Estimation of the Basin of Attractions in CNN's

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Abstract—A technique will be discussed in order to find an estimation of the basin of attraction for each stable equilibrium point in cellular neural networks. The technique is based on the determination of the direction of the trajectories for each of the regions in which the state space is divided via the piecewise linear output function of each cell.

Index Terms—Basin of attraction, cellular neural networks.

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I. INTRODUCTION

In 1988, Chua and Yang introduced the cellular neural network (CNN) as a local connected network of nonlinear dynamic cells [1], [2]. CNN’s are nonlinear dynamic systems which give rise to complex dynamics. Since the invention of the CNN system, many properties of the CNN have been derived. Due to the complexity of the problems, they are mainly valid for special cases. Generally, the problems related to the dynamics and the basin of attraction of CNN’s are not solved thoroughly. The dynamic properties of dynamic systems with saturated nonlinearity’s (e.g., neural networks) are considered in the book written by Liu and Michel [3] and in the references given there. This work is primarily concerned with the existence and the location of equilibrium points, with the qualitative properties of these points, and with the extension of the basin of attraction of asymptotically stable equilibrium.

The subject of this brief is to consider the case when the initial patterns are given at instance zero to the state nodes of the CNN, and to find an estimation of the basin of attractions of the stable equilibrium points when no stable periodic orbits or chaotic attractors (considered in [5]) are available. This problem is of great importance for CNN implementation because it is connected with the robustness of the CNN. The problem is related to stability and convergence analysis of CNN’s. A complete description on the basin of attractions for a simple two-cell CNN is given in [4], and we try to extend the approach to higher order CNN’s. A method for estimation of the basin of attractions of Hopfield neural networks, based on Lyapunov functions, is given in [6]. The method holds especially when the output functions are sigmoidal, and hence there is no extension for the piecewise linear output functions.

The brief is outlined as follows. Section II outlines the proposed method, which in Section III will be discussed with two examples. In Section IV some more details will be treated, and we will end with some conclusions.

II. ESTIMATION OF THE BASIN OF ATTRACTIONS

The state space of the given CNN is divided by regions via the piecewise linear output function of each cell. These are linear region (where all variables are saturated), partial saturation regions (where some of the variables are saturated), and saturation regions (where all variables are saturated) [5].

The basic idea of the suggested approach is to determine the so-called trees of regions connected to each saturation region with a stable equilibrium point.

Definition: The tree of regions connected to a given saturation region with an equilibrium point is a tree of regions where:

1) the root region is the saturation region with a given stable equilibrium point, and
2) each branch of the tree consists of regions ensuring a path of the state trajectories leading to the root region.

The tree can be obtained by following all possible branches. For the suggested approach, it is important to consider the behavior of the CNN’s state trajectories on the boundaries between each of the regions. Without loss of generality we will consider the boundary $S P$ between the regions $A$ and $B$. We assume that $x_{1}$ is the separation variable (e.g., $S P$ is determined by $x_{1} = 1$ or $x_{1} = -1$) and $A$ is on the left side of the $B$ with respect to $x_{1}$. There are three possible cases for the direction of the state trajectories while passing this boundary:

1) all state trajectories are directed from region $A$ into region $B$; 2)
all state trajectories are directed from region $B$ into region $A$, and 3) there is an area on $S$ where the state trajectories are directed from $A$ to $B$, and another area where the state trajectories are directed from $B$ to $A$. In order to check which of the above cases takes place, we simply have to check the sign of $dx_1/dt$ in each point in $S$. The following can occur:

- If for all points on the boundary $S$ $\text{sign}(dx_1/dt) = 1$, then the state trajectories are directed from $A$ to $B$, i.e., we have case 1).
- If for all points on the boundary $S$ $\text{sign}(dx_1/dt) = -1$, then the state trajectories are directed from $B$ to $A$, i.e., we have case 2).
- If for the area $SP^1$ (a subset of $S$) $\text{sign}(dx_1/dt) = 1$ and for the other area $SP^2\setminus SP^1 = SP^2 = SP\setminus SP^1 = 0$ $\text{sign}(dx_1/dt) = -1$, we have the above case 3).

Note that if there is an area $S^0$, subset of $S$, where $\text{sign}(dx_1/dt) = 0$ and the measure of the set (area) $S^0$ is zero, then this case is not important. In a real implementation, the CNN will always avoid $S^0$ because of the thermal noise. If the measure of the set $S^0$ is not zero, two cases should be considered:

1) If the state trajectories remain in $S^0$, e.g., when the first derivatives of the other variables have appropriate signs and/or $S^0 = S$, then the analysis for the points near $S^0$ should be done and the state trajectories will follow the direction determined from this analysis.

