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Published: 01/01/1999

Document Version
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

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SUBORDINATION PRINCIPLE FOR FRACTIONAL EVOLUTION EQUATIONS

EMILIA BAZHLEKOVA

Abstract. The abstract Cauchy problem for the fractional evolution equation

\[ D^\alpha u = Au, \quad \alpha > 0, \]

where \( A \) is a closed densely defined operator in a Banach space, is investigated. A generalization of the abstract Weierstrass formula is presented and used for studying the relationship between the problems for different \( \alpha \).

Key words. fractional integral, fractional derivative, one parameter semigroup, cosine family, abstract Weierstrass formula, Mittag-Leffler function, Mellin transform

AMS subject classification. 26A33, 45N05, 47D06, 47D09

1. Introduction. Consider a linear closed operator \( A \) densely defined in a Banach space \( X \). Let \( \alpha > 0 \) and \( n \in \mathbb{N} \) is such that \( n-1 < \alpha \leq n \). Given \( x \in X \), we investigate the following Cauchy problem in \( X \):

\[ D^\alpha u(t) = Au(t), \quad u(0) = x, \quad u^{(k)}(0) = 0, \quad k = 1, \ldots, n-1. \quad (FE_\alpha) \]

By \( D^\alpha \) we denote the regularized fractional derivative of order \( \alpha \) (see e.g., [Ma]):

\[ D^\alpha u(t) = \frac{d^n}{dt^n} J^\alpha T(t) \left( u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0) \right), \]

where \( J^\alpha T(t) \) is the fractional integration operator

\[ J^\alpha T(t) x = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) \, ds, \quad \alpha > 0; \quad J^0 T(t) = I. \]

For the cases \( \alpha = 1 \) and \( 2 \) the theories of strongly continuous semigroups and cosine families, respectively, are developed (see e.g., [Pa] and [Fa2]). For integer \( \alpha \) the following results are classical:

- If \( \alpha = 3, 4, \ldots \), the Cauchy problem \((FE_\alpha)\) is well-posed if and only if \( A \) is bounded ([Fa1]);
- If \( A \) generates a cosine family \( S_2(t) \) then \( A \) generates a \( C_0 \)-semigroup \( S_1(t) \) and they are related by the abstract Weierstrass formula ([Fa1], [Fa2, p.169]):

\[ S_1(t)x = \frac{1}{\sqrt{\pi t}} \int_0^\infty e^{-s^2/(4t)} S_2(s) x \, ds, \quad t > 0, \quad (1.1) \]

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which incidentally shows that $S_1(t)$ is a holomorphic semigroup of angle $\pi/2$. However, the converse is false: there are infinitesimal generators of holomorphic semigroups of angle $\pi/2$ which do not generate cosine families. A classical example is the Laplacian $\Delta$ in $L^p(\mathbb{R}^n)$ if $p \neq 2$, $n \neq 1$ (see e.g. [H]).

In this work, the classical results presented above, are generalized to non-integer values of $\alpha$. It is proven in Section 2, that if $\alpha > 2$, $(FE_\alpha)$ has a solution of exponential growth if and only if $A$ is bounded. In Section 3, we generalize the abstract Weierstrass formula and prove the so called subordination principle: if $(FE_\beta)$ has an exponentially bounded solution operator, then for any $\alpha < \beta$, $(FE_\alpha)$ has an exponentially bounded and holomorphic solution operator. In particular, if $1 < \beta < 2$, then $A$ generates a holomorphic semigroup of angle $(\beta - 1)\pi/2$.

An example shows that the converse is false again. In Section 4, an inversion of the generalized abstract Weierstrass formula provided $-A$ has bounded imaginary powers, is given. For convenience the properties of two special functions, arising in the study of fractional equations, are collected in an appendix.

2. Preliminary results. Throughout this paper $\sigma(A)$ and $\varrho(A)$ are the spectrum and the resolvent set of $A$, $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator of $A$, $\mathcal{B}(X)$ is the space of all bounded operators from $X$ into itself; $z^0$ denotes the principal branch of $z^n$ in $\mathbb{C}$ cut along the negative real axis.

Similarly to the ordinary differentiation and integration, we have (see e.g. [Ma])

$$D_t^\alpha J_t^\alpha u(t) = u(t); \quad J_t^\alpha D_t^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^k, \quad n - 1 < \alpha \leq n.$$ 

Therefore the Cauchy problem $(FE_\alpha)$ is equivalent to the Volterra integral equation

$$u(t) = x + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} Au(s) ds \quad (IE_\alpha)$$

and we shall use the terminology given in [Pr, Section 1].

**Definition 2.1.** A family $\{S_\alpha(t)\}_{t \geq 0} \subset \mathcal{B}(X)$ is called a solution operator for $(FE_\alpha)$ if the following conditions are satisfied:

a) $S_\alpha(t)$ is strongly continuous for $t \geq 0$ and $S_\alpha(0) = I$;

b) $S_\alpha(t)D(A) \subset D(A)$ and $AS_\alpha(t)x = S_\alpha(t)Ax$ for all $x \in D(A)$, $t \geq 0$;

c) $S_\alpha(t)x$ is a solution of $(FE_\alpha)$ for all $x \in D(A)$, $t \geq 0$.

**Definition 2.2.** The solution operator $S_\alpha(t)$ is called exponentially bounded if there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|S_\alpha(t)\| \leq Me^{\omega t}, \quad t \geq 0. \quad (2.1)$$

**Definition 2.3.** An operator $A$ is said to belong to $\mathcal{C}^\alpha(M, \omega)$ if the problem $(FE_\alpha)$ has a solution operator $S_\alpha(t)$ satisfying (2.1).
Denote $C^\alpha(\omega) = \bigcup \{C^\alpha(M, \omega); \ M \geq 1\}$, $C^\alpha = \bigcup \{C^\alpha(\omega); \ \omega \geq 0\}$. In these notations $C^1$ and $C^2$ are the sets of all infinitesimal generators of $C_0$ semigroups and cosine families, respectively.

