulae analogous to (3.14).

3.5 \( \Gamma_1 = \Gamma(1), \Gamma_2 = \Gamma_0(p), \) 2r = l an odd integer, \( Q = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \).

In 3.1 we proved that if \( l \) is an odd integer, we cannot define \( T(p) \) as in (3.03). The operator \( T_K^\lambda(Q) \) we now define, resembles \( T(p) \) very much and is therefore of interest. We shall call this operator \( T_0(p) \).

For \( \begin{pmatrix} a & b \\ cp & d \end{pmatrix} \in \Gamma_0(p) \), we define \( v^* \left( \begin{pmatrix} a & b \\ cp & d \end{pmatrix} \right) = \left( \frac{d}{p} \right) v^* \left( \begin{pmatrix} a & b \\ cp & d \end{pmatrix} \right) = v \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). By (1.06) this is a multiplier system for \( \Gamma_0(p) \). In fact we have: if \( f(\tau) \in \{ \Gamma(1), -\tau, v \} \), then \( f(p\tau) \in \{ \Gamma_0(p), -\tau, v^* \} \).

Furthermore we have \( v^* = v|Q \) on \( \Gamma_0(p) = \Gamma_2 \cap \Gamma_1 | Q \). Therefore \( T_K^\lambda(Q) = T_0(p) \) can be defined. In this case we cannot find a normal form.

For \( \{ \Gamma_0(p) : \Gamma_0(p) \} \) we take \( \begin{pmatrix} 1 & 24b \\ 0 & 1 \end{pmatrix}, \) \( (b = 0, \ldots, p-1) \). We then have: If \( f(\tau) \in \{ \Gamma(1), -\tau, v \} \), 2r + l is an odd integer and \( v^* \) is the multiplier system of \( f(p\tau) \), then \( f(\tau) \mid T_0(p) \in \{ \Gamma_0(p), -\tau, v^* \} \).

where

\[ f(\tau) \mid T_0(p) = \sum_{b=0}^{p-1} f \left( \frac{\tau + 24b}{p} \right), \]  

(3.22)

and

\[ f(\tau) \mid T_0(3) = \sum_{b=0}^{24b} f \left( \frac{\tau + 8b}{3} \right) \text{ if } 3|\tau \text{ and } \lambda = \pm e^\pi i \]  

(3.23)
Remark: As was said in (1.13), we can also define $T^K_\Lambda(Q^*)$ in the case considered here. After some calculations we find:

$$g(\tau)|T^K_\Lambda(Q^*) = \sum_{b=0}^{p^2-1} g(\tau)|\left( \begin{array}{c} p \\ 24b \\
1 \\ 24bp \end{array} \right) + \mu^{-1}\sum_{b=0}^{p-1} g(\tau)|\left( \begin{array}{c} 0 \\ 1 \\
0 \\ 0 \end{array} \right).$$

(3.24)

An interesting property is:
If $f(\tau) \in \Gamma(1)$, then

$$f(\tau)|T^K_\Lambda(Q^*) = f(\tau)|T(p^2).$$

(3.25)

3.6 $\Gamma_1 = \Gamma(1)$, $\Gamma_2 = \Gamma^0(p)$, $2r = l$ an odd integer, $Q = \left( \begin{array}{c} p \\ 0 \\
0 \\ 1 \end{array} \right)$.

This case is suggested by the previous one. It is a bit more difficult in the calculations. That $T^\Lambda_K(Q)$ can be defined is proved in exactly the same way as was done in 3.5. We shall call the operator $T^0(p)$.

For $\{ \Gamma^0(p) : \Gamma^0(p) \}$ we choose $\left( \begin{array}{c} 1 \\ 0 \\
24b \\
1 \end{array} \right)$, $(b = 0, \ldots, p-1)$.

This gives us:

$$f(\tau)|T^\Lambda_K(Q) = f^p(\tau) + p^r \sum_{b=1}^{p-1} f\left( \frac{\tau}{24b + 1} \right)(24b\tau + 1)^{-r}.$$

Let $b'$, $d'$ be integers, so that $24^2b^2 \equiv 1 \pmod{p}$ and $pd' + 24b^2b' = 1$.

Then we have:

$$p^r . f\left( \frac{\tau}{24b\tau + 1} \right)(24b\tau + 1)^{-r} =$$

\[
p^r . f \left[ \frac{p \left( \frac{\tau + 24b'}{p} \right) - 24b'}{24b \left( \frac{\tau + 24b'}{p} \right) + d'} \right] (24b\tau + 1)^{-r} =
\]

\[
= \left( \frac{-24b'}{p} \right) . e^{-\frac{2im(p-1)}{2}} . f\left( \frac{\tau + 24b'}{p} \right).
\]
So we have: If \( f(\tau) \in \{ \Gamma(1), -r, \nu \} \), where \( 2r = l \) is an odd integer and \( \nu^* \) is the multiplier system of \( f\left( \frac{\tau}{p} \right) \),
then \( f(\tau) | T^0(p) \in \{ \Gamma^0(p), -r, \nu^* \} \),
where \( f(\tau) | T^0(p) = \)
\[
= p^r f(p \tau) + \left( -\frac{24}{p} \right) e^{\frac{-\pi i (p-1)}{2}} \sum_{b=1}^{p-1} \left( \frac{b}{p} \right) \frac{f\left( \frac{\tau + 24b}{p} \right)}{p}.
\] (3.26)

The connection between (3.22) and (3.26) is given by the following relation:

\[
f(\tau) | T_0(p) | T = \mu \ f(\tau) | T^0(p).
\] (3.27)

The operators we have defined are of interest because of their applications which will be discussed in § 4.

§ 4. Applications.

We shall now discuss some applications of the operators defined in § 3, especially to powers of \( \eta(\tau) \). Many of the results we find, have already been found by M. Newman (cf. Newman [1], [3], [4]) and H. Rademacher (cf. Rademacher [1]). The interesting fact is here, that all these results are consequences of the existence of Hecke-operators.

If \( f(\tau) \) is an integral modular form \( \in \{ \Gamma(1), -r, \nu \} \), where \( 2r \) is an integer, we see from (1.03) and (2.02) that the Fourier expansion of \( f(\tau) \) has the form:

\[
\sum_{n=0}^{\infty} a(n) e^{\frac{2\pi in\tau}{24}}.
\] (4.01)

We can also write \( f(\tau) \) as \( \sum_{n=0}^{\infty} a'(n) e^{\frac{2\pi in\tau}{N}} \), with \( N/24 \) and \( a'(n) = 0 \), if \( (n, N) > 1 \).

The Dirichlet series connected with this Fourier series is:

\[
\varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s}.
\] (4.02)

If, in (4.01) we have \( (n, 24) = k \) for all \( n \) with \( a(n) \neq 0 \), we shall write:

\[
\bar{\varphi}(s) = \sum_{n=1}^{\infty} a(nk) n^{-s}.
\] (4.03)
For the first applications we consider:

4.1 Even powers of $\eta(\tau)$.

If $l = 2r$ is even, then $\eta^l(\tau) \in \{ \Gamma(1), -r, v_i \}$, where $v$ is determined by $\lambda = e^{\pi i \theta}$, $r$ an integer.

First we remark that these functions could also be discussed with Hecke's theory, applied to $\{ \Gamma(12), -r, v_i \}$ (cf. theorem 4, corollary). The results would not follow as easily as they do now. In this case we find for Hecke's operator $T_p$, the relation:

$$T_p = (-1)^{\frac{r(p-1)}{2}} p^{-1} T(p). \quad (4.04)$$

Now consider $f(\tau) | T(p) \in \{ \Gamma(1), -r, v_i \}$, where $v^*$ is determined by $\lambda^* = e^{\pi i \theta}$. By (3.03) we have:

$$f(\tau) | T(p) = p^r \sum_{n=0}^{\neqln} a(n) e^{\frac{2\pi i x n}{p}} + (-1)^{\frac{r(p-1)}{2}} p^{\frac{r}{2}} \sum_{b=0}^{\neq b} a(n) e^{\frac{2\pi i (x+24b) n}{p}}.$$  

(As usual, we define $a(x) = 0$ if $x$ is not an integer.)

If $3 \mid l$, formula (4.05) is also true for $p = 3$.

We are interested in knowing if the Dirichlet series $\varphi(s)$, connected with $\eta^l(\tau)$, are Euler products in regard to certain primes. In the next paragraph we shall prove: If $f(\tau)$ is eigenfunction for $T(p)$ with eigenvalue $c$, then:

$$\varphi(s) = \sum_{n \equiv 0 (mod p)} a(n) n^{-s} \left( 1 - (-1)^{\frac{r(p-1)}{2}} c \cdot p^{-s-1} + \right. \left. + (-1)^{\frac{r(p-1)}{2}} p^{r-1-2s-1} - 1 \right). \quad (4.06)$$

We now prove:

**Theorem 9.** If $0 < l \leq 24$, $l$ is even and $l(p-1) \equiv 0 \ (mod 24)$, then $\eta^l(\tau)$ is eigenfunction of $T(p)$.

Remark: This is theorem 1 and theorem 1' of Newman [3].
Proof: If \( l(p - 1) \equiv 0 \pmod{24} \), we have by 3.1: \( \lambda^* = \lambda \) and therefore \( g(\tau) = \frac{\eta^l(\tau) | T(p)}{\eta^l(\tau)} \in \{ \Gamma(1), 0, 1 \} \). Using the properties of \( \eta(\tau) \) stated in § 1, we see that \( g(\tau) \) has no poles in the fundamental region of \( \Gamma(1) \) except possibly in \( \tau = i \infty \). In the Fourier expansion of \( \eta^l(\tau) | T(p) \) we have for the first coefficient which is not zero: \( n \equiv l \pmod{24} \) and therefore \( n \geq l \). Hence \( g(\tau) \) has no pole in \( \tau = i \infty \) and so \( g(\tau) \) has no poles at all. We now use the fact that an automorphic function \( f \in \{ \Gamma, 0, 1 \} \), which has no poles in the fundamental region of \( \Gamma \), is a constant (cf. L. R. Ford, Automorphic Functions, page 94). We have proved that \( g(\tau) \) is a constant. It is easily seen that this constant is \( p \cdot a(lp) \) if we write: \( \eta^l(\tau) = \sum_{0 < n \pmod{24}} a(n) e^{24} \). (We shall use this notation for the rest of this paragraph.)