2) If the state trajectories could leave $S^0$ due to the appropriate signs of the first derivatives of the other variables, then $S^0$ (respectively, $S$ if $S = S^0$) should not be considered.

The above considerations can be formalized. With each region, let us associate a so-called “state trajectory destination vector.” Each of the components of this vector corresponds to a separation hyperplane for the region and could have as value 1 if the state trajectories are directed out of the region, $-1$ if the state trajectories are directed into the region, or 0 if there are state trajectories directed out of the region and state trajectories directed into the region. Now the following two theorems can be applied.

**Theorem 1:** If there is an equilibrium point in a given saturation region, then all the components of the $n$-dimensional state trajectories destination vector are $-1$.

**Proof:** If there is an equilibrium point in a given saturation region, then the whole saturation region (the boundaries are included) belongs to the basin of attraction of the equilibrium point, and each trajectory started from the boundary will asymptotically reach it [5]. Therefore, each state trajectory started from the boundary hyperplane has to be directed into the region, e.g., the corresponding component of the state trajectories destination vector is $-1$.

**Theorem 2:** Consider the components of the state trajectories destination vector of each of the partial saturation regions, linear region, and saturation regions without equilibrium points. If the condition $a_{ii} > 1$ holds (where $a_{ii}$ are the diagonal elements of the matrix $A$ in a CNN state equation $dx/dt = -x + Ax + Bu + i$), then there is at least one component of the destination vector with value 0 or 1.

**Proof:** If the condition $a_{ii} > 1$ holds, then any equilibrium points in a partial saturation regions and in the linear region are unstable [5]. In this case, the state trajectories should leave this region and also the saturation regions without equilibrium points. Therefore, at least one component of the state trajectories destination vector from the above-mentioned regions has to be 0 or 1.

The procedure for obtaining the basin of attraction of stable equilibrium points consists of constructing all branches of a tree. The root of the tree (for a given equilibrium point) should be the saturation region containing this equilibrium point. On each level, an additional region is added on the corresponding branch if the following simple criterion holds: if the state trajectories on the boundary between a certain region and some of the regions from the branches of the tree can leave this particular region (i.e., there is at least one component value in the state trajectory destination vector 0 or 1), then this region should be added to tree. In this way, the path of the state trajectories directed into the root region (where a stable equilibrium point is located) is ensured. Therefore, in each of the regions belonging to the tree, there is a part of the basin of attraction of the equilibrium point defined in the root region.

After the trees corresponding to all stable equilibrium points are constructed, the following remarks for estimating the basin of attractions of the stable equilibrium points hold.

- **If** in a given tree there do not exist regions which also belong to other trees, then this tree represents the basin of attraction of the corresponding stable equilibrium point (in the root region).
- **If** several trees have a common region, then this region belongs to the basin of attractions of the corresponding equilibrium points (related to the root regions in each of the trees), e.g., the boundaries between the basin of attractions of these equilibrium points are in the region and in the regions from next levels in the mentioned trees.

### III. Examples

In this section we will discuss two examples to clarify the outlined method.

**Example 1:** We will consider the two-cell CNN

\[
\begin{align*}
\frac{dx_1}{dt} &= -x_1 + 1.5y_1 + y_2 + 1 \\
\frac{dx_2}{dt} &= -x_2 + y_1 + 1.5y_2 + 1
\end{align*}
\]

where $y_i = f_i(x_i) = 0.5|x_i + 1| - |x_i - 1|$, $i = 1, 2$ are well known piecewise linear functions.

The above CNN has two stable equilibrium points, namely, $E_1 = (x_1, x_2) = (3.5, 3.5)$ and $E_2 = (x_1, x_2) = (-1.5, -1.5)$. Due to the piecewise linear functions $y_i = f_i(x_i)$, the CNN can operate in nine linear regions, numerated in the way presented in Fig. 1. The regions are separated by the lines $x_1 = 1, x_1 = -1, x_2 = 1$, and $x_2 = -1$. The linear region is denoted by $D_0 = [5]$, the set of saturation regions is expressed as $DS = \{1, 3, 7, 9\}$, and the set of partial saturation regions is given by $DP = \{2, 4, 6, 8\}$ (Fig. 1). For each separation hyperplane (in our case, lines), the destinations of the state trajectories can be given, corresponding to the component determining the hyperplane. These destinations could be found following the simple analysis given below for the line $x_1 = -1$. ($y_1 = -1$):