Assume $A \in C^\alpha(M, \omega)$ and let $S_\alpha(t)$ be the corresponding solution operator. For $\text{Re}\lambda > \omega$ we define $H(\lambda)x = \int_0^\infty e^{-\lambda t}S_\alpha(t)x \ dt$, $x \in X$. In view of (2.1) $H(\lambda) \in \mathcal{B}(X)$. Using properties b) and c) of Definition 2.1, and the identity (see [Ma])

$$\mathcal{L}\{D^\alpha u\}(\lambda) = \lambda^\alpha \mathcal{L}\{u\}(\lambda) - \lambda^{\alpha-1}u(0),$$

where $\mathcal{L}$ denotes the Laplace transform, we obtain

$$\lambda^\alpha H(\lambda)x - \lambda^{\alpha-1}x = AH(\lambda)x, \ x \in X; \ \lambda^\alpha H(\lambda)x - \lambda^{\alpha-1}x = H(\lambda)Ax, \ x \in D(A).$$

Hence the operator $\lambda^\alpha I - A$ is invertible and $H(\lambda) = \lambda^{\alpha-1}R(\lambda^\alpha, A)$, that is

$$\lambda^{\alpha-1}R(\lambda^\alpha, A)x = \int_0^\infty e^{-\lambda t}S_\alpha(t)x \ dt, \ \text{Re}\lambda > \omega, \ x \in X. \quad (2.2)$$

In particular

$$\{\lambda^\alpha : \text{Re}\lambda > \omega\} \subseteq \varrho(A). \quad (2.3)$$

We apply these results to prove the next

**Theorem 2.1.** Assume $\alpha > 2$. Then $A \in C^\alpha$ iff $A \in \mathcal{B}(X)$.

**Proof.** If $A \in C^\alpha$ for some $\alpha > 2$ then (2.3) implies that $\Omega_{\alpha, \omega} = \{\lambda^\alpha : \text{Re}\lambda > \omega, \ |\arg \lambda| \leq \pi/\alpha < \pi/2\} \subseteq \varrho(A)$. Hence $\varrho(A)$ consists of the entire complex plane with the exception of some bounded set containing the origin. If $\mu \in \mathbb{C}$ with $|\mu|$ large enough then $\mu = \lambda^\alpha \in \Omega_{\alpha, \omega}$. Equations (2.1) and (2.2) imply

$$\|\mu R(\mu, A)\| \leq \frac{M|\lambda|}{\text{Re}\lambda - \omega} \leq \frac{M|\lambda|}{|\lambda|\cos(\pi/\alpha) - \omega} \to \frac{M}{\cos(\pi/\alpha)}, \ |\mu| \to \infty.$$

Then the following lemma [Go, Lemma 5.2.] shows that $A$ is bounded.

**Lemma 2.1.** If $\sigma(A)$ is a bounded subset of $\mathbb{C}$ and $\|R(\mu, A)\| = O(1/|\mu|)$ as $|\mu| \to \infty$ then $A \in \mathcal{B}(X)$.

Conversely, let $A \in \mathcal{B}(X)$ and fix $\alpha > 2$. Then

$$S_\alpha(t) = E_\alpha(At^\alpha) = \sum_{n=0}^\infty \frac{A^n t^{\alpha n}}{\Gamma(\alpha n + 1)}.$$

We shall prove that $S_\alpha(t)$ is exponentially bounded. The asymptotic expansion (A.3) and the continuity of the Mittag–Leffler function in $t \geq 0$ imply that if $\alpha \in (0, 2)$, $\omega \geq 0$, there is a constant $C$ such that

$$E_\alpha(\omega t^\alpha) \leq C \exp(\omega^{1/\alpha} t), \ t \geq 0. \quad (2.4)$$
Fix $k \in \mathbb{N}$ such that $k \geq \alpha$. Since $\Gamma(c) > 0$ for $c > 0$, using (2.4) we obtain the estimate
\[
\left\| S_\alpha(t) \right\| \leq \sum_{n=0}^{\infty} \frac{\| A \|^{n \alpha n}}{\Gamma(\alpha n + 1)} = \sum_{n=0}^{\infty} \frac{\| A \|^{n(1/k)n^2 + n}}{\Gamma((\alpha/k)k n + 1)} \leq \\
\sum_{n=0}^{\infty} \frac{\| A \|^{n(1/k)n^2 + n}}{\Gamma((\alpha/k)k n + 1)} = E_{\alpha/k}(\| A \|^{1/k^2 \alpha / k}) \leq C_{\alpha} \| A \|^{1/\alpha}, \; t \geq 0.
\]
Hence (2.1) is satisfied with $\omega = \| A \|^{1/\alpha}$ and $A \in C^\alpha$. \hfill \Box

In view of Theorem 2.1, further we will consider mainly $\alpha, \; 0 < \alpha \leq 2$. The following generation theorem is a particular case of a result, essentially due to Da Prato and Iannelli [DP-I]. We use the version given in [Pr, Theorem 1.3.], that in our case reads:

**Theorem 2.2.** Let $0 < \alpha \leq 2$. Then $A \in C^\alpha(M, \omega)$ iff $(\omega, \infty) \subset \rho(A)$ and
\[
\| (\lambda^{\alpha-1} R(\lambda^{\alpha}, A))^{(n)} \| \leq Mn! (\lambda - \omega)^{-(n+1)}, \; \lambda > \omega, \; n \in \mathbb{N}_0.
\]

The following result is an immediate consequence of Theorem 2.2.

