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Toroidal flows in resistive magnetohydrodynamic steady states

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We consider the resistive steady states of a uniformly conducting magnetofluid inside a toroidal boundary. The problem becomes tractable in the limit of slow flow: i.e., low Reynolds number, which may be in turn justified when the viscous Lundquist number is small. Previous calculations are extended to apprehend the toroidal component of the necessary flow. The emerging pattern is one of helical vortices which seem likely to be ubiquitous in toroidal geometry, and which disappear in the “straight-cylinder approximation.” © 1998 American Institute of Physics. [S1070-6631(98)00906-4]

I. INTRODUCTION

The properties of a duct or channel flow of some current-carrying fluid in the presence of externally applied electric and/or magnetic fields are among the basic problems in magnetohydrodynamics.1 This type of magnetohydrodynamic flow has been studied now for several decades (see, e.g., Stieglitz et al.,2 Molokov et al.,3 Buhler and Molokov,4 and Branover et al.,5,6 and references therein). The results of these investigations are relevant to the flow of, for example, liquid metals in magnetic fields (see, e.g., Cuevas et al.,7 Sidorenkov et al.,8 Dolgikh,9 Walker and Picologlou,10 Molokov and Buhler,11 and Moon et al.12) and to magnetogasdynamics. In this paper, we calculate the mechanical motion of a viscous magnetofluid that is contained in a closed axisymmetric toroid, nonintrusively involving time-independent externally supported electric and magnetic fields. This motion is a purely geometric effect that disappears in the “straight-cylinder” approximation.

In a previous paper,13 hereafter referred to as I, the generation of toroidal vorticity in a steady-state, viscous, current-carrying magnetofluid was calculated, assuming the magnetofluid to be uniformly conducting and contained in an axisymmetric toroid with a current maintained by supplying a toroidal voltage. The toroidal vorticity resulted from the fact, not true in the “straight-cylinder” approximation, that allowable current densities J that are consistent with static magnetic fields B are of such a nature as to give rise to a nonvanishing curl of the Lorentz force, J×B. The statement applies when the electric field is demanded to be curl-free and the electrical resistivity is assumed to be spatially uniform. It is insisted that not only mechanical force-balance, but also Ohm’s law and Faraday’s law, be taken equally seriously. Since J×B has a finite curl, it cannot be balanced by the gradient of any scalar pressure. Thus the entire question of what constitutes a magnetohydrodynamic (MHD) steady state must be reconsidered. Static solutions without velocity fields and vorticity are possible, in simple one-fluid MHD, only by assuming a rather unnatural spatial dependence for the resistivity (the conductivity could not be made uniform on a magnetic surface).14

The resulting toroidal vorticity found in I was treated analytically in the limit of small velocities or low Reynolds number, and apparently the problem remains analytically intractable without this assumption. The low Reynolds number, in turn, was justifiable by the assumption of a small viscous Lundquist number, which amounts to considering the magnetofluid as highly viscous. The point of the assumption is to be able to neglect the inertial term, (v·grad)v, in the equation of motion, and to satisfy mechanical force balance by means of the viscous term alone. A secondary benefit is to be able to ignore, consistently to lowest significant order, the v×B term in Ohm’s law, where v is the fluid velocity. The motion becomes related to the electrically induced forced flows involved in metallurgical applications of magnetohydrodynamics.15 The calculation reported in I may be regarded as the first step in a perturbation procedure in which either the Reynolds number or (more basically) the viscous Lundquist number is regarded as small. High Reynolds number solutions may also exist, but so far they have proved intractable.

What is of interest in the present paper is to take this perturbation procedure a step further by returning to the Ohm’s law and correcting v and B. A significant consequence is that a toroidal velocity field results. Also, unlike the calculations in I, an externally supported toroidal magnetic field is crucial in calculating the flow field that results. The calculation in I was insensitive to whether or not a toroidal vacuum magnetic field was present.

In order to recapitulate the logical structure of the calculation in I, we summarize it as follows. A toroidal electric field is the only external agency applied to the magnetofluid, and it produces a toroidal electric current. This toroidal current is responsible for the lowest-order poloidal magnetic field. The resulting J×B Lorentz volume force on the magnetofluid creates a poloidal flow. All this is to lowest order in the Reynolds number. In the present extension, we proceed

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to higher order and show that the poloidal flow interacts with the toroidal magnetic field, if there is one, to produce a poloidal electric current. Finally, the poloidal current and the poloidal magnetic field interact to produce a toroidal flow. We note that the toroidal magnetic field is essential in this calculation, in a way that it was not in I.