- $x_2 \leq -1, \frac{dx_1}{dt} = -0.5 < 0$: direction toward region 7
- $-1 < x_2 < 1, \frac{dx_1}{dt} = x_2 + 0.5 > 0$ or $\frac{dx_1}{dt} / dt > 0$ when $x_2 > -0.5$: direction toward region 5 and $\frac{dx_1}{dt} / dt < 0$ when $x_2 < -0.5$: direction toward region 4
- $x_2 \geq 1, \frac{dx_1}{dt} = 1.5 > 0$: direction toward region 2

and a similar analysis for the other lines can be done. The trees for the stable equilibrium points $E_1$ (with root region 3) and $E_2$ (with root region 7) are given in Figs. 2 and 3, respectively, where dotted lines show that one region can appear on several branches. The conclusions from the above trees connected with the two equilibrium points $E_1$ and $E_2$ are as follows. The basin of attraction of the equilibrium point $E_1$ (in region 3) consists of regions 3, 2, 6, 1, 9, and parts of regions 4, 8, and 5. The basin of attraction of the equilibrium points $E_2$ (in region 7) consists of regions 7, 3, 1, 5, and parts of regions 6, 8, and 4.
However, from the example, it may become clear that the method generates only a first-order estimation of the basin of attraction. For initial conditions chosen in the regions 4, 8, and 5, it is not clear to which stable equilibrium the trajectories will tend. Those regions have to be examined in more detail.

Example 2: As a second example, we consider a three-cell CNN described by the templates $A = [1 2 1], B = 0$, $i = 1$, yielding the system of equations

$$\begin{align*}
    dx_1/dt &= -x_1 + 2y_1 + y_2 + 1 \\
    dx_2/dt &= -x_2 + y_1 + 2y_2 + y_3 + 1 \\
    dx_3/dt &= -x_3 + y_2 + 2y_3 + 1
\end{align*}$$

where $y_i = f_i(x_i) = 0.5(|x_i + 1| - |x_i - 1|), i = 1, 2, 3$ are piecewise linear functions.

The CNN has two stable equilibrium points $E1 = (4, 5, 4)$ and $E2 = (-2, -5, -2)$. For each of the cases $x_3 \geq 1$, $-1 < x_3 < 1$, and $x_3 \leq -1$, the numbering of the regions follows that used in Fig. 1. When $x_3 \geq 1$, the regions are numbered from 1 to 9; for $-1 < x_3 < 1$, the regions are numbered from 10 to 18; and for $x_3 \leq -1$, the regions are numbered from 19 to 27. The regions are separated by the planes $x_3 = 1$, $x_3 = -1$, $x_2 = 1$, $x_2 = -1$, $x_3 = 1$, and $x_3 = -1$.

The linear region is given by $D0 = \{14\}$, the sets of saturation regions is expressed as $DS = \{1, 3, 7, 9, 19, 21, 25, 27\}$, and the set of partial saturation regions is $DP = \{2, 4, 5, 6, 8, 10, 11, 12, 13, 15, 16, 17, 18, 20, 22, 23, 24, 26\}$.

The tree for the equilibrium point $E1$ (in the root region 3) contains all the regions without region 25 and the tree for the equilibrium point $E2$ (in the root region 25) contains the regions 25, 16, 22, 26, 13, 17, 23, and 14.

IV. MORE DETAILED ESTIMATION OF THE BASIN OF ATTRACTIONS

Generally, a more detailed description of the basin of attractions should be done for those regions which belong to several basin of attractions. In order to find an estimation for the basin of attraction of a given equilibrium point in a given region, we should find the separation manifold which satisfies the conditions concerning the state trajectory directions.

Let us consider region 5 of the above Example 1. First we will search separation lines for the part (sub region) that fully belongs to the basin of attraction of $E1$. Therefore, our separation manifold must satisfy the following condition: the manifold divides the region into two subregions; the state trajectories on all manifold boundaries of one subregion are directed into this subregion and therefore toward the regions 2 or 6; hence this subregion is a part of the basin of attraction of $E1$.

We can search for one line which fulfills this condition. It is easy to check that the line $x_1 + x_2 + 1 = 0$ will do this task. However, it is a coarse subdivision of region 5. We can also search for a system of two lines, and one can check that the part which is on the right
side of the lines \(0.5x_1 + x_2 + 1 = 0\) and \(x_1 + 0.5x_2 + 1 = 0\) fully belong to the basin of attraction of \(E1\). The above lines correspond to \(dx_1/dt = 0\) and \(dx_2/dt = 0\) (in region 5), and for the points above these lines \(dx_1/dt > 0\) and \(dx_2/dt > 0\), i.e., the state trajectories are directed toward the region 5. The separation lines in regions 8 and 4 are \(x_1 = 0\) and \(x_2 = 0\), respectively, because they correspond to \(dx_1/dt = 0\) and \(dx_2/dt = 0\), and the state trajectories are directed toward the regions 8 and 4. This set of lines is drawn in Fig. 1 in bold.