**Theorem 2.3.** Let $0 < \alpha \leq 2$. Then $A \in C^\alpha(M, \omega)$ iff $(\omega, \infty) \subset \rho(A)$ and there is a strongly continuous operator-valued function $S(t)$ satisfying $\| S(t) \| \leq M e^{\omega t}, \; t \geq 0$, and such that
\[
\lambda^{\alpha-1} R(\lambda^{\alpha}, A)x = \int_0^\infty e^{-\lambda t} S(t)x \; dt, \; \lambda > \omega, \; x \in X.
\]

If this is the case $S(t) = S_\alpha(t)$.

**Proof.**

Let a function $S(t)$ with the properties mentioned above exists. After easily justified differentiation under the integral sign in (2.6) we obtain the inequalities (2.5). Applying Theorem 2.2, it results that $A \in C^\alpha(M, \omega)$. Let $S_\alpha(t)$ be the solution operator of $(FE_\alpha)$. Then (2.6) holds for both $S_\alpha(t)$ and $S(t)$ and $S_\alpha(t) = S(t)$ follows from the uniqueness of the Laplace transform. The converse has already been proven before Theorem 2.1. \hfill \Box

**3. The subordination principle.** Further we study the relationship between the problems $(FE_\alpha)$ for different values of $\alpha$. The so called *subordination principle* (see also [Pr, Section 4.]) is presented in Theorems 3.1. and 3.2.

**Theorem 3.1.** Let $0 < \alpha < \beta \leq 2, \; \gamma = \alpha/\beta$. If $A \in C^\beta(\omega)$ then $A \in C^\alpha(\omega^{1/\gamma})$ and the following representation holds
\[
S_\alpha(t)x = t^{-\gamma} \int_0^\infty \Phi_{\gamma}(st^{-\gamma}) S_\beta(s)x \; ds, \; t > 0, \; x \in X,
\]
where $\Phi_{\gamma}(z)$ is defined by (A.5).
Proof. A proof can be found in [B, Theorem 4.2.], but we present it here for completeness. Set \( \varphi_{t,\gamma}(s) = t^{-\gamma}\Phi_{\gamma}(st^{-\gamma}) \) and define

\[
S(t)x = \int_0^\infty \varphi_{t,\gamma}(s)S_\beta(s)x\,ds, \quad t > 0.
\]  
(3.2)

Our aim is to prove that \( S(t) \) satisfies the conditions of Theorem 2.3, with \( \omega^{1/\gamma} \) instead of \( \omega \). Since \( A \in C^2(\omega) \), then \( (\omega^\beta, \infty) \subset \varrho(A) \) and there is a constant \( M \geq 1 \) such that

\[
\|S_\beta(s)\| \leq Me^{\omega s}, \quad s \geq 0.
\]  
(3.3)

Then the condition \( (\omega^{1/\gamma})^\alpha, \infty) \subset \varrho(A) \) is trivially fulfilled. Further (3.3) together with (A.7) and (A.6) implies

\[
\|S(t)\| \leq \int_0^\infty \varphi_{t,\gamma}(s)\|S_\beta(s)\|\,ds \leq M \int_0^\infty \varphi_{t,\gamma}(s)e^{\omega s}\,ds = ME_\gamma(\omega t^\gamma), \quad t \geq 0.
\]  
(3.4)

This together with (2.4) gives

\[
\|S(t)\| \leq MC\exp(\omega^{1/\gamma}t), \quad t \geq 0.
\]  
(3.5)

Next we prove the strong continuity of \( S(t) \) at the origin on the basis of the dominated convergence theorem, using (A.8), (3.5) and the strong continuity of \( S_\beta(t) \) at the origin:

\[
s - \lim_{t \downarrow 0} S(t)x = s - \lim_{t \downarrow 0} \int_0^\infty \Phi_{\gamma}(\sigma)S_\beta(\sigma t^\gamma)x\,d\sigma = \int_0^\infty \Phi_{\gamma}(\sigma)x\,d\sigma = x.
\]

For \( \lambda > \omega^{1/\gamma} \), using (3.2) and interchanging the order of integration, we have

\[
\int_0^\infty e^{-\lambda S(t)x}\,dt = \int_0^\infty S_\beta(s)x\int_0^\infty e^{-\lambda \varphi_{t,\gamma}(s)}\,dt\,ds.
\]  
(3.6)

Substituting \( \mu = t\tau \) in (2.1) and shifting the new contour \( \Gamma' = \Gamma/t \) to \( \Gamma \) we get the integral representation

\[
\varphi_{t,\gamma}(s) = \frac{1}{2\pi i} \int_{\Gamma} \tau^{\gamma-1}\exp(\pi t - \tau^\gamma s)\,d\tau.
\]

Therefore

\[
\int_0^\infty e^{-\lambda \varphi_{t,\gamma}(s)}\,dt = \lambda^{\gamma-1}\exp(-\lambda^\gamma s).
\]  
(3.7)

Using now (2.2) for \( S_\beta(s) \) we obtain from (3.6) and (3.7)

\[
\int_0^\infty e^{-\lambda S(t)x}\,dt = \lambda^{\gamma-1}\int_0^\infty \exp(-\lambda^\gamma s)S_\beta(s)x\,ds = \lambda^{\alpha-1}R(\lambda^\alpha, A)x.
\]

So we have proved the conditions of Theorem 2.3. Therefore \( A \in C^\alpha(\omega^{1/\gamma}) \) and the corresponding solution operator \( S_\alpha(t) = S(t) \).

Note that (3.4) implies the following additional result:
Corollary 3.1. Assume \( \alpha \in (0, 1) \), \( \omega \in \mathbb{R} \) and \( A \) is the infinitesimal generator of a \( C_0 \) semigroup \( S_t(t) \) satisfying \( \| S_t(t) \| \leq Me^{\omega t}, \ t \geq 0 \). Then \( A \in C^\alpha \) and \( \| S_\alpha(t) \| \leq ME_\alpha(\omega t^\alpha), \ t \geq 0 \).

