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On the Complexity of the Robust Stability Problem for Linear Parameter Varying Systems

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Abstract—In this paper, it is shown that the problem of checking robust stability of linear parameter varying (LPV) systems is $\mathcal{NP}$-hard, and therefore, it is rather unlikely to find polynomial-time solution procedures for this problem. In the frequency-domain structured uncertainty case, it is known that the robust stability problem is $\mathcal{NP}$-hard (Toker and Özay, 1995; Toker, 1995; Poljak and Rohn, 1993; Nemirovski, 1993; Braatz et al., 1994), but allowing the uncertain blocks to be time varying gives a computationally tractable problem (Shamma, 1994; Poolla and Tikku, 1995), which can be solved by convex optimization techniques. In the parameteric uncertainty case, $\mathcal{NP}$-hardness of the robust stability problem has been shown in (Poljak and Rohn, 1993; Nemirovski, 1993). The results of this paper show that, allowing the uncertain parameters to be time varying, does not give a computationally simpler problem, i.e., it remains $\mathcal{NP}$-hard, and hence it is rather unlikely to find computationally tractable solution procedures for this problem. On the other hand, as far as the existence of an algorithm is concerned, there is still no known (non-polynomial time) algorithm for the robust stability problem of linear parameter varying systems (Lagarias and Wang, 1995), and the well-known Tarski’s theorem (Tarski, 1951) does not provide a solution procedure (Kozyakin, 1990). Recently, there has been some developments in the direction of constructing non-polynomial-time algorithms for a related problem, called the joint spectral radius (JSR) computation problem (Lagarias and Wang, 1995). We also comment on the use of these results for developing a non-polynomial time algorithm for testing robust stability of linear parameter varying systems. © 1997 Elsevier Science Ltd.

Notation

- $\mathbb{Z}$: The set of integers
- $\mathbb{Q}$: The set of rational numbers
- $\mathbb{R}$: The set of real numbers
- $\mathbb{C}$: The set of complex numbers
- $\rho(A)$: Spectral radius of $A$
- $\|v\|_\infty$: Infinity norm of $v$
- $\mathcal{B}(\mathcal{F}^\omega)$: The set of all bounded sequences over $\mathcal{F}$
- Mat$(n,\mathbb{R})$: The set of all $n \times n$ matrices over $\mathbb{R}$

1. Introduction

In this paper, linear parameter varying systems are considered, and it is shown that the problem of checking robust stability is an $\mathcal{NP}$-hard problem. Based on this result, it is rather unlikely to find polynomial-time solution procedures for this problem. For the frequency-domain structured uncertainty case, it is well known that the problem of checking robust stability is $\mathcal{NP}$-hard. However, recently it has been shown that allowing the uncertain blocks to be time varying gives a computationally simpler problem which can be solved by convex optimization techniques (Shamma, 1994; Poolla and Tikku, 1995). Furthermore, the $\mathcal{NP}$-hardness of the robust stability problems for the parameteric uncertainty case is also known (Poljak and Rohn, 1993; Nemirovski, 1993). Therefore, it is natural to consider the computational complexity of this problem when parameters are allowed to be time varying. The results of this paper show that, this does not give a computationally simpler problem, i.e., the problem remains $\mathcal{NP}$-hard. Based on this result, it is rather unlikely to find polynomial-time solution procedures for the robust stability problem of linear parameter varying systems, however, in the rest of this paper, we also comment on the recent results towards developing non-polynomial-time algorithms.

