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Vortex Statistics for Turbulence in a Container with Rigid Boundaries

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The evolution of vortex statistics for decaying two-dimensional turbulence in a square container with rigid no-slip walls is compared with a few available experimental results and with the scaling theory of two-dimensional turbulent decay as proposed by Carnevale et al. Power-law exponents, computed from an ensemble average of several numerical runs, coincide with some experimentally obtained values, but not with data obtained from numerical simulations of decaying two-dimensional turbulence with periodic boundary conditions.

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The emergence and the temporal evolution of a hierarchy of coherent vortices in decaying two-dimensional (2D) turbulence has been subject to many analytical, numerical, and experimental studies. One of the most remarkable theoretical results of the last decade is the scaling theory as proposed by Carnevale et al. [1,2]. In this theory it is supposed that the average number density of vortices decreases algebraically, \( \rho(t) \propto t^{-\zeta} \), with \( \zeta \) so far undetermined. Numerical simulations with a simple, punctuated-Hamiltonian, dynamical model for the evolution of a system of coherent vortices [1] and also numerical simulations of the Navier-Stokes equations, although hyperviscosity has been used, show that \( \zeta \) is approximately 0.72 \pm 0.03 [3]. This value of the decay exponent suggests that the straightforward scaling theory by Batchelor [4] is incorrect. His theory is based on the assumption that for nearly inviscid 2D turbulent flows the energy might still be considered as conserved, but the enstrophy not, because viscosity is preferentially dissipative on the smallest scales of the flow. Dimensional analysis then results in \( \rho(t) \propto t^{-2} \), which shows a large discrepancy with the Carnevale approach. Beside energy conservation Carnevale et al. also assumed that the extremum of vorticity, \( \omega_{\text{ext}} \), is conserved. Dimensional analysis then yields for the average number density \( \rho(t) \), the average enstrophy \( Z(t) \), the average vortex radius \( a(t) \), and the average mean vortex separation \( r(t) \)

\[
\rho(t) \propto L^{-2}(t/T)^{-\zeta}, \quad Z(t) \propto T^{-2}(t/T)^{-\zeta/2}, \quad a(t) \propto L(t/T)^{\zeta/4}, \quad r(t) \propto L(t/T)^{\zeta/2}. \tag{1}
\]

The characteristic length scale \( L \) and time scale \( T \) are defined by \( L = \omega_{\text{ext}}^{-1}\sqrt{\nu} \) and \( T = \omega_{\text{ext}}^{-1} \), respectively. As mentioned before, the power-law exponent is a free parameter which has to be predicted on the basis of numerical simulations. Another remarkable consequence of this scaling theory is the rather small decay exponent for the enstrophy, \( Z(t) \propto t^{-0.36} (\zeta \approx 0.72) \), compared to \( Z(t) \propto t^{-2} \) derived by Batchelor [4].

Measurements of the decay exponent \( \zeta \) have been performed in experiments of decaying 2D turbulence in thin electrolyte solutions in a rectangular container [5,6]. A 2D array of magnets is located just below the bottom of the container, and in combination with an electric current the fluid is forced electromagnetically. In the experiments arrays of \( 6 \times 6, 8 \times 8, \) and \( 10 \times 10 \) magnets, where nearest neighbors have oppositely signed magnetic polarity, have been used. The interaction of the magnetic field with the electric current results in a regular array of counterrotating vortices. After the current is switched off, the flow will relax and the process of decaying 2D turbulence can be investigated. The initial value of the Reynolds number, based on the scale of the container, is typically \( \text{Re} = 2000 \). The experiments yielded power laws for the average number density of vortices and associated quantities which could be correlated with the scaling theory introduced by Carnevale. These experiments [5,6] are considered as a confirmation of the validity of this scaling theory despite the differences between the theoretical and experimental setup such as the presence of no-slip walls and finite Reynolds number effects in the latter.