Remark 3.1. Since \( \Phi_{1/2}(z) = \pi^{-1/2}e^{-z^2/4} \), the formula (3.1) coincides with the abstract Weierstrass formula (1.1) when \( \alpha = 1, \beta = 2 \).

The next example illustrates how Theorem 3.1. can be applied to obtain solutions of \((FE_\alpha)\) for non-integer \( \alpha \) from known solutions for \( \alpha = 1 \) or 2.

Example 3.1. Consider the fractional diffusion problem (see [Ma]):

\[
D_\alpha^\tau u = \partial^2 u / \partial x^2, \ -\infty < x < \infty, \ t \geq 0, \ 0 < \alpha \leq 2; \ u(\pm\infty, t) = 0, \ u(x, 0) = f(x).
\]

Let \( X = L^p(\mathbb{R}), \ A = f'' \) with \( D(A) = W^{2,p}(\mathbb{R}) \). For \( \alpha = 2 \) the solution is given by d’Alembert’s formula:

\[
S_2(t)f(x) = \frac{1}{2}(f(x + t) + f(x - t)).
\]

Applying formula (3.1) we then obtain the solutions for all \( \alpha \in (0, 2) \):

\[
S_\alpha(t)f(x) = t^{-\alpha/2} \int_0^\infty \Phi_{\alpha/2}(s) f(x - s) ds = (1/2)t^{-\alpha/2} \int_{-\infty}^\infty \Phi_{\alpha/2}(s) f(x - s) ds.
\]

This representation coincides with a result given in [Ma].

Further we denote by \( \Sigma(\theta) \) the open sector \( \Sigma(\theta) = \{ t \in \mathbb{C} \setminus \{0\}; |\arg t| < \theta \} \).

Theorem 3.2. Under the hypotheses of Theorem 4.1, the solution operator \( S_\alpha(t) \) has the following additional properties where \( \theta(\gamma) = (1/\gamma - 1)\pi/2 \)

(a) \( S_\alpha(t) \) admits an analytic extension to the sector \( \Sigma(\min\{\theta(\gamma), \pi\}) \);

(b) If \( \omega = 0, \| S_\alpha(t) \| \leq C \) for \( t \in \Sigma(\min\{\theta(\gamma), \pi\} - \varepsilon) \) where \( C = C(\gamma, \varepsilon) \);

(c) If \( \omega > 0, \| S_\alpha(t) \| \leq Ce^{\delta t} \) for \( t \in \Sigma(\min\{\theta(\gamma), \pi/2\} - \varepsilon) \) where \( \delta = \delta(\gamma, \varepsilon), \ C = C(\delta, \gamma, \varepsilon) \).

Proof. Let \( \Sigma_\alpha \) denotes the sector \( \Sigma_\gamma = \Sigma(\min\{\theta(\gamma), \pi\}) \). The function under the integral sign in (3.1) is analytic in \( t \in \Sigma_\gamma \). In view of the asymptotic expansion (A.9) and the inequality (3.3) and, noting that \( \text{Re}(t^{-\gamma/(1-\gamma)}) > 0 \) when \( t \in \Sigma_\gamma \) and \( 1/(1 - \gamma) > 1 \) for \( 0 < \gamma < 1 \), it follows that the integral in (3.1) is absolutely and uniformly convergent on compact subsets of \( \Sigma_\gamma \). Therefore, \( S_\alpha(t) \) given by (3.1) is analytic in \( \Sigma_\gamma \) (see [Mar, p. 32, Th. 7]). This implies a).

In what follows \( c_n \) denote positive constants, not depending on \( t \). Let \( t \in \Sigma(\min\{\theta(\gamma), \pi\} - \varepsilon) \). Then (3.1) and (3.3) imply

\[
\| S_\alpha(t) \| \leq M|t|^{-\gamma} \int_0^\infty |\Phi_\gamma(st^{-\gamma})| e^{\omega s} ds = M \int_0^\infty |\Phi_\gamma(se^{i\varphi})| e^{\alpha|t|^{1-\gamma}} ds, \quad (3.8)
\]
where \( \varphi = -\gamma \arctan t \), that is \( |\varphi| \leq (1 - \gamma)\pi/2 - \gamma\varepsilon \). Thanks to the expansion (A.9) there exists \( S > 0 \) such that for any \( s > S \)

\[
|\Phi_s(se^{i\varphi})| \leq c_6 s^{(\gamma^{-1}-1)/(1-\gamma)} \exp \left\{ -c_1 \sin \frac{\gamma^{\varepsilon}}{1-\gamma} s^{1/(1-\gamma)} \right\}.
\]  

(3.9)

Divide the last integral in (3.8) in two parts: \( I_1 \) from 0 to \( S \) and \( I_2 \) from \( S \) to \( \infty \). If \( \omega = 0 \) then \( I_1 \) and \( I_2 \) are uniformly bounded: \( I_1 \)- since \( \Phi_s(z) \) is an entire function and \( I_2 \)- in view of (3.9). Hence \( b) \) is proven.

Assume now \( \omega > 0 \) and \( t \in \Sigma(\min\{\theta(\gamma), \pi/2\} - \varepsilon) \). Denote \( r = |t| \). Then

\[
I_1 \leq c_2 \exp(c_3 r^\gamma) \leq c_4 \exp(c_5 r), \quad \text{since } 0 < \gamma < 1.
\]

Applying (3.9) we have

\[
I_2 \leq M c_6 \int_0^\infty s^{(\gamma^{-1}-1)/(1-\gamma)} \exp \left\{ -c_1 \sin \frac{\gamma^{\varepsilon}}{1-\gamma} s^{1/(1-\gamma)} + \omega r^{\gamma} \right\} ds.
\]

Using now the asymptotic expansion (see [Evg, p. 141])

\[
\int_0^\infty s^p \exp(-s^q + ys) ds \simeq Ay(2q-p+2)/!(2q-2) \exp\{ay^q/(q-1)\}, \quad q > 1, \ y \to +\infty,
\]

where \( A = A(p, q), \ a = a(q) \) are positive constants, we obtain \( I_2 \leq c_5 r^\delta \exp(c_6 r) \leq c_7 \exp(c_7 r) \). Then \( I_1 + I_2 \leq C \exp(c_7 r) \). Since \( |\arg t| < \pi/2 - \varepsilon \), we can take \( \delta = c_{10}/\sin\varepsilon \) and then \( \|S_\alpha(t)\| \leq C \exp(\delta\Re t) \).