In this paper, we consider linear parameter varying systems of the form,

$$x(k + 1) = (A_0 + \sum_{i=1}^{n} r(k)A_i)x(k),$$

where $r_1, ..., r_\omega \in \mathcal{B}(\mathcal{F}^\omega)$. By Theorem 1a of (Berger and Wang, 1992), the following conditions are equivalent:

(S1) \[ \forall r_1, ..., r_\omega \in \mathcal{B}(\mathcal{F}^\omega), \sup \left\{ \left| \prod_{j=1}^{n} (A_0 + \sum_{i=1}^{n} r(j)A_i) \right| \right\} < \infty. \]

(S2) \[ \sup_{\omega \in \mathcal{B}(\mathcal{F}^\omega)} \left\| \prod_{j=1}^{n} (A_0 + \sum_{i=1}^{n} r(j)A_i) \right\| < \infty. \]

The linear parameter varying system of equation (1) is said to be stable, i.e., one of the above equivalent conditions hold. Note that, this stability definition corresponds to the boundedness of the state for all possible initial conditions. Similarly, by Theorem 1b of (Berger and Wang, 1992) the following conditions are equivalent:

(AS1) \[ \forall r_1, ..., r_\omega \in \mathcal{B}(\mathcal{F}^\omega), \lim_{n \to \infty} \left\| \prod_{j=1}^{n} (A_0 + \sum_{i=1}^{n} r(j)A_i) \right\| = 0, \]

(AS2) \[ \lim_{n \to \infty} \sup_{\omega \in \mathcal{B}(\mathcal{F}^\omega)} \left\| \prod_{j=1}^{n} (A_0 + \sum_{i=1}^{n} r(j)A_i) \right\| = 0, \]

(AS3) \[ \exists M > 0, and \rho < 1, \text{ such that} \]

$$\forall r_1, ..., r_\omega \in \mathcal{B}(\mathcal{F}^\omega), \left\| \prod_{j=1}^{n} (A_0 + \sum_{i=1}^{n} r(j)A_i) \right\| < M\rho^n.$$
polynomial-time solution procedures for these problems. The well known Tarski’s theorem (Tarski, 1951) does not provide a (non-polynomial time) solution procedure for these problems. Recently, Lagarias and Wang proposed an algorithm based on the so-called Finiteness Conjecture (Lagarias and Wang, 1995; Gurvits, 1995), and computational experiments support their conjecture (Dogruel, 1995). If the Finiteness Conjecture is proved, this will provide a non-polynomial-time solution procedure for the joint spectral radius computation problem, which also can be used to test robust asymptotic stability of linear parameter varying systems. Furthermore, the proposed algorithm will be a non-algebraic decision test, and this is consistent with the recent results of Kozyakin (Kozyakin, 1990) which basically says that, if there is an algorithm, it should be non-algebraic. But to the best of the author’s knowledge, the Finiteness Conjecture is still open, and there is still no known non-polynomial time solution procedure for the above stability problems.  

2. $\mathcal{NP}$-hardness of robust stability and robust asymptotic stability problems for LPV systems

In this section, $\mathcal{NP}$-hardness of stability and asymptotic stability problems for linear parameter varying systems, are proved. Our results are based on some observations from Nemirovski (1993) and Heil and Colella (1993).

Lemma 1 (Karp, 1972). For a given $a \in \mathbb{Z}^n$, the problem of checking the existence of $1, \ldots, t, \in [-1, 1]$ such that

$$\sum_{j=1}^{t} a_j = 0,$$

is $\mathcal{NP}$-hard.

Lemma 2 (Heil and Colella, 1993). For a Hermitian matrix $A \in \mathbb{C}^{n \times n}$, $\rho(A) = \lambda_{\max}(A)$.

Proof. Consider the problem of Lemma 1 with the same notation, and define $B = (I_\mathbb{C} - e^{-\alpha t}A)^{-1}$, $z = e^{(1+|\alpha|)t}$, $\mu = n - \frac{1}{1+|\alpha|}$, and $M_z = \begin{bmatrix} B & z \\ z^T & \mu \end{bmatrix}$, $z \in \mathbb{R}^n$, $\|z\|_\infty \leq 1$.

similar to the definitions given in Nemirovski (1993). Then $M_z$ is positive definite if $z B z^T < \mu$. Furthermore, $z B z^T = \|z\|_2^2 (I_\mathbb{C} - e^{-\alpha t}A)^{-1} z < n - e^{-\alpha t} \|z\|^2_2$, and if the problem of Lemma 1 has no solution, we have $z B z^T \leq n - e^{-\alpha t} \|z\|^2_2 \leq n - \frac{1}{|\alpha|}$. On the other hand, if the problem of Lemma 1 has a solution, then $z B z^T \leq n - \frac{1}{|\alpha|}$. Therefore, if the problem of Lemma 1 has no solution, then all of the matrices

$$M_z = \begin{bmatrix} B & z \\ z^T & \mu \end{bmatrix},$$

are positive definite. Otherwise, $z \in \mathbb{R}^n$, $\|z\|_\infty \leq 1$ such that $M_z$ has a negative eigenvalue.