We have proved:

\[
\eta^l(\tau) | T(p) = \begin{cases} p \cdot a(lp) \cdot \eta^l(\tau) & \text{if } l \text{ is even, } l(p - 1) \equiv 0 \pmod{24}. \end{cases} \tag{4.07}
\]

**Theorem 10.** Define \( l_1 = pl - 24 \left\lfloor \frac{pl}{24} \right\rfloor \). If \( l \) is even, \( 0 < l \leq 24 \) and \( l_1 > l \), then \( \eta^l(\tau) \) is eigenfunction of \( T(p) \).

In this case \( v^* \neq v \), and therefore the eigenvalue can only be 0.


Proof: In this case \( g(\tau) = \frac{\eta^l(\tau) | T(p)}{\eta^l(\tau)} \in \{ \Gamma(1), 0, v^*, v^{*-1} \} \),

\[
(\lambda^* \cdot \lambda^{-1})^6 = e^{\pi i \tau (p - 1)} = 1. \text{ Hence } g^6(\tau) \in \{ \Gamma(1), 0, 1 \}.
\]

\( g^6(\tau) \) has no poles in the fundamental region of \( \Gamma(1) \) except possibly in \( \tau = i \infty \). In the Fourier expansion of \( \eta^l(\tau) | T(p) \),

the first coefficient which is not zero is the coefficient of \( e^{24} \), and as \( l_1 > l \) we find that \( g^6(\tau) \) has a zero in \( \tau = i \infty \). Because \( g^6(\tau) \) has no poles in the fundamental region of \( \Gamma(1) \), \( g^6(\tau) \) must be a constant. Therefore we have \( g(\tau) = 0 \). We have found: If \( l \) is even, \( 0 < l \leq 24 \) and \( pl - 24 \left\lfloor \frac{pl}{24} \right\rfloor > l \), then

\[
\eta^l(\tau) | T(p) = 0. \tag{4.08}
\]
Theorem 11. If $0 < l \leq 24$, $l$ even, then $\eta^l(\tau)$ is eigenfunction of $T(p^2)$ for all $p > 3$, and for $p = 3$ if $3|l$.

Proof: This theorem is proved in the same way as theorem 9 was proved.

From theorems 9 and 10 we find that $\eta^l(\tau)$ is eigenfunction of $T(p)$ for all $p > 3$ if $l = 2, 4, 6, 8, 12$ or 24, i.e. if $l$ is an even divisor of 24. Denoting the Dirichlet series connected with $\eta^l(\tau)$ by $\varphi_l(s)$ we have

\[
\varphi_2(s) = \prod_{p > 3} \left( 1 - \tau_{12}(p) \cdot p^{-s} + \left( \frac{-1}{p} \right) p^{-2s} \right)^{-1} \tag{4.09}
\]

\[
\varphi_4(s) = \prod_{p > 3} \left( 1 - \tau_6(p) \cdot p^{-s} + p^{1-2s} \right)^{-1} \tag{4.10}
\]

\[
\varphi_6(s) = \prod_{p > 2} \left( 1 - \tau_4(p) \cdot p^{-s} + \left( \frac{-1}{p} \right) p^{2-2s} \right)^{-1} \tag{4.11}
\]

\[
\varphi_8(s) = \prod_{p > 3} \left( 1 - \tau_3(p) \cdot p^{-s} + p^{3-2s} \right)^{-1} \tag{4.12}
\]

\[
\varphi_{12}(s) = \prod_{p > 2} \left( 1 - \tau_2(p) \cdot p^{-s} + p^{5-2s} \right)^{-1} \tag{4.13}
\]

\[
\varphi_{24}(s) = \prod_{p} \left( 1 - \tau(p) \cdot p^{-s} + p^{11-2s} \right)^{-1} \tag{4.14}
\]

For the coefficients $\tau_n(p)$, the following expressions are known (cf. Schoeneberg [2], [3]). Let $\pi_1$ be a Gaussian prime, $\pi'_1$ its conjugate.

\[
\tau_3(p) = \begin{cases} 
\pi_1^3 + \pi'_1^3 & \text{if } p = \pi_1\pi'_1 \text{ and } \pi_1 \equiv 1 \pmod{\sqrt{-3}} \\
0 & \text{if } p \equiv 1 \pmod{3}
\end{cases}
\]

\[
\tau_4(p) = \begin{cases} 
\pi_1^2 + \pi'_1^2 & \text{if } p = \pi_1\pi'_1 \text{ and } \pi_1 \equiv 1 \pmod{\sqrt{-4}} \\
0 & \text{if } p \equiv 1 \pmod{4}
\end{cases}
\]

\[
\tau_6(p) = \begin{cases} 
\pi_1 + \pi'_1 & \text{if } p = \pi_1\pi'_1 \text{ and } \chi(\pi_1) = 1, \text{ where } \chi \text{ is the character mod } 2 \sqrt{-3} \text{ of order 6, for which } \chi(e^{2\pi i}) = e^{-2\pi i/6} \\
0 & \text{if } p \equiv 1 \pmod{6}
\end{cases}
\]

\[
\tau_8(p) = \begin{cases} 
\pi_1 + \pi'_1 & \text{if } p = \pi_1\pi'_1 \text{ and } \chi(\pi_1) = 1, \text{ where } \chi \text{ is the character mod } 2 \sqrt{-4} \text{ of order 6, for which } \chi(e^{2\pi i/4}) = e^{-2\pi i/6} \\
0 & \text{if } p \equiv 1 \pmod{4}
\end{cases}
\]
These expressions and an analogous expression for $\tau_{12}(p)$ are given in Schoeneberg [3].

By (1.08) we have

$$\tau_{12}(p) = \sum_{\substack{n^2 + m^2 = 2p \\ n > 0, \ m > 0}} \left(\frac{12}{n}\right) \left(\frac{12}{m}\right) =
$$

$$= \begin{cases} \pm 2 & p \equiv 1 \pmod{12}, \\ 0 & p \equiv \pm 1 \pmod{12}. \end{cases}$$

Expressions for $\tau_2(p)$ and $\tau(p)$ are given in Schoeneberg [2].

(Remark: for some of the functions we have discussed, operators $\mathcal{T}(2)$ and $\mathcal{T}(3)$ can also be defined as we see from the products (4.09) to (4.14).)

We now discuss the even powers of $\eta(\tau)$ which are not eigenfunction for all $T(p)$, ($1 \leq 24$). From Hecke's theory of the $T_\rho$ (Hecke [1]) and the formula (4.04) we see that $\eta^i(\tau) = \sum_i f_i(\tau)$, where the modular forms $f_i(\tau)$ are eigenfunctions of all $T(p)$.

We can take the $f_i(\tau)$ linearly independent. Applying $T(p)$ we find $\eta^i(\tau) | T(p) = \sum_i c_i(p)f_i(\tau)$, where the $c_i(p)$ are the eigenvalues.

Applying $T(p)$ once more we find

$$\eta^i(\tau) | T(p) | T(p) = \sum_i c_i^2(p)f_i(\tau).$$

By (3.08) and theorem 11:

$$\eta^i(\tau) | T(p) | T(p) = c(p)\eta^i(\tau) + \left(1 - \frac{1}{2} \right) \eta^i(\tau) = c'(p)\eta^i(\tau).$$

Because the $f_i(\tau)$ are linearly independent, we have:

$$c_i^2(p) = c'(p) \text{ for all } i.$$

Taking the functions $f_i(\tau)$ with $c_i(p) = \sqrt{c}(p)$ and those with $c_i(p) = -\sqrt{c}(p)$ together, we find: $\eta^i(\tau)$ is the sum of 2 functions which are eigenfunction for all $T(p)$. We refer to this as (4.15)

We now consider $\eta^{10}(\tau)$. This is an eigenfunction of $T(p)$ for $p \equiv \pm 5 \pmod{12}$ by theorems 9 and 10. We write $\eta^{10}(\tau) | T(p) = \sum_{n=0}^{\infty} b(n) e^{2\pi i n}$. For $p \equiv 5 \pmod{12}$ we find $b(0) = b(1) = 0$, $b(2) = p \cdot a(2p)$. 

From this we see that the function \( g(\tau) = \frac{\eta_{10}(\tau)|T(p)}{\eta^2(\tau)} e^{\frac{i}{2} \Gamma(1, -4, 1)} \)

has no poles in the fundamental region of \( \Gamma(1) \). Hence \( g(\tau) \) must be a multiple of \( G_4(\tau) \). Comparing coefficients we find:

\[ \eta_{10}(\tau)|T(p) = p \cdot a(2p) G_4(\tau) \eta^2(\tau) \text{ for } p \equiv 5 \pmod{12}. \]  

(4.16)

The two functions \( f_i(\tau) \) mentioned in (4.15) are linear combinations of \( \eta_{10}(\tau) \) and \( G_4(\tau) \eta^2(\tau) \). If we write

\[ G_4(\tau) \eta^2(\tau) = \sum_{n=0}^{\infty} c(n) e^{\frac{2\pi in}{24}}, \]

we have for \( p \equiv 5 \pmod{12}, p \not\equiv 5:\)

\[ G_4(\tau) \eta^2(\tau)|T(p) = p \cdot c(10p) \eta_{10}(\tau). \]

If \( \eta_{10}(\tau) \) is eigenfunction for all \( p \) then \( p \cdot a(2p) = x^2 p \cdot c(10p) \).