**Definition 3.1.** A solution operator \( S_\alpha(t) \) of \( (FE_\alpha) \) is called analytic in \( \Sigma(\theta_0), \theta_0 \in (0, \pi/2] \), if \( S_\alpha(t) \) admits an analytic extension to \( \Sigma(\theta_0) \) and for any \( \theta \in (0, \theta_0) \)

\[
\|S_\alpha(t)\| \leq C e^{\delta \Re t}, \ t \in \Sigma(\theta),
\]

where \( \delta = \delta(\theta), \ C = C(\delta, \theta) \).

**Remark 3.2.** We proved in Theorem 3.2, that \( S_\alpha(t) \) is an analytic solution operator. Note that if \( \omega = 0 \) and \( \gamma < 1/2 \), \( S_\alpha(t) \) is in fact analytic in a sector with opening angle \( \theta > \pi \) and if \( \gamma < 1/3 \), it is analytic in all of \( C \) except of a neighbourhood of the negative real axis.

**Remark 3.3.** To complete the picture, we note in addition to Theorem 3.2, that if \( S_\beta(t) \) is analytic in \( \Sigma(\theta_0) \) and \( \gamma = \alpha/\beta < 1 \) then \( S_\alpha(t) \) is analytic in \( \Sigma(\min\{\theta(\gamma) + \theta_0/\gamma, \pi/2\}) \). Indeed, if for \( |\phi| < \theta_0 \) we consider the path \( \Gamma_R = [0, R], R \exp(i[0, \phi]), [R, 0]e^{i\phi}, \) then by Cauchy’s theorem \( \int_{\Gamma_R} \varphi_{t,\gamma}(z)S_\beta(z) \, dz = 0 \), i.e. for \( R \to \infty \) we obtain

\[
S_\alpha(t) = \int_0^\infty \varphi_{t,\gamma}(s)S_\beta(s) \, ds = \int_0^\infty \varphi_{t,\gamma}(se^{i\phi})S_\beta(se^{i\phi}) \, de^{i\phi} \, ds
\]

and the desired result follows as in the proof of Theorem 3.2.

**Corollary 3.2.** If \( A \in C^\alpha(1, 0) \) for some \( \alpha \in (0, 1] \), then \( A \in C^\alpha(1, 0) \) for all \( \alpha \in (0, 1] \).
Proof. If \( A \in C^0(1,0) \) for some \( \alpha \in (0,1] \), then according to Theorem 2.2, \( (0,\infty) \subseteq g(A) \) and (2.5) holds with \( \omega = 0 \), \( M = 1 \) and \( n = 0 \), that is \( \|\lambda^{n-1}R(\lambda^\alpha, A)\| \leq 1/\lambda \), \( \lambda > 0 \). This is exactly \( \|R(\mu, A)\| \leq 1/\mu \), \( \mu > 0 \). Hence \( A \in C^1(1,0) \) and Theorem 3.1. implies \( A \in C^\alpha(1,0) \) for all \( \alpha \in (0,1] \). □

Combining Theorem 3.2. and Corollary 3.2., we obtain the next result.

**Corollary 3.3.** Let \( \alpha \in (0,1) \) and \( S_\alpha(t) \) is a contraction: \( \|S_\alpha(t)\| \leq 1 \), \( t \geq 0 \). Then it is necessarily analytic.

Set now \( \alpha = 1 \) in Theorem 3.2. If \( A \in C^\beta \) for some \( \beta \in (1,2] \), then \( A \) generates an analytic semigroup \( S_1(t) \) of angle \( (\beta-1)\pi/2 \). We are interested in the opposite: whether the analyticity of \( S_1(t) \) in \( \Sigma((\beta-1)\pi/2) \) suffices for existing of \( S_\beta(t) \)?

The next example shows that the answer is negative for \( \beta \in (1,2) \).

**Example 3.2.** Fix \( \beta \in (1,2) \) and, denoting \( D^2_x = d^2/dx^2 \), consider the operator \( A_\beta = e^{i(2-\beta)\pi/4}D^2_x \) in the space \( X = L^1(\mathbb{R}) \). Since \( D^2_x \) generates a bounded analytic semigroup of angle \( \pi/2 \) then \( \|R(\lambda, D^2_x)\|_1 \leq M/|\lambda| \), \( \lambda \in \Sigma(\pi-\varepsilon) \).

Therefore
\[
\|R(\lambda, A_\beta)\|_1 = \|R(\lambda e^{-i(2-\beta)\pi/4}, D^2_x)\|_1 \leq M/|\lambda| \text{, } \lambda \in \Sigma(\beta\pi/2 - \varepsilon),
\]

i.e. \( A_\beta \) generates a bounded holomorphic semigroup of angle \( (\beta-1)\pi/2 \). Let now consider the problem
\[
D^2_x u(x,t) = A_\beta u(x,t), \text{ } u(x,0) = f(x), \text{ } u'_t(x,0) = 0,
\]

with \( f \in \mathcal{S}(\mathbb{R}) \)- the space of rapidly decreasing functions of Schwartz. The solution is then given by
\[
u(x, t) = \frac{1}{2} e^{-i(2-\beta)\pi/4} \int_{-\infty}^{\infty} \varphi_{1,\beta/2}(e^{-i(2-\beta)\pi/4}|s|) f(x-s) \, ds.
\]