In Sec. II, we revisit the perturbation procedure to make it susceptible to a higher-order development. Section III contains the algebraic details of the first-order corrections. It is possible only to go so far with an analytical calculation and, past a certain point, numerical solutions of the resulting higher-order partial differential equations are required if it is desired to determine quantitatively the nature of the toroidal flow. These numerical computations are explicitly supplied in Sec. IV, with an emphasis on extracting the toroidal part of the flow pattern. Section V presents some discussion and conclusions, and speculates on possible nonlinear helical states with flow, which while still out of reach, may be being apprehended by the first few orders of the perturbation theory.

The analytical development is rather difficult, and, to keep it manageable, we have assumed boundary conditions that are as simple as are compatible with the effect under consideration. For example, the boundary of the toroidal cross section has been assumed to be a rectangle rather than only the first of these, as is conventionally done in ideal Ohm’s law, and Faraday’s law, as given in I: $J = \nabla \times B$, $E = -\nabla \phi$, where $\phi$ is the magnetic field, $E$ is the electric field, and $J$ is the magnetic field. The boundary conditions on the conducting walls are $\mathbf{E} = 0$, $\mathbf{n} \times \mathbf{B} = 0$, and $\mathbf{v} \cdot \mathbf{n} = 0$. These conditions are satisfied on the faces of the conducting walls. The boundary conditions on the outer and inner walls of the toroidal cross section are $\mathbf{E} = 0$, $\mathbf{n} \times \mathbf{B} = 0$, and $\mathbf{v} \cdot \mathbf{n} = 0$. These conditions are satisfied on the faces of the conducting walls.

II. ZERO-ORDER REVISITED

The starting point for the next-order calculations is the dimensionless MHD equations of motion (in the familiar “Alfvénic” units) for a uniform-density, incompressible, conducting, steady-state fluid, Ohm’s law, Ampere’s law, and Faraday’s law, as given in I:

\begin{align*}
(\mathbf{E} + \mathbf{v} \times \mathbf{B}) &= \nabla \times \mathbf{B} - \nabla p + \nu \nabla^2 \mathbf{v}, \\
\mathbf{E} + \mathbf{v} \times \mathbf{B} &= \eta \mathbf{J}, \\
\nabla \times \mathbf{B} &= \mathbf{J}, \\
\nabla \times \mathbf{E} &= 0.
\end{align*}

Here, $p$ is the pressure, $\nu$ is the kinematic viscosity, and $\eta$ is the reciprocal of the electrical conductivity. In the dimensionless units used, $\nu$ is the reciprocal of the viscous Lundquist number, $M$ (expressing $M$ in laboratory units, $M = \nu^{-1} = C_a \sqrt{L} / \bar{v}$, where $C_a$ is an Alfvén speed based on the rms magnetic field, $L$ is a characteristic length scale, $\bar{v}$ is the kinematic viscosity), and $\eta$ is the reciprocal of the resistive Lundquist number $S$ ($S = \eta^{-1} = \mu_0 C_a \sqrt{\bar{v} \sigma}$, in SI units, where $\mu_0$ is the magnetic permeability of vacuum, $\sigma$ is the SI electrical conductivity).

As in I we assume that the viscous Lundquist number $M$ is small so that the inertial term in Eq. (1) is of higher order. In effect this means that $|\mathbf{v}| = O(M)$, $M \rightarrow 0$.

Replacing the fluid velocity $\mathbf{v}$ in Eqs. (1) and (2) by $M \mathbf{u}$ results in

\begin{align*}
\mathbf{J} \times \mathbf{B} &= -\nabla p + \nabla^2 \mathbf{u} = M^2 (\mathbf{u} \cdot \nabla) \mathbf{u} = 0, \\
\mathbf{J} \times \mathbf{E} &= \nabla^2 \mathbf{u}.
\end{align*}

Here, $H^2$ is the square of the Hartmann number, which is given by

\begin{equation}
H^2 = \frac{\mu_0 C_a^2 \sqrt{\bar{v}}}{\bar{v} \eta}.
\end{equation}