For the decomposition of \( \eta_{10}(\tau) \) into eigenfunctions of all \( T(p) \) we find:

\[ \eta_{10}(\tau) = \frac{1}{2} \eta_{10}(\tau) + \frac{1}{8} G_4(\tau) \eta^2(\tau) \]
\[ + \frac{1}{2} \eta_{10}(\tau) - \frac{1}{8} G_4(\tau) \eta^2(\tau) \]

(4.17)

The eigenvalues are:

\[ p \cdot a(10p) \text{ for } p \equiv 1 \pmod{12}, \]
\[ 48 p \cdot a(2p) \text{ and } -48 p \cdot a(2p) \text{ resp. for } p \equiv 5 \pmod{12}, \]
\[ 0 \text{ for } p \equiv 7 \pmod{12} \text{ and } p \equiv 11 \pmod{12}. \]

The Dirichlet series connected with the two functions, have Euler products. Combining these we find with (4.17):

\[ \varphi_{10}(s) = \frac{1}{96} \prod_{p \equiv 7, 11 \pmod{12}} (1 - p^{4-2s})^{-1} \cdot \prod_{p \equiv 5 \pmod{12}} (1 - a(10p)p^{-s} + p^{4-2s})^{-1}, \]

\[ \prod_{p \equiv 5 \pmod{12}} (1 - 48 a(2p) p^{-s} + p^{4-2s})^{-1} \]

(4.18)

Another problem we are interested in, is finding the zeros of \( p_i(n) \) where \( \prod_{n=1}^{\infty} (1 - x^n)^i = \sum_{n=0}^{\infty} p_i(n) x^n. \)
Writing \( \psi'(\tau) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n \tau / 24} \) we have \( p_{1}(n) = a(24n + 1) \).

Tables of the \( p_{1}(n) \) were given by Newman (Newman [1]). If \( l/24 \), the zeros of \( p_{1}(n) \) are found easily from the well known Euler products (4.9) to (4.14). For \( l = 10 \) we have by (4.18): \( p_{10}(n) \) is zero if at least one of the primes \( \equiv 7 \) or \( 11 \) (mod 12) divides \( 24n + 10 \) in an odd power, and also if all primes \( \equiv 5 \) (mod 12) divide \( 24n + 10 \) in even powers. This last condition does not give us any zeros not covered by the first condition.

We shall now consider \( \psi_{14}(\tau) \). By theorems 9 and 10 this is an eigenfunction of \( T(p) \) for \( p \equiv 1 \) (mod 12) and \( p \equiv 5 \) (mod 12). \( \psi_{14}(\tau) \) is also eigenfunction of \( T(p) \) for \( p \equiv 11 \) (mod 12).

Proof: \( g(\tau) = \frac{\psi_{14}(\tau) | T(p)}{\psi_{10}(\tau)} \) is a modular form \( e \{ \Gamma(1), -2, 1 \} \) without poles in the fundamental region of \( \Gamma(1) \). The only function satisfying these conditions is \( g(\tau) = 0 \). We remark that this same proof can be applied in the case of \( \psi_{25}(\tau) \) and \( p \equiv 11 \) (mod 12).

By the same method as was used for \( \psi_{10}(\tau) \) we find:
\[
\psi_{14}(\tau) | T(p) = -p . a(2p) G_{6}(\tau) \psi^{2}(\tau) \text{ for } p \equiv 7 \text{ (mod 12).} \tag{4.19}
\]

\[
\psi_{14}(\tau) = \frac{i}{360 \sqrt{3}} \frac{G_{6}(\tau) \psi^{2}(\tau)}{2} + \frac{i}{360 \sqrt{3}} G_{6}(\tau) \psi^{2}(\tau) \tag{4.20}
\]

\[
= f_{1}(\tau) + f_{2}(\tau) \text{ with } f_{1}(\tau) \text{ eigenfunction for all } T(p), \ (p > 3).
\]

From this we find \( \tilde{\psi}_{14}(s) \) as sum of two Euler products:
\[
\tilde{\psi}_{14}(s) = \frac{i}{720 \sqrt{3}} \prod_{p \equiv 5, 11 \text{ (mod 12)}} \left( 1 + (-1)^{\frac{p-1}{2}} p^{6-2s} \right)^{-1} \cdot \prod_{p \equiv 1 \text{ (mod 12)}} \left( 1 - a(14p) p^{-s} + p^{6-2s} \right)^{-1}.
\]

\[
\prod_{p \equiv 7 \text{ (mod 12)}} \left( 1 + 360 i \sqrt{3} . a(2p) p^{-s} \right) - p^{6-2s} \right)^{-1}
\]

\[
\prod_{p \equiv 7 \text{ (mod 12)}} (1 - 360 i \sqrt{3} . a(2p) p^{-s} - p^{6-2s})^{-1} \right).
\]

\[
(4.21)
\]
So \( p_{14}(n) = 0 \) if one of the primes \( \equiv 5 \) or \( 11 \) (mod 12) divides \( 24n + 14 \) in an odd power, or if all primes \( \equiv 7 \) (mod 12) divide \( 24n + 14 \) in even powers. Again this second condition does not give us any zeros not covered by the first condition.

To discuss \( \eta^{16}(\tau) \) completely we must introduce an operator \( T(2) \). To do this we choose \( \Lambda = \{ \Gamma(1), -8, v^* \} \), where \( v^* \) is the multiplier system of \( \eta^8(\tau) \). By using one of the known explicit expressions for the multiplier system of \( \eta(\tau) \), it is easily seen that we have \( v^* = v \mid Q \) if \( Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \). Therefore \( T(2) \) can be defined. In the same way as (4.16) and (4.19) we now find:

\[
\eta^{16}(\tau) \mid T(2) = 2 G_4(\tau) \eta^8(\tau),
\]

and for all primes \( p \equiv 2 \) (mod 3) the analogous result:

\[
\eta^{16}(\tau) \mid T(p) = p \cdot a(8p) G_4(\tau) \eta^8(\tau).
\]

By theorem 9, \( \eta^{16}(\tau) \) is eigenfunction of \( T(p) \) for \( p \equiv 1 \) (mod 3). The decomposition into eigenfunctions of all \( T(p) \) is:

\[
\eta^{16}(\tau) = \frac{1}{2} \left\{ \eta^{16}(\tau) + \frac{1}{6\sqrt[5]{10}} G_4(\tau) \eta^8(\tau) \right\} + \\
+ \frac{1}{2} \left\{ \eta^{16}(\tau) - \frac{1}{6\sqrt[5]{10}} G_4(\tau) \eta^8(\tau) \right\}.
\]

From (4.25) we find: \( p_{16}(n) = 0 \) if all primes \( \equiv 2 \) (mod 3) divide \( 3n + 2 \) in even powers, but this is not possible. In this way we do not find zeros of \( p_{16}(n) \). It is not known if \( p_{16}(n) \) has zeros. By the tables in Newman [1], \( p_{16}(n) \neq 0 \) for \( n \leq 400 \).

The functions \( \eta^{18}(\tau), \eta^{20}(\tau) \) and \( \eta^{22}(\tau) \) can also be treated as was done for the other even powers of \( \eta(\tau) \). For \( \eta^{22}(\tau) \) the formulae become much more complicated. We shall give the results for \( \eta^{18}(\tau) \) and \( \eta^{20}(\tau) \).

\[
\eta^{18}(\tau) \mid T(p) = \begin{cases} 
p \cdot a(18p) \eta^{18}(\tau) & \text{for } p \equiv 1 \text{ (mod 4)}, \\
- p \cdot a(6p) G_6(\tau) \eta^6(\tau) & \text{for } p \equiv 3 \text{ (mod 4)}. \end{cases}
\]
\[
\bar{\varphi}_{18}(s) = \frac{i}{48\sqrt[3]{35}} \prod_{p \equiv 1 \pmod{4}} (1 - a(18p) p^{-s} + p^{8-2s})^{-1}.
\]

\[
\bar{\varphi}_{20}(s) = \frac{1}{576\sqrt[3]{70}} \prod_{p \equiv 1 \pmod{6}} (1 - a(20p) p^{-s} + p^{9-2s})^{-1}.
\]

As was the case for \(\gamma^{16}(\tau)\), this formula gives us no zeros of \(p_{18}(n)\).

\[
\gamma^{20}(\tau) \mid T(p) = p \cdot a(20p) \gamma^{20}(\tau) \quad \text{for } p \equiv 1 \pmod{6},
\]

\[
\gamma^{20}(\tau) \mid T(p) = p \cdot a(4p) G_8(\tau) \gamma^{4}(\tau) \quad \text{for } p \equiv 5 \pmod{6}.
\]

We find no zeros of \(p_{20}(n)\).

### 4.2 Odd, positive powers of \(\gamma(\tau)\).

Consider \(\gamma^r(\tau) \in \{\Gamma(1), -r, \nu\}\) where \(2r \equiv l\) is odd, \(v\) determined by \(\lambda = e^{\frac{i\pi}{6}}\). As we have seen, operators \(T(p)\) generally cannot be defined for these functions. We write

\[
\gamma^r(\tau) = f(\tau) = \sum_{n=1}^{\infty} a(n) e^{\frac{2\pi in}{24}}.
\]

We find for \(p > 3\) from (3.13) for every \(f(\tau) \in \{\Gamma(1), -r, \nu\}\):

\[
f(\tau) \mid T(p^2) = p^{2r} \sum_{n=1}^{\infty} a(n) e^{\frac{2\pi in}{24}} +
\]

\[
+ \gamma_p \cdot p^r \sum_{n=1}^{\infty} a(n) \left( \sum_{b=1}^{p-1} \left( \frac{3b}{p} \right) e^{\frac{2\pi ibn}{p}} \right) e^{\frac{2\pi in}{24}} +
\]

\[
+ \sum_{n=1}^{\infty} a(n) \left( \sum_{b=0}^{p^2-1} e^{\frac{2\pi ib}{p^2}} \right) e^{\frac{2\pi in}{24p^2}} =
\]

\[
= \sum_{n=1}^{\infty} \left( p^{2r} a\left( \frac{n}{p^2} \right) + \left( \frac{-1}{p} \right)^{\frac{l-1}{2}} \left( \frac{3n}{p} \right) p^{-\frac{l}{2}} a(n) + p^2 a(np^2) \right) e^{\frac{2\pi in}{24}}.
\]
If $3/l$ we have $f(\tau) \mid T(3^2) =$
\[
= \sum_{n=0}^{\infty} \left\{ 3^{2r} a \left( \frac{n}{9} \right) + (-1)^{\frac{l-1}{2}} \left( \frac{n/3}{3} \right)^{3r+\frac{1}{2}} a(n) + 9a(9n) \right\} e^{2\pi i n}. \tag{4.31}
\]

**Theorem 12.** If $0 < l < 24, l$ odd, then $\eta^l(\tau) \text{ is eigenfunction of } T(p^2)$ for all $p > 3$, and for $p = 3$ if $3/l$.