Recent numerical simulations of 2D turbulence [7,8] have elucidated the role of no-slip boundaries in general, and vortex-wall interactions in particular. It is therefore necessary to carry out numerical simulations of decaying 2D turbulence in containers with no-slip boundaries with analogous initial conditions as those employed in the experiments discussed above (i.e., an array of \( N \times N \) vortices). Numerical experiments with the same initial flow field have also been repeated with periodic boundary conditions. The ensemble averaged power-law exponents of the no-slip and the Fourier runs are compared with each other and with the experimental results. It should be emphasized that in the experiments as well as in the numerical simulations isotropy and homogeneity are absent when no-slip walls confine the flow. A direct comparison with the Carnevale theory is therefore hampered because this theory assumes isotropy and homogeneity.

The numerical simulations of the 2D Navier-Stokes equations on a bounded domain were performed with a 2D dealiased Chebyshev pseudospectral method [9], with a maximum of 513 Chebyshev modes in each direction.
for \( Re = 20000 \) (361 modes for \( Re = 10000 \) and 257 modes for \( Re = 5000 \)). The numerical computations of decaying 2D Navier-Stokes turbulence with periodic boundary conditions were carried out with a standard 2D Fourier pseudospectral method with a maximum of 341 active Fourier modes in each direction. In both cases neither hyperviscosity nor any other artificial dissipation has been used. The integral-scale Reynolds number of the flow is defined as \( Re = UW/\nu \) with \( U \) the rms velocity of the initial flow field, \( W \) the half width of the container, and \( \nu \) the kinematic viscosity of the fluid. Time has been made dimensionless by \( W/U \) and vorticity by \( U/W \).

The initial microscale Reynolds number is defined as \( Re_{\text{micro}} = 2Re/\omega_0 \) with \( \omega_0 \) the (dimensionless) initial rms vorticity. In our numerical experiments \( \omega_0 = 38.0 \pm 0.5 \), thus corresponding with \( Re_{\text{micro}} \approx 263, 526, \) and 1052, respectively. The time \( \tau \) is defined as \( \tau = \frac{\omega_0 t}{N^2} \), with \( t \) the dimensionless time and \( N^2 \) the number of vortices, and \( \tau = 1 \) corresponds approximately with the (initial) eddy turnover time.

The initial condition for the velocity field consists of 100 nearly equal-sized Gaussian vortices (thus \( N = 10 \)). The vortices have a dimensionless radius of 0.05 and a dimensionless absolute vortex amplitude of \( |\omega_0| \approx 100 \). Half of the vortices have positive circulation, and the other vortices have negative circulation. The vortices are placed on a regular lattice, well away from the boundaries, with a random displacement of the vortex centers equal to approximately 6% of the dimensionless lattice parameter \( \lambda \), with \( \lambda \approx 0.17 \). A smoothing function, similar to the one employed in Refs. [7,8], has been used in order to ensure the no-slip condition exactly. The initial conditions for the simulations with periodic boundary conditions are the same. Figure 1 shows two snapshots of the vorticity distribution of decaying turbulence in a container with no-slip walls (a) and for the case with periodic boundary conditions (b).

The power laws discussed in this Letter are computed from ensemble averages of 12 runs for \( Re = 5000 \) and 8 runs for \( Re = 10000 \) [2 runs have been performed for \( Re = 20000 \) and only power-law exponents for \( Z(\tau) \) could be computed]. The ensemble averaged decay exponents are also representative for the power laws obtained for each individual run. The slight variation observed in the set of individual decay exponents will be indicated by the error margin. The error margins are typically somewhat smaller for the runs with periodic boundary conditions compared to those for the runs with no-slip walls.