The steady-state toroidal vortex states as considered in I obey Eqs. (6) and (7) with terms of order $M^2$ and $H^2$ neglected, i.e.,

\begin{align*}
\mathbf{J}^{(0)} \times \mathbf{B}^{(0)} - \nabla \phi^{(0)} + \nabla^2 \mathbf{u}^{(0)} &= 0, \\
\mathbf{J}^{(0)} - \mathbf{E}^{(0)} &= 0, \\
\nabla \times \mathbf{B}^{(0)} &= \mathbf{J}^{(0)}, \\
\nabla \times \mathbf{E}^{(0)} &= 0.
\end{align*}

The geometry of the model (see Fig. 1) for which these equations have been solved numerically in I consists of an axisymmetric toroid, the axis of symmetry of which coincides with the $z$ axis in a set of cylindrical polar coordinates $(r, \phi, z)$. The midplane of the toroid is the plane $z = 0$. The boundaries of the toroidal cross section are taken to be straight lines. Doing so leads to calculational simplicity, but is not believed to be necessary for the effects we shall describe. The upper and lower boundaries are at $z = L$ and $z = -L$, respectively, and the inner and outer boundaries are at the radii $r = r_-$ and $r = r_+$, respectively. This geometry will also be adopted in the present paper. The boundary con-
dictions that have been imposed upon the solutions of the zeroth-order set of Eqs. (9)–(12) are that any tangential stress, and the normal components of \( \mathbf{u}^{(0)}, \mathbf{J}^{(0)}, \) and \( \mathbf{B}^{(0)} \) should vanish at the walls. As in the case of the planar-boundary assumption, these boundary conditions are believed not to be uniquely important ones, and are chosen mainly for calculational convenience. One may idealize such a boundary as a perfectly smooth dielectric coating on a perfect conductor.

The curl-free zeroth-order electric field is assumed to be generated externally and to be purely toroidal, i.e.,

\[
\mathbf{E}^{(0)}(r,z) = E_0 \frac{r_0}{r} \hat{\phi},
\]

where \( E_0 \) is a reference value of the electric field at radius \( r = r_0 \), and \( \hat{\phi} \) is a unit vector in the toroidal (azimuthal) direction. To the purely toroidal field in Eq. (13) might be added the gradient of an additional, more general, scalar potential. This scalar potential would obey the Laplace equation obtained from taking the divergence of the Ohm’s law, and would be determined by boundary conditions. It would be unique, and, with the (Neumann) boundary condition we impose, it may be chosen to vanish in the zeroth order, though more elaborate boundary conditions are imaginable. As mentioned before, we will also assume the existence of an externally supported, curl-free toroidal magnetic field according to

\[
\mathbf{B}^{(0)}(r,z) = B_0 \frac{r_0}{r} \hat{\phi},
\]

where \( B_0 \) is a reference value of the magnetic field at radius \( r = r_0 \). It is easily verified that this toroidal magnetic field does not affect the zeroth-order steady states as described in I. Due to the curl-free nature of \( \mathbf{B}^{(0)} \), the zeroth-order current density (which comes from Ohm’s law) is of the form

\[
\mathbf{J}^{(0)}(r,z) = SE_0 \frac{r_0}{r} \hat{\phi}.
\]

In the present paper we will correct these toroidal vortex states by using Ohm’s law to one higher order; i.e., in Eq. (7), we keep the right-hand side, which is linearly proportional to velocity, while consistently neglecting the quadratic term in Eq. (6). So far the perturbation procedure we assume that

\[
M^2 \ll H^2 \ll 1,
\]

which is tantamount to

\[
\frac{1}{\nu} \ll \eta \ll \nu.
\]

The geometry of interest in connection with Eqs. (6) and (7) and Eqs. (3) and (4) is in the present paper identical to that in I.