Proof: Consider $g(\tau) = \frac{\eta^l(\tau) \mid T(p^2)}{\eta^l(\tau)}$. By the same reasoning as was used for even powers of $\eta(\tau)$, we see that $g(\tau) \in \{\Gamma(1), 0, 1\}$ and that $g(\tau)$ has no poles in the fundamental region of $\Gamma(1)$. Therefore $g(\tau)$ is a constant. By computing $g(\tau)$ in $\tau = i \infty$ we find
\[
g(\tau) = \left( \frac{-1}{p} \right)^{\frac{l-1}{2}} \left( \frac{24}{3} \right)^{3r+\frac{1}{2}} a(\tau) + p^2 a(lp^2) \quad \text{for } p > 3 \quad \text{and}
\]
\[
g(\tau) = (-1)^{\frac{l-1}{2}} \left( \frac{l/3}{3} \right)^{3r+\frac{1}{2}} + 9a(9l) \quad \text{for } p = 3 \text{ if } 3/l.
\]

With this theorem we can deduce relations for the numbers $p_l(n)$. We have
\[
p^{2r} a \left( \frac{n}{p^2} \right) + \left( \frac{-1}{p} \right)^{\frac{l-1}{2}} \left( \frac{3n}{p} \right) . p^{r+\frac{1}{2}} a(n) + p^2 . a(np^2) =
\]
\[
= \left\{ \left( \frac{-1}{p} \right)^{\frac{l-1}{2}} \left( \frac{3l}{p} \right) . p^{r+\frac{1}{2}} + p^2 . a(lp^2) \right\} a(n). \tag{4.32}
\]

We now write $n = 24m + l, \Delta_p = \frac{p^2 - 1}{24}$ and we use $a(n) = p \left( \frac{n - l}{24} \right)$.

Then (4.32) becomes:
\[
p^{2r} \cdot p_l \left( \frac{m - l\Delta_p}{p^2} \right) +
\]
\[
+ \left\{ \left( \frac{-1}{p} \right)^{\frac{l-1}{2}} \left( \frac{3}{p} \right) . p^{r+\frac{1}{2}} \left[ \left( \frac{24m + l}{p} \right) - \left( \frac{l}{p} \right) \right] - p^2 . p_l(l\Delta_p) \right\} p_l(m) +
\]
\[
+ p^2 . p_l(mp^2 + l\Delta_p) = 0 \quad \text{for } p > 3. \tag{4.33}
\]
For instance: \( l = 5, p = 5, m = 5n \) gives us

\[ 5^3 p_5 \left( \frac{n - 1}{5} \right) + 6 p_5(5n) + p_5(125n + 5) = 0. \]

In the same way: \( l = 5, p = 7, m = 7n + 3 \) gives us

\[ 7^3 p_5 \left( \frac{n - 1}{7} \right) - 16 p_5(7n + 3) + p_5(343n + 157) = 0. \]

These, and many other relations are mentioned in Newman [1]. As we have seen they are all special cases of the general formula (4.33).

For \( 3/l, p = 3 \), formula (4.33) is true if we replace \( \left( \frac{l}{p} \right) \) by \( \left( \frac{1/3}{3} \right) \). The author did not succeed in finding zeros of \( p_1(n) \) by using these formulae as was done for even \( l \). Of course if a zero is known, others can be found from (4.32). We have by induction: If \( a(n) = 0 \) and \( p^2 \times n \ (p > 3) \) then \( a(np^{2k}) = 0 \).

If \( 3/l, a(n) = 0 \) and \( 27 \times n \) then \( a(3^{2k}n) = 0 \). It has been shown that \( p_{15}(53) = 0 \) (cf. Newman [1]). So we find:

\[ p_{15} \left( 53n^{2k} + 15 \frac{n^{2k} - 1}{24} \right) = 0 \text{ for all odd } n. \quad (4.34) \]

Whether there are other zeros than these and whether the fact that \( p_{15}(53) = 0 \) has a deeper significance or not remains an unsolved problem.

The formulae we can find by applying the operators \( T_0(p) \) and \( T^0(p) \) are generally too complicated to be of any use in finding relations for the coefficients \( p_l(n) \). To illustrate this, we shall give one example (here the result is still relatively simple).

Consider \( g(\tau) = \frac{\chi^5(\tau) T_0(7)}{\chi^5(7\tau)} \). By (3.22) we have:

\[ g(\tau) = \frac{1}{\sum_{n=0}^{\infty} a(n) e^{2\pi i n.7}} \sum_{n=0}^{\infty} a(7n) e^{\frac{2\pi i n}{24}} \in \{T_0(p), 0, 1\}. \]
Applying the operator $T$ and using (3.27) we find:

$$g(\tau) = \sum_{n=0}^{\infty} a(n) e^{-2\pi n \tau} = 2\pi i \sum_{n=0}^{\infty} a(n) e^{-2\pi n \tau} + 75 \sum_{n=0}^{\infty} a(n) e^{-\pi n \tau} \cdot \frac{49n}{24}$$

We see that the function $g(\tau) - 7^3$ has a pole of order 1 in $e^{2\pi n \tau}$ in $\tau = i \infty$ and a zero of order 1 in $e^{-\pi n \tau}$ in $\tau = 0$ and no other poles or zeros in the fundamental region of $T_0(p)$. The function $\left\{ \frac{\eta(\tau)}{\eta(7\tau)} \right\}^4$ has the same properties and therefore we have

$$g(\tau) - 7^3 = c \left\{ \frac{\eta(\tau)}{\eta(7\tau)} \right\}^4.$$

Comparing coefficients we find $c = 7 a(77) = 70$, and so:

$$\eta^5(\tau) \mid T_0(7) = 70 \eta^4(\tau) \eta(7\tau) + 7^3 \eta^5(7\tau).$$

(4.35)

Even this simple result does not give us linear relations for the $p_5(n)$.

4.3 Applications to $\eta^{-1}(\tau)$.

The operators we defined can also be used for negative values of $\tau$. An interesting case is the function $\eta^{-1}(\tau)$.

$$\eta^{-1}(\tau) = e^{\frac{1}{12} \sum_{n=0}^{\infty} p(n) e^{2\pi n \tau}},$$

where $p(0) = 1$ and $p(n)$ is the number of partitions of $n$. By proving formulae as $\eta^{-1}(\tau) \mid T_0(5) = 5^2 \eta^5(5\tau) \eta^{-6}(\tau)$ and

$$\eta^{-1}(\tau) \mid T_0(7) = 7^2 \eta^3(7\tau) \eta^{-4}(\tau) + 7^3 \eta^7(7\tau) \eta^{-3}(\tau),$$

we can give proofs of the Ramanujan congruences

$p(5n + 4) \equiv 0 \pmod{5}$ and $p(7n + 5) \equiv 0 \pmod{7}$ and others.

These proofs were given by Rademacher (Rademacher [1]). The theorems he used were special cases or applications of the theorems we have proved.

We could also apply the operators $T(p^2)$ to $\eta^{-1}(\tau)$. As $\eta^{-1}(\tau)$ is not eigenfunction of these operators, the results do not give us linear relations for $p(n)$. 
We give one example:
\[ g(\tau) = \frac{\eta^{-1}(\tau) | T(p^2) \rangle}{\eta^{-\tau^2}(\tau)} e \left\{ \Gamma(1), -\frac{p^2-1}{2}, 1 \right\} \]
and has no poles in the fundamental region of \( \Gamma(1) \). Therefore \( g(\tau) \) can be written as
\[ \sum_{n=0}^{k} x_n \Delta^n(\tau) G_{12(k-n)}(\tau), \left( k = \frac{p^2-1}{24} \right). \]

For instance
\[ \frac{\eta^{-1}(\tau) | T(25) \rangle}{\eta^{-25}(\tau)} = -\frac{17250}{691} \Delta(\tau) + \frac{1}{2} G_{12}(\tau). \]

4.4 Applications to \( \mathcal{S}(\tau) \).

We can also use the methods of this paragraph to prove some results of Hurwitz [1] and Sandham [1] on the representation of a square as a sum of squares. We apply \( T(p^2) \) to \( \mathcal{S}(\tau) \), \( \langle \Gamma_{\theta}, -\frac{l}{2}, v_{\theta} \rangle \), where \( v_{\theta} \) is the multiplier system of \( \mathcal{S}(\tau) \).

\( \mathcal{S}(\tau) | T(p^2) \) has no poles in the fundamental region of \( \Gamma_{\theta} \), because \( \mathcal{S}(\tau) \) has this property. Therefore the behaviour in \( \tau = -1 \) is described by:
\[ (\tau + 1)^{l/2} | \mathcal{S}(\tau) | T(p^2) \rangle = e^{2\pi i \left( \frac{\tau}{l+1} \right)} \sum_{r=-k}^{\infty} c_r e^{2\pi i \left( \frac{\tau}{l+1} \right) r}, \]
where \( k \leq \frac{l}{8} \). (4.36)

Let \( l \) be 3, 5 or 7. Consider \( g(\tau) = \frac{\mathcal{S}(\tau) | T(p^2) \rangle}{\mathcal{S}(\tau)} \).

We have: \( g(\tau) \in \{ \Gamma_{\theta}, 0, 1 \} \). As \( \theta(\tau) \) has no zeros in the fundamental region of \( \Gamma_{\theta} \) except in the limit points \( \tau = \pm 1 \), we see that \( g(\tau) \) has no poles in the fundamental region of \( \Gamma_{\theta} \) except possibly in \( \tau = \pm 1 \). From (1.10) and (4.36) we see that \( g(\tau) \) is regular in \( \tau = \pm 1 \). Therefore \( g(\tau) \) has no poles in the fundamental region of \( \Gamma_{\theta} \). Hence \( g(\tau) \) is a constant.