Note that, $M_z$ is Hermitian, and for $x \in \mathbb{R}^n$, $x_\mathbb{C} \in \mathbb{C}^n$ we have

$$\begin{bmatrix} x \mathbb{R} \\ x_\mathbb{C} \end{bmatrix} \begin{bmatrix} B & z \\ z^T & \mu \end{bmatrix} \begin{bmatrix} x \mathbb{R} \\ x_\mathbb{C} \end{bmatrix} = x^T B x + 2x^T (z x_\mathbb{C}) + \mu z^T z \leq (\|B\| + \mu + n) \begin{bmatrix} x \mathbb{R} \\ x_\mathbb{C} \end{bmatrix}^2.$$

Therefore, $\rho(M_z) = |M_z| < \|B\| + \mu + n$. Since $\|B\| < 1/(1 - e^{-\alpha t} \|A\|^T_2)$, $1/(1 - 0.01) < 2$, we obtain $\rho(M_z) = |M_z| < \mu + n + 2$. Now, define

$$N_z = -I_n + \frac{M_z}{\mu + n + 2},$$

Then, if the problem of Lemma 1 has no solution, $M_z$ is positive definite for all $z \in \mathbb{R}^n$ with $\|z\|_\infty \leq 1$, and $\rho(N_z) = |N_z| < 1$. Otherwise, $z \in \mathbb{R}^n$, $\|z\|_\infty \leq 1$ such that $\rho(N_z) > 1$, and the above LPV system is not robustly stable. The theorem follows by the $\mathcal{NP}$-hardness result of Lemma 1.

Theorem 1 implies that, both robust stability, and robust asymptotic stability problems, are $\mathcal{NP}$-hard for LPV systems. In fact, it shows that, even if we restrict our attention to only Hermitian matrices, these robust stability problems still remain $\mathcal{NP}$-hard. Hence, it is rather unlikely to find polynomial-time solution procedures for these problems. In the case of structured frequency-domain uncertainty, it is known that the robust stability problem (analytical is $\mathcal{NP}$-hard (Toker and Özbay, 1995; Toker, 1995). (See also Braatz et al., 1994; Poljak and Rohn, 1993, Nemirovski, 1993.) However, if one allows the uncertainty to be time varying, the problem becomes significantly easier to solve (Shamma, 1994; Poolla and Tukku, 1995). The $\mathcal{NP}$-hardness of the robust stability problem for the structured parametric uncertainty case is also well known (Poljak and Rohn, 1993, Nemirovski, 1993), but the results of this paper show that, allowing the uncertain parameters to be time varying does not give a computationally simpler problem (i.e. the problem remains $\mathcal{NP}$-hard).

In the next section, some recent results in the direction of developing non-polynomial-time algorithms are discussed. But to the best of the author’s knowledge, there is no known non-polynomial-time algorithm for these problems.

3. The finiteness conjecture

In this section, we summarize some recent results in the direction of developing non-polynomial-time algorithms for checking robust stability and robust asymptotic stability of LPV systems.

First of all, for a given set of matrices $\Sigma = \{A_1, \ldots, A_n\}$, define

$$\rho_{\Sigma}(k) = \sup\{\rho(A_1, \ldots, A_n) : A_j \in \Sigma, j = 1, \ldots, k\},$$

and

$$\rho_{\Sigma}(k) = \limsup_{k \to \infty} \rho_{\Sigma}(k)^{1/k},$$

as in Lagarias and Wang (1995). Then, $\rho_{\Sigma}(k) \leq \rho_{\Sigma}(k)^{1/k} \leq \rho_{\Sigma}(k)$,

and

$$\rho_{\Sigma}(k) = \liminf_{k \to \infty} \rho_{\Sigma}(k)^{1/k} = \liminf_{k \to \infty} \rho_{\Sigma}(k)^{1/k},$$

see Lagarias and Wang (1995). In Berger and Wang (1992), it is shown that $\rho_{\Sigma}(k) = \rho_{\Sigma}(k)^{1/k}$.