**III. FIRST-ORDER CALCULATION**

In what follows we will be interested mainly in those first-order components of field variables that are absent in the zeroth-order calculations of I. To the zeroth-order it has been found that the internally supported magnetic field is purely poloidal, as generated by the purely toroidal current density that is given by Eq. (15). Also the zeroth-order fluid velocity is purely poloidal (for details see I). The zeroth-order electric field in both calculations is purely toroidal [see Eq. (31)]. Hence the field components that we will evaluate hereafter are \( \mathbf{u}^{(1)} = u_\phi^{(1)} \hat{\phi}, \mathbf{B}^{(1)} = B_\phi^{(1)} \hat{\phi}, \mathbf{J}^{(1)} = J_\phi^{(1)} \hat{\phi}, \mathbf{E}^{(1)} = E_\phi^{(1)} \hat{\phi}, \) where \( \hat{\phi} \) is a unit vector in the radial direction and \( \hat{\phi} \times \hat{\phi} \) is the unit vector in the \( z \) direction. These first-order field variables have to satisfy the following equations:

\[
\begin{align*}
\mathbf{J}^{(1)} \times \mathbf{B}^{(0)} + \mathbf{J}^{(0)} \times \mathbf{B}^{(1)} - \nabla p^{(1)} + \nabla^2 \mathbf{u}^{(1)} &= 0, \\
\mathbf{J}^{(1)} + \nabla \Phi^{(1)} &= H^2 (\mathbf{u}^{(0)} \times \mathbf{B}^{(0)}), \\
\nabla \times \mathbf{B}^{(1)} &= \mathbf{J}^{(1)}.
\end{align*}
\]

Here, we replaced the first-order static electric field \( \mathbf{E}^{(1)} \) by the gradient of a first-order electric potential \( \Phi^{(1)} \) according to

\[
\mathbf{E}^{(1)} = - \frac{1}{S} \nabla \Phi^{(1)}.
\]

For later reference we recall that the zeroth-order quantities \( \mathbf{B}^{(0)} \) and \( \mathbf{u}^{(0)} \) can be expressed as

\[
\begin{align*}
\mathbf{B}^{(0)} &= B_0 \frac{r_0}{r} \hat{\phi} + \nabla \chi \times \nabla \phi, \\
\mathbf{u}^{(0)} &= \nabla \psi \times \nabla \phi.
\end{align*}
\]

Here \( \chi(r,z) \) is a magnetic flux function and \( \psi(r,z) \) is a velocity stream function.

To find an equation for the first-order poloidal current density \( \mathbf{J}^{(1)}_p \), we take the curl of the poloidal part of Eq. (19). This yields

\[
\nabla \times \mathbf{J}^{(1)}_p = H^2 \nabla \times (\mathbf{u}^{(0)} \times \mathbf{B}^{(1)}_t) = -H^2 (\mathbf{u}^{(0)} \times \nabla) \mathbf{B}^{(1)}_t.
\]

Since \( \nabla \cdot \mathbf{J}^{(1)}_p = 0 \) and we have axial symmetry, \( \mathbf{J}^{(1)}_p \) can be expressed in terms of a function \( \alpha(r,z) \) according to

\[
\mathbf{J}^{(1)}_p = \nabla \alpha \times \nabla \phi.
\]

Substituting this as well as Eq. (23) and \( \mathbf{B}^{(1)}_t = (B_0 r_0 / r) \hat{\phi} \) into Eq. (24) yields

\[
\begin{align*}
\Delta \alpha &= \frac{r}{\partial r} \left( \frac{1}{\nu} \frac{\partial \alpha}{\partial r} + \frac{\partial^2 \alpha}{\partial z^2} \right) \\
&= 2 H^2 B_0 \frac{r_0}{r} \frac{\partial \psi}{\partial z} - 2 H^2 B_0 \frac{r_0}{r} u^{(0)}_r.
\end{align*}
\]

Note that the first-order toroidal magnetic field \( \mathbf{B}^{(1)}_t = B^{(1)}_\phi \hat{\phi} \) is obtained from the function \( \alpha \) by dividing by \( r \): \( B^{(1)}_\phi = \alpha / r \). Any solution of Eq. (26) should obey the boundary condition \( \mathbf{J} \cdot \mathbf{n} = 0 \), where \( \mathbf{n} \) is the unit vector normal to the wall of the toroid.