Let \( r_1(n) \) be the number of representations of \( n \) as a sum of \( l \) squares. Then \( \theta^l(\tau) = \sum_{n=0}^{\infty} r_1(n) e^{2\pi in \tau} \) and
\[ \mathcal{S}(\tau) | T(p^2) \rangle = \sum_{n=0}^{\infty} \left( p^l r_1 \left( \frac{n}{p^2} \right) + \left( \frac{p}{n} \right)^{l+1} \right) \cdot r_1(n) + p^2 r_1 (np^2) e^{2\pi in \tau}. \]
By comparing the constant terms in these series we find
\[ g(\tau) = p^i + p^j. \] This gives us the relation:
\[ p^i r_1 \left( \frac{n}{p^2} \right) + (-1)^{\frac{i-1}{2}} \cdot p^{\frac{j+1}{2}} \left( \frac{n}{p} \right) r_1(n) + p^j r_1(np^2) = (p^i + p^j) r_1(n). \] (4.37)

Taking \( n = 1 \) we find:
\[ r_3(p^2) = 6 \left\{ p + 1 - \left( \frac{-1}{p} \right) \right\}, \]
\[ r_5(p^2) = 10 \left\{ p^3 - p + 1 \right\}, \]
\[ r_7(p^2) = 14 \left\{ p^5 + 1 - \left( \frac{-1}{p} \right) p^2 \right\}. \] (4.38)

§ 5. Linear relations and Euler products.

In this paragraph we shall study modular forms for which the associated Dirichlet series have an Euler product. Hecke [1, 3] and Petersson [2] studied this problem for modular forms of integral dimension. For the many results concerning these forms, we refer to their papers. We are interested in the problem for nonintegral dimension. Some special cases were discussed by Hecke [2]. First we shall prove some generalizations of theorems in Hecke [1], § 9.

**Theorem 13.** Let \( f(\tau) \) be a modular form \( \in \{ \Gamma(Q), -r, v \} \), and suppose that for some primitive transformation \( S_n \) of order \( n \), with \( (n, Q) = 1 \), we have \( f(\tau) \mid S_n \in \{ \Gamma(Q), -r, v^* \} \). Then \( f(\tau) \) must be identically zero.

**Proof:** The proof is nearly the same as in Hecke [1]. Without loss of generality we may take \( S_n \) to be \( \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} \). Now we have
\[ f \mid S_n U^Q = v^* (U^Q) f \mid S_n, \]
and therefore
\[ f \mid S_n U^Q S_n^{-1} = f \mid S_n U^Q \mid S_n^{-1} = v^* (U^Q) f \mid S_n \mid S_n^{-1} = v^* (U^Q) f, \]
and so:
\[ f \mid \begin{pmatrix} n & Q \\ 0 & n \end{pmatrix} = n^e v^* (U^Q) f. \]
Choose \( A = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \in \Gamma(Q) \), with \( a_1 \equiv c_1 \equiv 1 \mod n \).

Then \( V = A \begin{pmatrix} n & Q \\ 0 & n \end{pmatrix} \equiv \begin{pmatrix} 0 & Q \\ 0 & Q \end{pmatrix} \mod n \), and \( f \mid V = \)
\[ f \mid A \begin{pmatrix} n & Q \\ 0 & n \end{pmatrix} = v(A)v^*(UQ)^{n^t}f. \]

All powers of \( V \) are primitive matrices. Choose \( l \) so that \( n^l \equiv 1 \mod Q \), and find matrices \( B, C \in \Gamma(1) \) so that \( V^l = BS_n^{2l}C \). Then \( V^l \equiv BC \equiv I \mod Q \). For \( g(\tau) = f(\tau) \mid B \in \Gamma(Q) \), \( -r, v \mid B \) we now have:
\[ g \mid S_n^{2l} = f \mid B \mid S_n^{2l} = f \mid BS_n^{2l} = \sigma (V^l, C^{-1}) f \mid V^l \mid C^{-1} = \]
\[ = \sigma (V^l, C^{-1}) v^l (A) v^*(UQ)^{n^t} f \mid C^{-1} \quad \text{and} \]
\[ g = f \mid B = \sigma (BC, C^{-1}) f \mid BC \mid C^{-1} = \sigma (BC, C^{-1}) v (BC) f \mid C^{-1}. \]

Hence \( g \mid S_n^{2l} = \gamma n^{rl} g \),

where \( \gamma = \sigma (V^l, C^{-1}) \sigma^{-1} (BC, C^{-1}) v^{-1} (BC) v^l (A) v^{*-1} (UQ) \).

As \( g(\tau) \) only changes by a factor \( \gamma n^{rl} \) when transformed by \( S_n^{2l} \), \( g(\tau) \) must be identically zero. Hence \( f(\tau) \) is identically zero.

Corollary: If \( f(\tau) = \sum a(n) e^{-\frac{n}{N}} \in \Gamma(Q) \), \( -r, v \mid 1 \), and \( p \) is a prime for which \( (p, N) = 1 \), and \( a(n) = 0 \) if \( p \nmid n \), then \( f(\tau) \) is identically zero. Proof: We have \( f(\tau) = f(\tau + \frac{N}{p}) \), and the transformation matrix \( \begin{pmatrix} p & N \\ 0 & p \end{pmatrix} \) is primitive.

**Theorem 14.** If \( f(\tau) = \sum_{n=0}^{\infty} a(n) e^{-\frac{n^{2ln}}{N}} \), and a relation
\[ f(p\tau) + \alpha \sum_{b=0}^{p-1} f \left( \frac{\tau + Nb}{p} \right) = \beta \cdot f(\tau) \text{ holds, we have formally:} \]
\[ \sum_{n=1}^{\infty} a(n) n^{-s} = \]
\[ = \left( \sum_{(n,p)=1} a(n) n^{-s} \right) \left( 1 - \beta \cdot \alpha^{-1} p^{-1-s} + \alpha^{-1} p^{-1-2s} - 1 \right). \quad (5.01) \]

Proof: From the linear relation we find \( a \left( \frac{n}{p} \right) + \alpha p a(np) = \)
\[ = \beta a(n) \), and therefore we have for \((n,p) = 1\):
\[ a(np^k) = a(n)c(p^k) \text{ where } c(1) = 1, c(p) = \beta \cdot (np)^{-1}, \text{ and } \]
\[ \alpha p c(p^{k+1}) + c(p^{k-1}) = \beta \cdot c(p^{k}). \]

From this (5.01) follows by dividing by \( p^k \) and summing over \( k \).

Modular forms \( f \in \{ \Gamma(1), -r, v \} \) with integral \( r \), which are eigenfunctions of the operators \( T(p) \), satisfy the conditions of theorem 14. For nonintegral \( r \) we generally do not have an operator of the type of \( T(p) \). We can define these operators for functions \( f(\tau) \) for which the exception mentioned in theorem 2 holds. We shall try to find modular forms \( f(\tau) \) for which the transformed functions \( f(p\tau), f\left(\frac{\tau+b}{p}\right), (b = 0, \ldots, p-1) \), are linearly dependent.

We can consider the linear relation as an operator of order \( p \), mapping \( f(\tau) \) onto 0. We shall study this problem for the groups \( \Gamma(1), \Gamma_0(2) \) and \( \Gamma_0 \).

Let \( f(\tau) \) be an integral modular form \( f \in \{ \Gamma(1), -r, v \} \), and suppose a linear relation \( \alpha f(p\tau) + \sum_{b=0}^{p-1} \alpha_b f\left(\frac{\tau+b}{p}\right) = 0 \) (5.02)
holds. We may suppose \( \alpha \neq 0 \), for application of a unimodular transformation changes (5.02) into an other relation, in which the coefficients \( \alpha_b \) have been permuted and multiplied by certain multipliers \( v \). (If all the coefficients are zero, the relation is trivial). We may take \( \alpha = 1 \).

We now apply the transformation \( U^p \) to (5.02). This gives us:
\[ \lambda p^2 f(p\tau) + \sum_{b=0}^{p-1} \alpha_b f\left(\frac{\tau+b}{p}\right) = 0 (\text{where } \lambda = \nu(U^p)). \]
Combining this with (5.02) we find \( (\lambda p^2 - 1) f(p\tau) = 0 \). If \( f(p\tau) \) is not identically zero we have \( \lambda p^2 - 1 = 1 \), that is:
\[ \frac{r(p^2-1)}{12} \text{ is an integer. We have found:} \]

**Theorem 15.** If \( f(\tau) \in \{ \Gamma(1), -r, v \} \), and a linear relation (5.02) holds for almost all primes, then \( f(\tau) \) is identically zero or \( 2r \) is an integer.

As we are interested in functions \( f \in \{ \Gamma(1), -r, v \} \) for which the linear relations hold for all \( p \geq 3 \), we only have to consider the
case where $2r$ is an integer. First we shall simplify the form of the linear relation. We have seen in (4.01) that we can write $f(\tau)$ as $\sum_{n=0}^{\infty} a(n) e^{2\pi in\tau}/N$, where $N/24$, and $a(n) = 0$ if $(n, N) > 1$.