Because of the finiteness of the initial Reynolds number in our simulations, the domain averaged kinetic energy \( E \) of the flow decreases during the decay process. The initial value of the kinetic energy is \( E = 2 \), while at \( \tau = 80 \) and \( Re = 5000 \) the ensemble averaged kinetic energy of the flows with periodic boundary conditions has decreased to \( \langle E_{\text{periodic}} \rangle = 0.93 \) and for flows with no-slip boundaries to \( \langle E_{\text{no-slip}} \rangle = 0.55 \), with \( \langle \cdot \cdot \cdot \rangle \) denoting the ensemble average. For \( Re = 10000 \) we find \( \langle E_{\text{periodic}} \rangle = 1.25 \) and \( \langle E_{\text{no-slip}} \rangle = 0.83 \) (note that all quantities are in dimensionless units). The interaction time scale of the vortices (e.g., the time scale associated with merging processes which is of the order of a few initial eddy turnover times) is so short that the energy might be considered as constant during interaction. From the simulations with no-slip and with periodic boundary conditions it is also clear that the conservation of \( \omega_{\text{ext}} \) is strongly violated even when its value is normalized by \( \sqrt{E} \). Because of the production of very strong vortices in the boundary layers, the runs with no-slip boundaries show a slower algebraic decay compared to those with periodic boundary conditions; see Table I. It

<table>
<thead>
<tr>
<th>Re</th>
<th>( \langle \omega_{\text{ext}} \rangle /\sqrt{E} ) no-slip</th>
<th>( \langle Z \rangle /\sqrt{E} ) no-slip</th>
<th>( \langle Z/E \rangle /\sqrt{E} ) no-slip</th>
<th>( \langle \rho \rangle /\sqrt{E} ) no-slip</th>
<th>( \langle \tau \rangle /\sqrt{E} ) no-slip</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>-0.35 \pm 0.05</td>
<td>-1.00 \pm 0.1</td>
<td>-0.80 \pm 0.1</td>
<td>-0.85 \pm 0.1</td>
<td>0.45 \pm 0.07</td>
</tr>
<tr>
<td>10 000</td>
<td>-0.30 \pm 0.05</td>
<td>-0.85 \pm 0.1</td>
<td>-0.70 \pm 0.1</td>
<td>-0.75 \pm 0.1</td>
<td>0.40 \pm 0.07</td>
</tr>
<tr>
<td>20 000</td>
<td>-0.30 \pm 0.05</td>
<td>-0.80 \pm 0.1</td>
<td>-0.65 \pm 0.1</td>
<td>-0.65 \pm 0.1</td>
<td>0.65 \pm 0.07</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Re</th>
<th>( \langle \omega_{\text{ext}} \rangle /\sqrt{E} ) periodic</th>
<th>( \langle Z \rangle /\sqrt{E} ) periodic</th>
<th>( \langle Z/E \rangle /\sqrt{E} ) periodic</th>
<th>( \langle \rho \rangle /\sqrt{E} ) periodic</th>
<th>( \langle \tau \rangle /\sqrt{E} ) periodic</th>
</tr>
</thead>
<tbody>
<tr>
<td>5000</td>
<td>-0.47 \pm 0.04</td>
<td>-1.30 \pm 0.05</td>
<td>-1.13 \pm 0.05</td>
<td>-1.13 \pm 0.05</td>
<td>0.65 \pm 0.05</td>
</tr>
<tr>
<td>10 000</td>
<td>-0.38 \pm 0.04</td>
<td>-1.15 \pm 0.05</td>
<td>-0.98 \pm 0.05</td>
<td>-1.03 \pm 0.1</td>
<td>0.60 \pm 0.05</td>
</tr>
<tr>
<td>20 000</td>
<td>-0.97 \pm 0.05</td>
<td>-0.90 \pm 0.05</td>
<td>-0.80 \pm 0.05</td>
<td>-0.80 \pm 0.05</td>
<td>0.80 \pm 0.05</td>
</tr>
</tbody>
</table>
is interesting to note that few numerical runs with periodic boundary conditions have been carried out with hyperviscosity instead of Newtonian viscosity. A run with initial conditions as employed in the present study and a few runs with initial conditions as used by Weiss and McWilliams [3] show that $\omega_{\text{ext}} \propto \tau^{-\alpha}$, with $\alpha = 0.05-0.10$. Similar behavior has been observed in a few experiments [6], and in the computed final decay regime of evolving 2D Navier-Stokes turbulence [7]. However, the flows in these particular experiments and simulations are characterized by relatively small Reynolds numbers ($Re < 1000$) in this regime and any agreement with the theory proposed by Carnevale et al. is therefore rather accidental.