This can be verified proving that \( \int_0^\infty e^{-\lambda u(x,t)} \, dt = \lambda^{\beta-1} R(\lambda^\beta, A_\beta) f(x) \). Suppose \( A_\beta \in C^\beta \) for \( X = L^1(\mathbb{R}) \). Then given \( t > 0 \) there is a constant \( C \) such that
\[
\|u(x,t)\|_1 \leq C \|f\|_1 \text{ for any function of the form (3.10)}.
\]

Let \( \delta_n \) be a delta sequence (i.e. for any \( y \in L^1(\mathbb{R}) \), \( \delta_n \ast y \to y \) with respect to \( \|\|_1 \) as \( n \to \infty \) and \( \|\delta_n\|_1 = 1 \)). Setting \( f(x) = \delta_n \) in (3.10) and calling \( u_n(x,t) \) the function so obtained we see that
\[
u_n(x,t) \to k(x,t) = \frac{1}{2} e^{-i(2-\beta)\pi/4} \varphi_{1,\beta/2}(e^{-i(2-\beta)\pi/4}|x|)
\]

for all \( x \). Since \( \|\delta_n\|_1 = 1 \), it follows from Fatou’s lemma that \( k(\cdot,t) \) must be in \( L^1(\mathbb{R}) \) which is false in view of (A.9).

Next theorem gives a sufficient condition under which \( A \in C^\beta(0) \).

**Theorem 3.3.** If \( 1 < \beta < 2 \) and
\[
\|R(\lambda, A)\| \leq M/|\lambda| \text{, } \lambda \in \Sigma(\beta\pi/2),
\]

then \( A \in C^\beta(0) \).
Proof. The key to the proof is the following lemma due to Prüss [Pr, Proposition 0.1.1]:

**Lemma 3.1.** Let \( h(\lambda) \) be analytic for \( \text{Re}\lambda > 0 \) and \( C \geq 0 \) is such that
\[
\| \lambda h(\lambda) \| \leq C; \quad \| \lambda^2 h'(\lambda) \| \leq C, \quad \text{Re}\lambda > 0.
\]

Then
\[
\| r^{n+1} h^{(n)}(r)/n! \| \leq 2C, \quad r > 0, \quad n \in \mathbb{N}.
\]

Setting now \( h(\lambda) = \lambda^{\beta-1} R(\lambda^{\beta}, A) \) it easily follows from (3.11) that \( h(\lambda) \) satisfies (3.12) with \( C = \max\{ M, M(\beta - 1) + M^2\beta \} \). Then we have (3.13), that is exactly (2.5) with \( \alpha = \beta, \ \omega = 0 \). Hence \( A \in \mathcal{C}^0(0) \).

**Corollary 3.4.** If \( A \) is the infinitesimal generator of a bounded holomorphic semigroup of angle \( (\beta - 1)\pi/2 + \varepsilon \), where \( \varepsilon \in (0, (2 - \beta)\pi/2) \) can be chosen arbitrarily small, then \( A \in \mathcal{C}^0(0) \).

This corollary incidentally shows that if \( A \) generates a bounded holomorphic semigroup of angle \( \pi/2 \) (e.g., the Laplacian \( \Delta \) in \( L^p(\mathbb{R}^n) \)) then \( A \in \mathcal{C}^0 \) for any \( \beta < 2 \), moreover \( S_\beta(t) \) is holomorphic in \( \Sigma(\delta_\beta) \) for sufficiently small \( \delta_\beta \). Recall that in this case \( A \) does not necessarily generate a cosine family.

4. **Bounded imaginary powers and the inverse formula.** Let \( X \) be a complex Banach space. We give an inversion of formula (3.1) provided \( A \) satisfies some additional conditions. Recall the Mellin transform defined by
\[
\{ \mathcal{M} f \}(\varrho) = \int_0^\infty t^{\varrho-1} f(t) \, dt
\]
and its inverse
\[
f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \{ \mathcal{M} f \}(\varrho) t^{-\varrho} \, d\varrho.
\]
From the integral representation (A.5) of \( \Phi_\alpha(z), \ \gamma \in (0, 1) \), we obtain \( \{ \mathcal{M} \Phi_\alpha \}(\varrho) = \Gamma(\varrho)/\Gamma(1-\gamma+\gamma \varrho), \ \Re \varrho > 0 \). Taking \( c \in (0, 1) \) in (4.1) and making the substitution \( \sigma = 1 - \varrho \), we have
\[
\Phi_\gamma(t) = \frac{1}{2\pi i} \int_L \frac{\Gamma(1-\sigma)}{\Gamma(1-\gamma+\gamma \sigma)} t^{\sigma-1} \, d\sigma,
\]
where \( L = \{ c + iy, \ -\infty < y < \infty \} \) and \( c \in (0, 1) \) can be arbitrarily chosen thanks to Cauchy’s theorem. Substituting this representation in (3.1) we obtain formally
\[
S_\gamma(t)x = \frac{1}{2\pi i} \int_L \frac{\Gamma(1-\sigma)}{\Gamma(1-\gamma+\gamma \sigma)} t^{-\gamma\sigma} \{ \mathcal{M} S_\beta \}(\sigma)x \, d\sigma, \quad x \in X.
\]
We are looking now for conditions under which (4.2) holds for \( 0 < \beta \leq 1 < \alpha < 2, \ \gamma = \alpha/\beta > 1 \).
Definition 4.1. A closed densely defined operator $B$ is called positive if $(-\infty, 0] \subseteq \sigma(B)$ and there is $C \geq 1$ such that
\[(1 + s)\|B^{-1}\| \leq C, \quad s \geq 0.\]

In this case the fractional powers of $B$ are defined as (see [Kr, p. 110]):
\[B^{-\sigma} = \frac{\sin \pi \sigma}{\pi} \int_0^\infty s^{-\sigma} (s + B)^{-1} ds, \quad 0 < \Re \sigma < 1. \quad (4.3)\]

In particular, if $A$ generates an exponentially decaying $C_0$ semigroup $S_1(t)$, then $-A$ is positive and (see [Kr, p. 122]):
\[(-A)^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty s^{\sigma-1} S_1(s) ds, \quad \Re \sigma > 0. \]

Next theorem generalizes this result in case $0 < \Re \sigma < 1$.