We write the linear relation as:

$$f(p\tau) + \sum_{b=0}^{p-1} \alpha_b f\left(\frac{\tau + Nb}{p}\right) = 0. \quad (5.03)$$

We write this as:

$$\sum_{n=0}^{\infty} a(n) e^{2\pi inp\tau}/N + \sum_{n=0}^{\infty} a(n) \left(\sum_{b=0}^{p-1} \alpha_b e^{2\pi ibn\tau}/p\right) e^{2\pi inp\tau}/N = 0.$$

From this we find, with $\alpha = \sum_{b=0}^{p-1} \alpha_b$, the linear relation:

$$f(p\tau) + \alpha \sum_{b=0}^{p-1} f\left(\frac{\tau + Nb}{p}\right) = 0. \quad (5.04)$$

If a linear relation (5.02) holds, then the relation (5.04) holds. We shall now prove that if a relation (5.04) holds, no essentially different transformation of order $p$ can be defined. Suppose that (5.04) holds and also $f(p\tau) + \sum_{b=0}^{p-1} \alpha_b f\left(\frac{\tau + Nb}{p}\right) = g(\tau)$, with $g(\tau) \in J(\Gamma(1), -r, v^*)$. To this we apply the transformation $U^p_k$, and then sum over $k = 0, \ldots, p-1$. We find:

$$f(p\tau) + \beta \sum_{b=0}^{p-1} f\left(\frac{\tau + Nb}{p}\right) = g(\tau), \text{ where } \beta = \frac{1}{p} \sum_{b=0}^{p-1} \alpha_b. \text{ This combined with (5.04) gives us:}$$

$$(\alpha - \beta) f(p\tau) = \alpha g(\tau) \in J(\Gamma(1), -r, v^*).$$

If $f(\tau)$ is not identically zero we find, applying theorem 13, $\alpha = \beta$ and $g(\tau) = 0$. We have proved:

**Theorem 16.** If $f(\tau) \in J(\Gamma(1), -r, v^*)$, and a relation (5.04) holds, then a linear combination $f(p\tau) + \sum_{b=0}^{p-1} \alpha_b f\left(\frac{\tau + Nb}{p}\right)$ can only be a modular form $\in J(\Gamma(1), -r, v^*)$ if it is identically zero.
Corollary: If \( f(\tau) \in \{ \Gamma(1), -\tau, \nu \} \), \( \tau \) an integer, and a relation (5.04) holds, then we have \( f(\tau) | T(p) = 0 \). In § 4 we have seen examples in which \( f(\tau) | T(p) = 0 \) for some primes, but not for all \( p > 3 \). Hecke [2] proved that for the functions \( \eta(\tau) \) and \( \eta^3(\tau) \) relations (5.04) exist for all \( p > 3 \) (for \( \eta^3(\tau) \) also for \( p = 3 \)). We shall prove that there are no other integral modular forms \( \in \{ \Gamma(1), -\tau, \nu \} \), for which the relation (5.04) holds for all \( p \) with \( (p, N) = 1 \). For this proof we shall need a theorem proved by Hecke [4].

Definition: Let \( \lambda \) and \( \nu \) be positive constants, and \( \gamma = \pm 1 \). We say \( f(\tau) \) is an automorphic form with signature \( \{ \lambda, \nu, \gamma \} \) if:

1. \( f(\tau) \) is regular for \( \text{Im} \, \tau > 0 \), and \( f(\tau) \) is represented by a series \( \sum_{n=0}^{\infty} a(n) e^{-\lambda n} \), which implies \( f(\tau + \lambda) = f(\tau) \) if \( \text{Im} \, \tau > 0 \).
2. \( f\left(-\frac{1}{\tau}\right) = \gamma(-i\tau)^\nu f(\tau), \quad (\text{Im} \, \tau > 0) \).
3. \( a(n) = O(n^K) \) for some \( K > 0 \).

Definition: A complex function \( \varphi(s) \) is called a Dirichlet series with signature \( \{ \lambda, \nu, \gamma \} \) if:

1. In some right half-plane \( \varphi(s) \) is represented by a Dirichlet series \( \sum_{n=1}^{\infty} a(n) n^{-s} \).
2. \( (s - \nu) \varphi(s) \) is an integral function of \( s \) of finite genus.
3. The function \( R(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s) \varphi(s) \) satisfies the functional equation: \( R(s) = \gamma R(\nu - s) \).

Theorem 17. If \( f(\tau) = \sum_{n=0}^{\infty} a(n) e^{\frac{2\pi in}{\lambda}} \) is an automorphic form with signature \( \{ \lambda, \nu, \gamma \} \), then the Dirichlet series \( \sum_{n=1}^{\infty} a(n) n^{-s} \) converges somewhere and is a Dirichlet series with signature \( \{ \lambda, \nu, \gamma \} \).

For a proof we refer to Hecke's paper.
Let \( f(\tau) = \sum_{n=0}^{\infty} a(n) \, e^{2\pi i n \tau} \) be an integral modular form \( \in \{ \Gamma(1), -r, v \} \) satisfying (5.04) for all \( p \) with \((p, N) = 1\). (In the following we suppose that the functions are normed so that \( a(1) = 1 \).) Then, by theorem 14, we have:

\[
\sum_{n=1}^{\infty} a(n) n^{-s} = \prod_{(p, N) = 1} (1 + \chi^{-1}(p) p^{-1-2s})^{-1}.
\]  

(5.05)

We have to consider different cases separately.

First let \( 2r = l \) be an odd integer, and \( N = 24 \). Then, by (4.30),

\[
g(\tau) = f(\tau) | T(p^2) - \left( \frac{-1}{p} \right)^{l-1} \cdot \left( \frac{3}{p} \right) p^{s+1} - \chi^{-1}(p) p \cdot f(\tau) = \sum_{n=0}^{\infty} b(n) e^{2\pi i n \tau}
\]

is a form \( \in \{ \Gamma(1), -r, v \} \) with \( b(n) = 0 \) if \( p \not| n \), and therefore, by theorem 13 (corollary), we have \( g(\tau) = 0 \). From this and (4.30) we find:

\[
\chi(p) = -\left( \frac{-1}{p} \right)^{l-1} \cdot \left( \frac{3}{p} \right) p^{s+1}.
\]  

(5.06)

Therefore:

\[
\varphi(s) = \sum_{n=1}^{\infty} a(n) n^{-s} = \prod_{p > 3} \left( 1 - \left( \frac{-1}{p} \right)^{l-1} \cdot \left( \frac{3}{p} \right) p^{s-1-2s} \right)^{-1} = L(2s - r + \frac{1}{2}, \chi).
\]  

(5.07)

with \( \chi(p) = \left( \frac{-1}{p} \right)^{l-1} \cdot \left( \frac{3}{p} \right) \). Here the Dirichlet \( L \)-series with character \( \chi \) is defined as \( L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \). The function \( L(s, \chi) \) satisfies a functional equation. Let \( \chi \) be a proper character \( \mod m \). If \( \chi(-1) = 1 \) and \( g(s, \chi) = \left( \frac{m}{\pi} \right)^{s/2} \Gamma \left( \frac{s}{2} \right) L(s, \chi) \), we have

\[
g(s, \chi) = c_\chi \, g(1-s, \chi), \text{ where } c_\chi = \frac{1}{Vm} \sum_{k=1}^{m} \chi(k) e^{\frac{2\pi ik}{m}}.
\]  

(5.08)

If \( \chi(-1) = -1 \), and \( g(s, \chi) = \left( \frac{m}{\pi} \right)^{s/2} \Gamma \left( \frac{1+s}{2} \right) L(s, \chi) \) we have:

\[
g(s, \chi) = -i \cdot c_\chi \, g(1-s, \chi).
\]  

(5.09)
Using the functional equations (5.08), (5.09) we find a functional equation for \( \varphi(s) \):

\[
\left( \frac{2\pi}{24} \right)^{-s} \Gamma \left( s - \frac{r}{2} + \frac{1}{4} \right) \varphi(s) =
= \left( \frac{2\pi}{24} \right)^{s-r} \Gamma \left( -s + \frac{r}{2} + \frac{1}{4} \right) \varphi(r-s) \text{ if } l \equiv 1 \pmod{4}, \quad (5.10)
\]

\[
\left( \frac{2\pi}{12} \right)^{-s} \Gamma \left( s - \frac{r}{2} + \frac{3}{4} \right) \varphi(s) =
= \left( \frac{2\pi}{12} \right)^{s-r} \Gamma \left( -s + \frac{r}{2} + \frac{3}{4} \right) \varphi(r-s) \text{ if } l \equiv 3 \pmod{4}. \quad (5.11)
\]

By (5.05) and (5.06) we have for the coefficients:

\[ a(n) = O(n^{3/2 - \frac{1}{4}}). \]

Hence \( f(\tau) \) is an automorphic form with signature \( \{N, r, \gamma\} \).

By theorem 17 this gives us the following functional equation for \( \varphi(s) \):

\[
\left( \frac{2\pi}{24} \right)^{-s} \Gamma(s) \varphi(s) = \gamma \left( \frac{2\pi}{24} \right)^{s-r} \Gamma(r-s) \varphi(r-s). \quad (5.12)
\]

From (5.10) and (5.12) we find:

\[
\frac{\Gamma \left( s - \frac{r}{2} + \frac{1}{4} \right)}{\Gamma(s)} = \gamma \frac{\Gamma \left( -s + \frac{r}{2} + \frac{1}{4} \right)}{\Gamma(r-s)}. \quad (5.13)
\]

This has the solution \( \gamma = 1, r = \frac{1}{4} \), which gives us \( f(\tau) = \chi(\tau) \).

If \( r > \frac{1}{2} \) we can write for (5.13):

\[
\left( -s + \frac{r}{2} + \frac{1}{4} \right) \left( -s + \frac{r}{2} + \frac{1}{4} + 1 \right) \ldots \left( -s + r - 1 \right) =
= \gamma \left( s - \frac{r}{2} + \frac{1}{4} \right) \left( s - \frac{r}{2} + \frac{1}{4} + 1 \right) \ldots (s - 1). \quad (5.14)
\]

This has no solutions \( r, \gamma \).

From (5.11) and (5.12) we find:

\[
2^{-s} \frac{\Gamma \left( s - \frac{r}{2} + \frac{3}{4} \right)}{\Gamma(s)} = \gamma \cdot 2^{-s+r} \frac{\Gamma \left( -s + \frac{r}{2} + \frac{3}{4} \right)}{\Gamma(r-s)}, \quad (5.15)
\]

By considering the behaviour for \( s \to \infty \), we see that this has no solution \( r, \gamma \).
Now let $l$ be odd, and $N = 8$. In the same way as in the previous case we find:

$$f(\tau) | T(p^2) - \left\{ \left( \frac{-1}{p} \right)^{\frac{l-1}{2}} p^{r+i} - \kappa^{-1}(p)p \right\}_f(\tau) = 0,$$

and with this:

$$\kappa(p) = - \left( \frac{-1}{p} \right)^{\frac{l-1}{2}} p^{-r-i}.$$

This gives us:

$$\varphi(s) = \prod_{p \neq 2} (1 - \left( \frac{-1}{p} \right)^{\frac{l-1}{2}} p^{-i-2s})^{-1} = L(2s - r + \frac{1}{2}, \kappa), \quad (5.16)$$

with $\kappa(p) = \left( \frac{-1}{p} \right)^{\frac{l-1}{2}}$.