In Fig. 2 we have plotted $\langle Z(\tau) \rangle$ for $Re = 5000$ (Fig. 2a) and $Re = 10,000$ (Fig. 2b). From these results it is clear that the decay of the enstrophy for the no-slip runs is distinctly different from the results computed with periodic boundary conditions. It should be noted that our data for the runs with periodic boundary conditions are consistent with the numerical study by Chasnov [10] for decaying 2D turbulence with similar initial microscale Reynolds numbers. It is more difficult to find power-law behavior for the runs with no-slip boundaries. Because of strong vortex-wall interactions the enstrophy is very spiky (note that the plotted data already represent an average over many runs). The final stage ($\tau > 40$) indicates, at least for the runs with $Re = 10,000$ and $20,000$, that a regime with a significantly lower decay exponent is reached [$\langle Z(\tau) \rangle \propto \tau^{-0.5}$ for $Re = 10,000$].

The computation of the average number density of vortices $\langle \rho(\tau) \rangle$, the average vortex radius $\langle a(\tau) \rangle$, and the average mean vortex separation $\langle r(\tau) \rangle$ is based on a discrete wavelet packet transform (WPT) technique [11–13]. In order to consider a structure as a vortex the following conditions should be satisfied: The aspect ratio of the long and short axis of the (nearly) ellipsoidal patch should be smaller than two. The (absolute) value of the vorticity extremum should always be larger than 20% of the absolute value of the vorticity extremum of the strongest vortex. The computation of the vortex strength and the vortex radius is performed by taking into account the 20% of the strongest vortices detected by the WPT algorithm only [except in the final stage when the number of vortices becomes too small, i.e., $\rho(\tau) < 10$]. An important condition for the present comparison is that the data from the simulations with no-slip and with periodic boundary conditions are treated in exactly the same way. In Fig. 3 we have plotted $\langle \rho(\tau) \rangle$ and $\langle r(\tau) \rangle$ (based on the distance of the vortex centers), respectively. It is obvious from these data that the no-slip boundaries result in a denser system of vortices: The number of vortices decreases more slowly and the average mean vortex separation increases more slowly in the no-slip case compared to the results obtained for the periodic runs (see Table I). The final stage ($\tau > 40$) indicates again a regime with a somewhat smaller decay exponent for the number of vortices (see Fig. 3): $\langle \rho(\tau) \rangle \propto \tau^{-0.5}$ (Re = 5000) and $\langle \rho(\tau) \rangle \propto \tau^{-0.45}$ (Re = 10,000).

![FIG. 2. Ensemble averaged enstrophy plotted as a function of $\tau$. Drawn lines represent the no-slip results and the dashed lines the results for periodic boundary conditions.](image1)

![FIG. 3. Ensemble averaged number density (a),(b) and mean vortex separation (c),(d) plotted as a function of $\tau$. Drawn lines (with +) represent the no-slip results and the dashed lines (with ×) the results for periodic boundary conditions.](image2)
average vortex radius grows like $\langle a(\tau) \rangle \propto \tau^{0.25 \pm 0.03}$ for the no-slip runs as well as the periodic case (both for Re = 5000 and 10 000). The data displayed in Fig. 3 suggest parallel evolution for the periodic and no-slip runs for $\tau \leq 20$. Inspection of the power-law compensated ensemble averages [e.g., for the vortex density with Re = 5000: $\langle \rho(\tau) \rangle_{\text{no-slip}} \propto \tau^{0.85}$ and $\langle \rho(\tau) \rangle_{\text{periodic}} \propto \tau^{1.13}$] and particularly the associated horizontal plateau-value regime, however, clearly supports the values of the exponents summarized in Table I. For $\tau \leq 10$ it is difficult to discriminate between the two cases. This is mainly due to the finite time necessary for boundary layers to develop and to become dynamically active. From our data it can be conjectured that this process takes some 4 to 8 initial eddy turnover times.