Theorem 4.1. Let $0 < \beta \leq 1$, $A \in C^\beta$ and there are constants $M \geq 1$ and $\mu > 0$ such that
\[\|S_\beta(t)\| \leq ME^{\mu t}, \quad t \geq 0. \quad (4.4)\]

Then
\[(-A)^{-\sigma} = \frac{\beta \Gamma(1 - \beta \sigma)}{\Gamma(\sigma) \Gamma(1 - \sigma)} \{MS_\beta\}(\beta \sigma), \quad 0 < \Re \sigma < 1. \quad (4.5)\]

Proof. From (4.4) it follows
\[\|\{MS_\beta\}(\beta \sigma)\| \leq M \int_0^\infty t^{\beta \Re \sigma - 1} E^{\mu t} dt, \quad 0 < \Re \sigma < 1,\]

where the integral on the right is convergent in view of the asymptotic expansion (A.4). Then $\{MS_\beta\}(\beta \sigma) \in B(X), 0 < \Re \sigma < 1$. According to (2.2)
\[R(s^\beta, A) = s^{1-\beta} \int_0^\infty e^{-st} S_\beta(t) dt\]

and thanks to (4.4) and (A.2)
\[\|R(s^\beta, A)\| \leq Ms^{1-\beta} \int_0^\infty e^{-st} E^{\mu t} dt = M / (s^\beta + \mu), \quad s > 0.\]

Therefore $-A$ is positive and formula (4.3) holds for it. Making the substitution $s = u^\beta$ in (4.3) with $B = -A$, applying (2.2) and the well-known formula $\Gamma(\sigma) \Gamma(1 - \sigma) = \pi / \sin \pi \sigma$, it follows by the Fubini’s theorem
\[(-A)^{-\sigma} = \frac{\beta}{\Gamma(\sigma) \Gamma(1 - \sigma)} \int_0^\infty u^{-\beta \sigma + \beta - 1} R(u^\beta, A) du = \]

...
\[
\frac{\beta}{\Gamma(\sigma)\Gamma(1-\sigma)} \int_0^\infty u^{-\beta \sigma} \int_0^\infty e^{-ut} S_\beta(t) \, dt \, du = \frac{\beta \Gamma(1-\beta \sigma)}{\Gamma(\sigma)\Gamma(1-\sigma)} \int_0^\infty S_\beta(t) t^{\beta \sigma-1} \, dt. \]

Substituting now (4.5) in (4.2), we obtain

\[
S_\alpha(t)x = \frac{1}{2\pi i} \int_L \frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(1-\alpha \sigma)} t^{-\alpha \sigma} (-A)^{-\sigma} x \, d\sigma \quad x \in X. \]

(4.6)

From the asymptotic expansion for the Gamma function (see [Mar, p. 49])

\[|\Gamma(c + iy)| = \sqrt{2\pi} |y|^{-1/2} \exp\left(-\frac{\pi}{2} |y|\right) (1 + O(1/y)), \quad |y| \to \infty,
\]

it follows for \(\sigma = c + iy, \quad |y| \to \infty\)

\[|\frac{\Gamma(\sigma)\Gamma(1-\sigma)}{\Gamma(1-\alpha \sigma)}| = \sqrt{2\pi} |y|^{\alpha - 1/2} \exp\left(- (2 - \alpha) \frac{\pi}{2} |y|\right) (1 + O(1/y)) \]

(4.7)

On the other hand if \(A\) satisfies the conditions of Theorem 4.1, then \(\|(-A)^{-\sigma}\| \leq \|(A)^{-\sigma}\| \leq C(\beta, c) \|(-A)^{-iy}\|\), which easily follows from (4.5) and (4.4). Therefore, for the convergence of the integral in (4.6) we need a condition on the imaginary powers of the operator \(-A\).

**Definition 4.2.** A linear operator \(B\) is said to have **bounded imaginary powers**, in symbols \(B \in \mathcal{BTP}(\theta)\), for some \(\theta \geq 0\), if \(B\) is positive, \(B^{iy} \in B(X)\) for each \(y \in \mathbb{R}\) and there is \(K \geq 1\) such that \(\|B^{iy}\| \leq Ke^{\theta|y|}, \quad y \in \mathbb{R}\).

The following lemma is proven in [Pr-S]:

**Lemma 4.1.** If \(-A \in \mathcal{BTP}(\theta)\) for some \(\theta \in [0, \pi/2)\) then \(A\) generates a **bounded analytic semigroup of angle \(\pi/2 - \theta\)**.

The main result of this section is given in the following

**Theorem 4.2.** Assume \(0 < \beta \leq 1 < \alpha < 2\) and

\[-A \in \mathcal{BTP}((2 - \alpha - \varepsilon)\pi/2) \]

(4.8)

for some \(\varepsilon \in (0, 2 - \alpha)\). Then \(A \in \mathcal{C}^\alpha\) and the representations (4.2) and (4.6) hold.