From this we find the following functional equation for $\varphi(s)$:

$$\left( \frac{2\pi}{8} \right)^{-s} \Gamma(s - \frac{r}{2} + \frac{1}{4}) \varphi(s) =$$

$$= \left( \frac{2\pi}{8} \right)^{s-r} \Gamma(-s + \frac{r}{2} + \frac{1}{4}) \varphi(r-s) \text{ for } l \equiv 3 \pmod{4}, \quad (5.18)$$

$$\pi^{s-r} \Gamma(s - \frac{r}{2} + \frac{1}{4})(1 - 2^{-2s+r-i})^{-1} \varphi(s) =$$

$$= \pi^{s-r} \Gamma(-s + \frac{r}{2} + \frac{1}{4})(1 - 2^{2s-r-i})^{-1} \varphi(r-s) \text{ for } l \equiv 1 \pmod{4}. \quad (5.19)$$

Now $f(\tau)$ is an automorphic form with signature $\{8, r, \gamma \}$. By theorem 17 this gives us:

$$\left( \frac{2\pi}{8} \right)^{-s} \Gamma(s) \varphi(s) = \gamma \left( \frac{2\pi}{8} \right)^{s-r} \Gamma(r-s) \varphi(r-s). \quad (5.20)$$

When we combine (5.18) and (5.20) we find:

$$\frac{\Gamma(s - \frac{r}{2} + \frac{1}{4})}{\Gamma(s)} = \gamma \frac{\Gamma(-s + \frac{r}{2} + \frac{1}{4})}{\Gamma(r-s)}. \quad (5.21)$$

The equation (5.21) has as only solution $r = \frac{3}{8}, \gamma = 1$, which gives us $f(\tau) = \varphi^3(\tau)$. The equation we get by combining (5.19) and (5.20) has no solutions.

We now consider integral $r$. In this case, $N$ can be any
divisor of 12. Using theorem 16 we find \( f(\tau) | T(p) = 0 \), and therefore:

\[
\alpha(p) = (-1)^{r(p-1)} p^{-r} \text{ if } (p, N) = 1. \tag{5.22}
\]

Hence we find:

\[
\varphi(s) = (1 + 2^{r-1-2s})^{s_2} (1 + (-1)^r 3^{r-1-2s})^{s_3} \prod_{p > 3} \left(1 + \left(-\frac{1}{p}\right)^r p^{r-1-2s}\right)^{-1}. \tag{5.23}
\]

Here \( s_2 = \begin{cases} 0 & \text{if } 2/N \\ -1 & \text{if } 2 \not\mid N \end{cases} \) and \( s_3 = \begin{cases} 0 & \text{if } 3/N \\ -1 & \text{if } 3 \not\mid N \end{cases} \).

We now use that \( f(\tau) \) is an automorphic form with signature \( \{N, r, \gamma, \nu\} \). Theorem 17 implies that \((s - r) \varphi(s)\) is an integral function of \( s \). From (5.23) we see that this cannot be true in this case as we have:

\[
\varphi(s) = \psi(s) \frac{\zeta(4s-2r+2)}{\zeta(2s-r+1)} \text{ or } \psi(s) \frac{\zeta(4s-2r+2)}{L(2s-r+1, \left(-\frac{1}{n}\right))},
\]

where \( \psi(s) \) is an integral function whose only zeros are on the imaginary axis.

As we have considered all cases, we have proved:

**Theorem 18.** \( \eta(\tau) \) and \( \eta^3(\tau) \) are the only integral modular forms \( \in \{ \Gamma(1), -r, \nu \} \) for which linear relations (5.04) hold for all primes \( p \) with \( (p, N) = 1 \).

When studying linear relations Hecke [2] found that \( \vartheta(\tau) \) and \( \eta^4(\tau) \vartheta^{-1}(\tau) \) satisfy relations (5.04). These functions are modular forms for the group \( \Gamma_0 \). Now we ask if these two are the only integral modular forms for this group, for which relations (5.04) hold. This is not so as it is easily seen that \( \eta(\tau) \), \( \eta^3(\tau) \), \( \eta\left(\frac{\tau + 1}{2}\right) \) and \( \eta^3\left(\frac{\tau + 1}{2}\right) \) also satisfy these conditions.

Consider integral modular forms \( \in \{ \Gamma_0, -r, \nu \} \), where \( 2r = l \) is an odd integer, and suppose a linear relation (5.02) holds for all \( p > 3 \). In the same way as for \( \Gamma(1) \) we see that this is only
possible if \( \nu(U^2) = \lambda^* \) is a 24-th root of unity. Therefore we can write the modular form as
\[
f(\tau) = \sum_{n=0}^{\infty} a(n) e^{\frac{2\pi i n \tau}{48}}.
\]

By (3.21) we then have:
\[
f(\tau)|T(p^2) = \sum_{n=0}^{\infty} \left( p^\alpha a\left( \frac{n}{p^2} \right) + \frac{(-1)^{l-1}}{p} \left( \frac{6n}{p} \right) p^{r+1} a(n) + p^2 a(np^2) \right) e^{\frac{2\pi i n \tau}{48}}.
\]

From now on we shall write
\[
f(\tau) = \sum_{n=0}^{\infty} a(n) e^{\frac{2\pi i n \tau}{N}},
\]
where \( N/48 \), \( a(1) = 1 \), and \( a(n) = 0 \) if \( \left( n, \frac{N}{2} \right) > 1 \). We shall demand that a relation (5.04) holds for \( f(\tau) \) and all \( p \) with \( p \not\equiv \frac{N}{2} \). We wish to find all integral modular forms satisfying these conditions. The calculations will run along the same lines as those for \( \Gamma(1) \). We have to consider the different possible values of \( N \) separately. In each case we determine \( \alpha(p) \) in the usual way and so find the function \( \varphi(s) \), for which a functional equation is known. This equation is then compared with the functional equation we find by using theorem 17. We then find equations of the type (5.13) or (5.14) which have one or no solutions.

For \( N = 48 \) and \( N = 12 \) we find:
\[
\alpha(p) = \left( \frac{-1}{p} \right)^{l-1} \frac{6}{p} p^{r-1}.
\]
This gives us \( \varphi(s) = L \left( 2s - r + \frac{1}{2}, \chi \right) \), where \( \chi(p) = \left( \frac{6}{p} \right) \) if \( l \equiv 1 \) (mod 4), and \( \chi(p) = \left( \frac{-6}{p} \right) \) if \( l \equiv 3 \) (mod 4).

For \( N = 48 \) the form \( f(\tau) \) has signature \( \{48, r, \gamma \} \). The functional equation this gives us for \( \varphi(s) \), combined with the one we have, gives us equations (5.13) and (5.21), each having one solution. We find:
\[
\varphi(s) = L \left( 2s, \left( \frac{6}{n} \right) \right) \text{ associated with } e^{\frac{-\pi i}{24}} \gamma \left( \frac{\tau + 1}{2} \right) = \gamma^4(\tau) \gamma^4(\tau) = \sum_{n=1}^{\infty} \left( \frac{6}{n} \right) e^{\frac{2\pi i n^2 \tau}{48}}.
\]

\[
(5.25)
\]
\( \varphi(s) = L(2s - 1, \left( \frac{6}{n} \right) ) \) associated with

\[
e^{-\frac{6}{24} \psi^5 \left( \frac{\tau + 1}{2} \right)} n^{-2}(\tau + 1) = \psi^4(\tau) S_{3/2}(\tau) = \sum_{n=1}^{\infty} \left( \frac{-6}{n} \right) n^{2\min\{2s-\tau\}}.
\]

(5.26)

For \( N = 12 \) the form \( f(\tau) \) has signature \( \{ 12, r, \gamma' \} \), which leads to the equation

\[
\Gamma(s - \frac{r}{2} + \frac{1}{4}) \frac{\Gamma(s)}{\Gamma(\tau)} = \gamma 4^{s-r} \frac{\Gamma(-s + \frac{r}{2} + \frac{1}{4})}{\Gamma(r - s)}
\]

for \( l \equiv 1 \text{ (mod 4)} \), and an analogous equation for \( l \equiv 3 \text{ (mod 4)} \). As we have seen, these have no solutions.

For \( N = 24 \) and \( N = 6 \) we find \( \varphi(p) = -\left( \frac{-1}{p} \right)^{l-1} \left( \frac{3}{p} \right) p^{-\frac{r}{2}} \).

This leads to the same problem we solved for \( \Gamma(1) \). We see that for \( N = 24, l \equiv 1 \text{ (mod 4)} \) we have the solution \( f(\tau) = \psi(\tau) \), and therefore for \( N = 6, l \equiv 1 \text{ (mod 4)} \) no solution. For \( N = 24, l \equiv 3 \text{ (mod 4)} \) there was no solution. This leaves us \( N = 6, l \equiv 3 \text{ (mod 4)} \). We then have a difficulty we have not encountered yet. We have to consider \( p = 2 \) for which \( \varphi(p) \) is not known. We could have defined an operator \( T(2^r) \) but, as we shall see, this is not necessary. We have:

\[
\varphi(s) = \left( \frac{1 + \varphi(2)^{-1} 2^{-1} 2^{-2s}}{1 + 2r - \frac{1}{2} - 2s} \right)^{-1} L \left( 2s - r + \frac{1}{3}, \left( \frac{n}{3} \right) \right).
\]

For \( \psi(s) = L \left( 2s - r + \frac{1}{3}, \left( \frac{n}{3} \right) \right) \), the functional equation is:

\[
\left( \frac{2\pi}{6} \right)^{-s} \Gamma(s - \frac{r}{2} + \frac{3}{4}) \psi(s) = \left( \frac{2\pi}{6} \right)^{s-r} \Gamma(-s + \frac{r}{2} + \frac{3}{4}) \psi(r - s).
\]

The form \( f(\tau) \) has signature \( \{ 6, r, \gamma' \} \). This leads to the equation:

\[
\left( \frac{1 + \varphi(2)^{-1} 2^{-1} 2^{-2s}}{1 + 2r - \frac{1}{3} - 2s} \right) \frac{\Gamma(s - \frac{r}{2} + \frac{3}{4})}{\Gamma(\tau)} = \gamma \left( \frac{1 + \varphi(2)^{-1} 2^{-2s - r - 1}}{1 + 2^{2s - r - \frac{1}{3}}} \right) \frac{\Gamma(-s + \frac{r}{2} + \frac{3}{4})}{\Gamma(r - s)}.
\]
It is easily seen that this has only the solution \( r = \frac{3}{2}, \gamma = 1, \alpha = 2 \). This gives us \( \varphi(s) = L\left(2s - 1, \left(\frac{n}{3}\right)\right) \) associated with \( \eta^4(\tau) \theta^{-1}(\tau) \).