Now we will search separation lines for the parts that belong to the basin of attraction of \(E2\). Reasoning similar to that above leads to the conclusion that in region 5 the same two lines \((0.5x_1 + x_2 + 1 = 0\) and \(x_1 + 0.5x_2 + 1 = 0\)) can be used. However, now all points below the two lines will fully belong to the basin of attraction of \(E2\) because the state trajectories are directed toward region 5. The separation line in regions 8 is \(x_1 = -0.5\) because in the left part of this line the state trajectories are directed toward region 8. Similarly, due to the symmetry, the separation line for the region 4 is \(x_2 = -0.5\).

From the above considerations, it has become apparent that there are still subregions which can belong to each of the basin of attractions. This part can be reduced using more separation lines. However, the general approach would be to search for some high-order manifold for the separation. However, so far it is not known how to find or construct this manifold. The use of hyperplanes (lines), as in the example, will only give a first-order estimation of the basin of attraction.

V. CONCLUSIONS

In this brief we suggest a method for determining an estimation of the basin of attractions for the stable equilibrium points in Cellular Neural Networks. It is valid also for continuous time Hopfield networks if the output functions are piecewise instead of sigmoid functions. The method is based on determining the so-called tree of regions connected with each stable equilibrium point, and gives more insight into how the basin of attractions are situated. The method is also useful when it is necessary to determine where (on which stable equilibrium point) a trajectory, started from a given (binary) initial condition, will settle after the transient decays, e.g., which basin of attraction this (binary) initial condition belongs to.

It should be stressed that the problem of the estimation of the basin of attractions is extremely complex and our approach also becomes complex for high-dimension CNN’s. Nevertheless, the calculations for determining the sign of \(dx_i/dt\) on the separation hyperplanes are not time consuming; the first-order estimation can simply be automated. The problem arises with the number of the regions which belong to several basins of attraction when the order of CNN increases. The approach in Section IV should be applied for each of these regions, and the improvement is an open question for further research.

REFERENCES


Lyapunov Stability of Two-Dimensional Digital Filters with Overflow Nonlinearities

Derong Liu

Abstract—In this short paper, the second method of Lyapunov is utilized to establish sufficient conditions for the global asymptotic stability of the trivial solution of zero-input two-dimensional (2-D) Fornasini–Marchesini state-space digital filters which are endowed with a general class of overflow nonlinearities. Results for the global asymptotic stability of the null solution of the 2-D Fornasini–Marchesini second model with overflow nonlinearities are established. Several classes of Lyapunov functions are used in establishing the present results, including vector norms and the quadratic form. When the quadratic form Lyapunov functions are considered, the present results involve necessary and sufficient conditions under which positive definite matrices can be used to generate Lyapunov functions for 2-D digital filters with overflow nonlinearities.

Index Terms—Finite wordlength effects, Lyapunov methods, 2-D systems.

I. INTRODUCTION

Consider zero-input two-dimensional (2-D) nonlinear digital filters described by

\[
x(k+1, l+1) = f[A_1x(k, l+1) + A_2x(k+1, l)],
\]

where \(x \in \mathbb{R}^n\), \(A_1 \in \mathbb{R}^{n \times n}\), \(A_2 \in \mathbb{R}^{n \times n}\), and \(f(\cdot)\) represents overflow nonlinearities. For system (1), assume a finite set of initial conditions, i.e., assume that there exist two positive integers \(K\) and \(L\), such that

\[
x(k, 0) = 0 \quad \text{for} \quad k \geq K \quad \text{and} \quad x(0, l) = 0 \quad \text{for} \quad l \geq L.
\]

System (1) is usually referred to as the zero-input Fornasini–Marchesini second model [10] implemented in finite word-length format.

In the implementation of digital filters, signals are usually represented and processed in a finite word-length format which gives rise to several kinds of nonlinear effects, such as overflow and quantization. Since finite word-length realizations of digital filters result in systems which inherently are nonlinear, the asymptotic stability of such filters (under zero-input) is of great interest in practice (cf., [1]–[9], [11]–[15], [17]–[21]). The global asymptotic

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