**Proof.** Lemma 4.1 implies that \(A\) generates a bounded analytic semigroup \(S_\varepsilon(t)\) of angle \((\alpha - 1 + \varepsilon)\pi/2\) and, applying Theorem 3.3., it follows that \(A \in \mathcal{C}^\alpha\). Moreover, since \(0 \in \rho(A)\), \(S_\varepsilon(t)\) is exponentially decaying: there are constants \(M \geq 1\) and \(\mu > 0\) such that \(\|S_\varepsilon(t)\| \leq Me^{-\mu t}, \quad t \geq 0\) (see [Pa, Theorem 4.3.]). Applying Corollary 3.1., it follows that \(A\) satisfies the conditions of Theorem 4.1. Therefore the representation (4.5) holds. In view of (4.7) and (4.8) the integral in (4.6) (respectively in (4.2)) is absolutely convergent. Set

\[
S(t)x = \frac{1}{2\pi i} \int_L \frac{\Gamma(1-\sigma)}{\Gamma(1-\gamma \sigma)} t^{-\gamma \sigma} \{MS_\sigma\}(\sigma)x \, d\sigma.
\]
Its Laplace transform for \( \lambda > 0 \) equals
\[
\int_0^\infty e^{-\lambda t} S(t) x \, dt = \frac{1}{2\pi i} \int_L \Gamma(1 - \sigma) \{\mathcal{L} S_\beta\}(\sigma) x \int_0^\infty e^{-\lambda \tau^{-\sigma} t} \, d\tau \, d\sigma = \\
\frac{1}{2\pi i} \int_L \Gamma(1 - \sigma) \lambda^{-\sigma-1} \{\mathcal{L} S_\beta\}(\sigma) x \, d\sigma = \int_0^\infty S_\beta(t) x \frac{1}{2\pi i} \int_L \Gamma(1 - \sigma) \lambda^{-\sigma-1} t^{-\sigma-1} \, d\sigma \, dt = \\
\lambda^{-1} \int_0^\infty e^{-\lambda^\gamma t} S_\beta(t) x \, dt = \lambda^{-1} R(\lambda^\alpha, A) x,
\]
where we apply Fubini’s theorem and the identity
\[
\frac{1}{2\pi i} \int_L \Gamma(1 - \sigma) z^{-\sigma-1} \, d\sigma = e^{-z}, \quad z > 0.
\]
Then \( S(t) = S_\alpha(t) \) follows from the uniqueness of the Laplace transform. □

In fact, (4.6) is an abstract version of a representation of the Mittag-Leffler function, given in [Mar, p.118].

**Appendix.** The Mittag-Leffler function (see [E])
\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} = \frac{1}{2\pi i} \int_C \frac{\mu^{\alpha-1} e^{\mu}}{\mu^\alpha - z} \, d\mu, \quad \alpha > 0, \, z \in \mathbb{C}, \quad (A.1)
\]
(C is a contour which starts and ends at \(-\infty\) and encircles the disc \(|\mu| \leq |z|^{1/\alpha}\) counter-clockwise) is an entire function which provides a simple generalization of the exponential function \( e^z = E_1(z) \). Similarly to the differential equation \((d/dt) e^{\omega t} = \omega e^{\omega t}\) the Mittag-Leffler function \( E_\alpha(z) \) satisfies more general differential relation
\[
D_t^\alpha E_\alpha(\omega t^\alpha) = \omega E_\alpha(\omega t^\alpha).
\]
The most important properties of the Mittag-Leffler function are associated with its Laplace integral:
\[
\int_0^\infty e^{-\lambda t} E_\alpha(\omega t^\alpha) \, dt = \frac{1}{\lambda^{\alpha-1} / (\lambda^\alpha - \omega)} \quad (A.2)
\]
and with their asymptotic expansions as \(|z| \to \infty\) when \( 0 < \alpha < 2 \):
\[
E_\alpha(z) = \frac{1}{\alpha} \exp(z^{1/\alpha}) + O(|z|^{-1}), \quad |\arg z| \leq \frac{1}{2} \alpha \pi, \quad (A.3)
\]
\[
E_\alpha(z) = O(|z|^{-1}), \quad |\arg(-z)| < (1 - \frac{1}{2} \alpha) \pi. \quad (A.4)
\]

Another entire function which appears in our study is
\[
\Phi_\gamma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\gamma n + 1 - \gamma)} = \frac{1}{2\pi i} \int_C \mu^{\gamma-1} \exp(\mu - z^\mu) \, d\mu, \quad \gamma < 1, \quad (A.5)
\]
where $\Gamma$ is a contour which starts and ends at $-\infty$ and encircles the origin once counter-clockwise (for more details see [Wt], [Ma], [M-T]). From the integral representations in (A.1) and (A.5) the identity

$$E_\gamma(z) = \int_0^\infty \Phi_\gamma(t) e^{zt} \, dt, \quad z \in \mathbb{C}, \quad 0 < \gamma < 1,$$

(A.6)

follows, that is $E_\gamma(-z)$ is the Laplace transform of $\Phi_\gamma(t)$ in the whole complex plane. Since $E_\gamma(-x)$ is completely monotonic for $x \geq 0$ and $0 < \gamma \leq 1$ (i.e. $(-1)^n(d^n/dt^n)E_\gamma(-x) \geq 0$), then from the Post–Widder inversion formula we obtain

$$\Phi_\gamma(t) \geq 0, \quad t > 0.$$  

(A.7)

Moreover, taking $z = 0$ in (A.6) it follows

$$\int_0^\infty \Phi_\gamma(t) \, dt = 1,$$  

(A.8)

i.e. $\Phi_\gamma(t)$ is a probability density function. The asymptotic expansion of $\Phi_\gamma(z)$, $0 < \gamma < 1$, as $|z| \to \infty$ in the sector $|\arg z| \leq \min\{(1 - \gamma)3\pi/2, \pi\} - \varepsilon$, is given in [Wt]:

$$\Phi_\gamma(z) = Y^{-\gamma-1/2} e^{-Y} \left( \sum_{m=0}^{M-1} A_m Y^{-m} + O(|Y|^{-M}) \right)$$  

(A.9)

with $Y = (1 - \gamma)(\gamma \gamma z)^{1/(1-\gamma)}$, where $A_m$ are certain real numbers.

**Acknowledgments.** The author is grateful to Professor J. de Graaf for the constant encouragement and helpful discussions. Thanks are also due to D.Sc. S. Yakubovich for the useful suggestions in the field of integral transforms and special functions.

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