For \( N = 16 \) and \( N = 4 \) we find:
\[
\alpha(p) = -\left(\frac{-1}{p}\right)^{\frac{t-1}{2}} \left(\frac{2}{p}\right) p^{-r-\frac{1}{2}}.
\]
This gives us \( \varphi(s) = L(2s - r + \frac{1}{2}, \chi) \) with \( \chi(p) = \left(\frac{2}{p}\right) \) for \( l \equiv 1 \) (mod 4), and \( \chi(p) = \left(\frac{-2}{p}\right) \) for \( l \equiv 3 \) (mod 4). The functional equations are:
\[
\begin{align*}
(2\pi)^{-\frac{1}{16}} \Gamma\left(s - \frac{r}{2} + \frac{1}{4}\right) \varphi(s) &= \left(\frac{2\pi}{16}\right)^{s-r} \Gamma\left(-s + \frac{r}{2} + \frac{1}{4}\right) \varphi(r - s) \\
& \text{for } l \equiv 1 \text{ (mod 4)},
\end{align*}
\]
\[
\begin{align*}
(2\pi)^{-\frac{1}{16}} \Gamma\left(s - \frac{r}{2} + \frac{3}{4}\right) \varphi(s) &= \left(\frac{2\pi}{16}\right)^{s-r} \Gamma\left(-s + \frac{r}{2} + \frac{3}{4}\right) \varphi(r - s) \\
& \text{for } l \equiv 3 \text{ (mod 4)}.
\end{align*}
\]

For \( N = 16 \) the form \( f(\tau) \) has signature \( \{16, r, \gamma\} \), which again leads us to the equations (5.13) and (5.21). So we find:
\[
\varphi(s) = L\left(2s, \left(\frac{2}{n}\right)\right) \text{ associated with}
\]
\[
\frac{1}{4} e^{-\frac{\pi i}{3}} \left\{ \mathcal{S}\left(\frac{r+1}{8}\right) - \mathcal{S}\left(\frac{r+9}{8}\right) \right\}, 
\]
\[
\varphi(s) = L\left(2s - 1, \left(\frac{-2}{n}\right)\right) \text{ associated with}
\]
\[
e^{-\frac{\pi i}{3}} \eta^3\left(\frac{r+1}{2}\right) = \eta^3(\tau) \mathcal{S}^{3/2}(\tau).
\]
We see that there will be no solutions for \( N = 4 \).

For \( N = 8 \) and \( N = 2 \) we find \( \alpha(p) = -\left(\frac{-1}{p}\right)^{\frac{t-1}{2}} p^{-r-\frac{1}{2}} \).

For \( N = 8 \) this is the same problem as for \( \Gamma(1) \). We have found the solution \( \eta^3(\tau) \). The only case in which another solution can be found is \( N = 2, l \equiv 1 \) (mod 4). Again we have to consider \( p = 2 \) separately.
We have $\varphi(s) = \left(1 + \alpha(2)^{-2s-1} - \frac{1}{2} - \frac{s}{2s} \right) \zeta(2s - r + \frac{1}{2})$.

We find one solution: $r = \frac{1}{2}$, $\alpha(2) = -\frac{1}{2}$, $\varphi(s) = \zeta(2s)$ associated with $\frac{1}{2} S(\tau) = \frac{1}{2} + \sum_{n=1}^{\infty} e^{\alpha n^2}$.

Remark: We do not have to consider $N = 1$ or $N = 3$. If $N$ could have one of these values, $f(\tau)$ would be a modular form for $\Gamma(1)$, and we have already discussed this case. The solutions found there were also found above.

Summarizing, we can give the following table of integral modular forms $e \mid \Gamma_0, -r, \nu \}$ (where $2r = l$ is an odd integer) for which the associated Dirichlet series has an Euler product of the special type (5.05).

<table>
<thead>
<tr>
<th>Modular form</th>
<th>Dirichlet series</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta^{1/2}(\tau) \mathcal{S}^{1/2}(\tau)$</td>
<td>$\sum_{n=1}^{\infty} \left(\frac{6}{n}\right) e^{\frac{2\pi i n^2}{48}} L\left(2s, \left(\frac{6}{n}\right)\right)$</td>
</tr>
<tr>
<td>$\eta^{1/2}(\tau) \mathcal{S}^{5/2}(\tau)$</td>
<td>$\sum_{n=1}^{\infty} \left(-\frac{6}{n}\right) e^{\frac{2\pi i n^2}{48}} L\left(2s-1, \left(-\frac{6}{n}\right)\right)$</td>
</tr>
<tr>
<td>$\eta(\tau)$</td>
<td>$\sum_{n=1}^{\infty} \left(\frac{12}{n}\right) e^{\frac{2\pi i n^2}{24}} L\left(2s, \left(\frac{12}{n}\right)\right)$</td>
</tr>
<tr>
<td>$\frac{1}{4} e^{-\pi i} \left(\mathcal{S}\left(\frac{\tau + 1}{8}\right) - \mathcal{S}\left(\frac{\tau + 9}{8}\right)\right)$</td>
<td>$\sum_{n=1}^{\infty} \left(\frac{2}{n}\right) e^{\frac{2\pi i n^2}{16}} L\left(2s, \left(\frac{2}{n}\right)\right)$</td>
</tr>
<tr>
<td>$\eta^{3/2}(\tau) \mathcal{S}^{3/2}(\tau)$</td>
<td>$\sum_{n=1}^{\infty} \left(-\frac{2}{n}\right) e^{\frac{2\pi i n^2}{16}} L\left(2s-1, \left(-\frac{2}{n}\right)\right)$</td>
</tr>
<tr>
<td>$\eta^3(\tau)$</td>
<td>$\sum_{n=1}^{\infty} \left(-\frac{4}{n}\right) e^{\frac{2\pi i n^2}{8}} L\left(2s-1, \left(-\frac{4}{n}\right)\right)$</td>
</tr>
<tr>
<td>$\eta^4(\tau) \mathcal{S}^{-1}(\tau)$</td>
<td>$\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) e^{\frac{2\pi i n^2}{6}} L\left(2s-1, \left(\frac{n}{3}\right)\right)$</td>
</tr>
<tr>
<td>$\frac{1}{2} S(\tau)$</td>
<td>$\frac{1}{2} + \sum_{n=1}^{\infty} e^{\frac{2\pi i n^2}{2}} \zeta(2s)$</td>
</tr>
</tbody>
</table>
If we replace \( \tau \) by \( 2\tau - 1 \) we find the integral modular forms for \( \Gamma_0(2) \) for which the associated Dirichlet series have a special Euler product.
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STELLINGEN

I.
Uit formule 4.30) van dit proefschrift volgt, dat, voor oneven positieve \( l < 24 \), de reeksen \( \sum_{n=1}^{\infty} a(ln^2) n^{-\frac{1}{2}} \) een Euler product hebben.

II.
Met behulp van stelling 4 van dit proefschrift kan men de kwadratische reciprociteitswet bewijzen.

III.
Euler product formules, zoals Sandham die geeft voor \( n \leq 7 \), gelden voor alle oneven positieve \( n \), indien men als coefficienten in de reeks van Dirichlet niet het aantal voorstellingen door de vorm \( \sum_{i=1}^{n} x_i^2 \) neemt, maar het gemiddelde aantal voorstellingen over het geslacht van deze vorm.


IV.
Als de zwak multiplicatieve symbolen \( a_i(n) \), \( (i = 1, 2, \ldots, k) \), lineair afhankelijk zijn, dan is er een getal \( Q \) zodat voor \( (n, Q) = 1 \) geldt: \( a_1(n) = a_2(n) = \ldots = a_k(n) \).

V.
Het bewijs, dat Titchmarsh geeft voor de continuïteit van een convexe functie, is niet juist. Een aanschouwelijk duidelijk bewijs kan zeer eenvoudig gegeven worden.

E. C. Titchmarsh: The Theory of Functions. 5. 31.
VI.
Als in een associatieve algebra $A$ over een lichaam $F$ ieder element van $A$ wortel is van een vergelijking $x^2 + ax + b = 0$ met $a \in F$, $b \in F$, dan is de algebra symmetrisch.


VII.
Bij zijn behandeling van invariante lineaire operatoren voor functies gedefinieerd op een zwaksymmetrische Riemannse ruimte, toont Selberg de verwisselbaarheid dezelfde operatoren aan. Beperkt men zich tot invariante integraaloperatoren, dan kan een eenvoudig direct bewijs gegeven worden.


VIII.
Bij het construeren van een ballistocardiograaf dient men er rekening mee te houden, dat de geestelijke gestelheid van de proefpersoon invloed kan uitoefenen op de hartslag.

IX.
Het verdient geen aanbeveling om 2e-jaars studenten in de wiskunde reeds colloquia te laten houden, zoals dat aan vele Duitse universiteiten geschiedt.

X.
Het is in het algemeen alleen voor gevorderde studenten zinvol om enige tijd aan een buitenlandse universiteit te studeren.

XI.
Dat bij de subsidiëring van de universitaire sportbeoefening de wedstrijdsporten in principe worden uitgesloten, is betreurenswaardig.