Recently, Lagarias and Wang conjectured the following:

Finiteness Conjecture (Lagarias and Wang, 1995). For each finite set $\Sigma$ of $n \times n$ matrices, there exists some finite $k_\Sigma$ such that

$$\rho_{\Sigma}(k) = \rho_{\Sigma}(k)^{1/k_{\Sigma}}.$$
Remark. Kozyakin's results (Kozyakin, 1990) show that, the above Finiteness Conjecture is false if $k_2$ is not allowed to be dependent on $\Sigma$. Similarly, in Lagarias and Wang (1995), an example is provided to show that $k_2$ can be arbitrarily large, and hence for the Finiteness Conjecture to be true, $k_2$ must depend on $\Sigma$.

In Lagarias and Wang (1995), it is shown that the Finiteness Conjecture is equivalent to the so-called normed Finiteness Conjecture, and the normed Finiteness Conjecture is proved for piecewise analytic norms. Furthermore, computational experiments support this conjecture (Dogruel, 1995). But to the best of the author's knowledge, the Finiteness Conjecture is still an open problem. If the Finiteness Conjecture is proved, then this will also prove that the following "program" is an algorithm for checking whether $p(\Sigma) < 1$ or not (Lagarias and Wang, 1995).

Step 1. Let $k = 1$.
Step 2. Compute $p_0(\Sigma)$ and $p_1(\Sigma)$.
Step 3. If $p_0(\Sigma) \geq 1$, then the condition $"p(\Sigma) < 1"$ does not hold. STOP.
Step 4. If $p_0(\Sigma) < 1$, then the condition $"p(\Sigma) < 1"$ does hold. STOP.
Step 5. Increase $k$ by 1. Go to Step 1.

It is not known whether the following "program" stops for all possible inputs $\Sigma$, or there exists an input $\Sigma$ for which the "program" runs forever. But, if the Finiteness Conjecture is proved, this will also imply that the above "program" stops for all possible inputs $\Sigma$, and is a polynomial-time algorithm for checking whether $p(\Sigma) < 1$ or not. At this point, we would like to mention that the well-known results of Brayton and Tong (1979, 1980) does not provide an algorithm for this problem. They suggest an iterative approach together with some conservative decision tests, but do not provide any non-conservative decision test that can be used as a stopping criterion for their iterative approach.

The Finiteness Conjecture based algorithm proposed by Lagarias and Wang, can be used to check the robust asymptotic stability of LPV systems of the form

$$x(k + 1) = \left(A_0 + \sum_{i=1}^n r_i(k)A_i\right)x(k), \quad r_1, \ldots, r_n \in \mathbb{R}^n.$$  

Because, robust asymptotic stability is equivalent to

$$p(\Sigma) < 1$$

for

$$\Sigma = \left\{A_0 + \sum_{i=1}^n r_iA_i : r_i \in \mathbb{R}^n, i = 1, \ldots, n \right\},$$

(Dogruel, 1995). Hence, if the Finiteness Conjecture is proved, then this will provide a polynomial-time algorithm for checking robust asymptotic stability of LPV systems. But to the best of the author's knowledge the Finiteness Conjecture is still open, and there is no known non-polynomial-time algorithm for checking robust stability and robust asymptotic stability of LPV systems.

4. Concluding remarks

In this paper, it has been shown that, stability and asymptotic stability problems for linear parameter varying systems, are $\mathcal{P}^\mathcal{P}$-hard. Therefore, it is rather unlikely to find polynomial-time solution procedures for these problems. Some recent results in the direction of constructing non-polynomial-time algo-