It is clear that a Reynolds number dependence is still present, and it is not yet known how the vortex statistics will be modified by increasing the Reynolds number substantially. However, the different power laws obtained for the no-slip and the periodic runs give strong evidence that the vortex statistics are modified by the presence of boundaries.

Comparison of our data from simulations with no-slip walls with some experimental power-law exponents obtained by Hansen et al. [6],

$$\rho(\tau) \propto \tau^{-0.70 \pm 0.1}, \quad Z(\tau) = \frac{E(\tau)}{a(\tau)} \propto \tau^{-0.47 \pm 0.06}, \quad a(\tau) \propto \tau^{0.21 \pm 0.06}, \quad r(\tau) \propto \tau^{0.38 \pm 0.08},$$

requires simulations with a smaller large-scale Reynolds number (Re = 1000 and 2000). The rather steep decay of $\langle \rho(\tau) \rangle$ and $\langle Z(\tau)/E(\tau) \rangle$, observed for the simulations with Re = 5000 and 10 000 for $\tau < 40$, are virtually absent for these relatively low Reynolds number simulations. The computed power-law exponents are $\langle \rho(\tau) \rangle \propto \tau^{-0.7}$, $\langle Z(\tau)/E(\tau) \rangle \propto \tau^{-0.5}$, $\langle a(\tau) \rangle \propto \tau^{0.2}$, and $r(\tau) \propto \tau^{0.5}$. Analogous power-law behavior is found during the final decay stage ($\tau > 40$) of the runs with substantially higher Reynolds number $\{\langle \rho(\tau) \rangle \propto \tau^{-0.5}$ and $\langle Z(\tau)/E(\tau) \rangle \propto \tau^{-0.4}\}$.

In the laboratory experiments with electromagnetically forced electrolytes the energy decay is enhanced by bottom drag. We have repeated several of the present simulations for Re = 5000 and 10 000 in bounded domains with an additional friction term representing the bottom drag. The numerical values for the bottom friction has been chosen in the same range as for the experiments. It is found that up to $\tau = 100$ the computed power-law exponents for $\omega_{\text{ext}}/\sqrt{E}$, $Z/E$, $\rho$, $a$, and $r$ are approximately the same for runs with and without bottom friction. For larger eddy turnover times the nonlinearity of the flow starts to become depleted due to bottom friction and a comparison between simulations with and without bottom friction is not opportune.

An alternative approach to analyze the numerical data is by normalizing $\rho(\tau), Z(\tau), a(\tau),$ and $r(\tau)$ with $T$ and $L$ [see Eqs. (1) and (2)]. Note that $L$ and $T$ depend on time, instead of being constants, due to finite Reynolds number effects. The power-law exponent $\zeta$ is then determined by computing $L^2 \rho(\tau), T^2 Z(\tau), a(\tau)/L,$ and $r(\tau)/L$. The power-law exponents calculated in this way show some agreement with the Carnevale approach for the runs with periodic boundary conditions. We were not able to observe power laws for the average number density and other quantities from the runs with no-slip boundaries, and computing a single power-law exponent describing their decay properties is impossible.

To summarize, we can conclude that vortex statistics for decaying 2D turbulence in containers with no-slip walls differ considerably from the case with periodic boundary conditions. We believe that any agreement between experimentally obtained power-law exponents and the Carnevale approach is rather accidental: The experimental data also agree with our numerical simulations with no-slip walls. Moreover, the power-law exponents obtained from our simulations with periodic boundary conditions, which is the closest numerical test for the scaling theory, disagree with the experimental data.

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