An equation for the first-order toroidal velocity \( \mathbf{u}^{(1)} \) is obtained by considering the toroidal part of Eq. (18). This yields the following Poisson vector equation:

\[
\nabla^2 \mathbf{u}^{(1)} = \mathbf{B}^{(0)}_t \times \mathbf{J}^{(1)}_p.
\]
Replacing \( u_1^{(1)} \) by \( u_1^{(1)} = \beta \nabla \phi \) and using the poloidal part of Eqs. (22) and (25), this Poisson equation may also be rewritten as

\[
\Delta^p \beta = \frac{1}{r} \left( \frac{\partial \chi}{\partial z} \frac{\partial \alpha}{\partial r} - \frac{\partial \chi}{\partial r} \frac{\partial \alpha}{\partial z} \right). \tag{28}
\]

Note that the first-order poloidal vorticity \( \omega_p^{(1)} \) that is associated with \( u_1^{(1)} \) is given by

\[
\omega_p^{(1)} = \nabla \times \beta. \tag{29}
\]

Any solution of Eq. (28) should satisfy the stress-free boundary conditions; that is, for this problem, \( \mathbf{n} \cdot \nabla (\beta/r^2) = 0 \) at the wall of the toroid.

The first-order scalar potential \( \Phi^{(1)} \) may be obtained by taking the divergence of Eq. (19). This yields

\[
\nabla^2 \Phi^{(1)} = H^2 \nabla \cdot (u^{(0)} \times B^{(0)}) = H^2 \omega_p^{(1)} \cdot B^{(0)}. \tag{30}
\]

Unfortunately this Poisson equation has to satisfy inhomogeneous boundary conditions. Since \( J \cdot n = 0 \) at the torroid wall, it is found from Eq. (19) that

\[
\frac{\partial \Phi^{(1)}}{\partial n} = H^2 (u^{(0)} \times B^{(0)}) \cdot n. \tag{31}
\]

Using Eqs. (22) and (23) it can be easily verified that Eq. (31) can also be written as follows:

\[
\frac{\partial \Phi^{(1)}}{\partial n} = -H^2 (B^{(0)} \cdot \nabla \phi) \cdot n = -H^2 B_0 \frac{r_0}{r} \frac{\partial \phi}{\partial n}. \tag{32}
\]

We now replace \( \Phi^{(1)} \) by a function \( \gamma \) according to

\[
\Phi^{(1)} = -H^2 (B^{(0)} \cdot \nabla \phi) \psi + \gamma. \tag{33}
\]

Inserting this in Eq. (30) yields

\[
\Delta \gamma = \Delta \Phi^{(1)} + H^2 \Delta [(B^{(0)} \cdot \nabla \phi) \psi] = -2H^2 B_0 \frac{r_0}{r} \frac{\partial \phi}{\partial r} \left( \frac{\psi}{r^2} \right). \tag{34}
\]

Using Eq. (32) with \( \Phi^{(1)} \) replaced by \( \gamma \) according to Eq. (33), and the fact that \( \psi = 0 \) at the wall of the toroid (see I), it is readily shown that \( \gamma \) has to satisfy the homogeneous boundary condition

\[
\frac{\partial \gamma}{\partial n} = 0 \tag{35}
\]

at the wall of the toroid.

Before computing the relevant first-order quantities explicitly, we return to Eqs. (26) and (28). By noting that the Laplacian of a vector in the toroidal direction that only depends on poloidal coordinates is also a vector in the toroidal direction that only depends on poloidal coordinates, we can in fact convert Eqs. (26) and (28) into Poisson equations for \( B_1^{(1)} = (a/r) \hat{\epsilon}_\phi \) and \( u_1^{(1)} = (\beta/r) \hat{\epsilon}_\phi \), respectively:

\[
\nabla^2 (B_1^{(1)} \hat{\epsilon}_\phi) = 2H^2 B_0 \frac{r_0}{r} \frac{\partial \psi}{\partial r} \hat{\epsilon}_\phi. \tag{36}
\]

\[
\nabla^2 (u_1^{(1)} \hat{\epsilon}_\phi) = \frac{1}{r^2} \left( \frac{\partial \chi}{\partial z} \frac{\partial \alpha}{\partial r} - \frac{\partial \chi}{\partial r} \frac{\partial \alpha}{\partial z} \right) \hat{\epsilon}_\phi. \tag{37}
\]

One way to solve these equations is by expanding \( B_1^{(1)} \) and \( u_1^{(1)} \) in vector eigenfunctions of the Laplacian,

\[
\nabla^2 (F \hat{\epsilon}_\phi) + \lambda^2 (F \hat{\epsilon}_\phi) = 0, \tag{38}
\]

where \( F \) denotes either \( B_1^{(1)} \) or \( u_1^{(1)} \). All components of all field variables are \( \phi \)-independent, and the solution of Eq. (38) is any one of the functions

\[
F_{mn} \hat{\epsilon}_\phi = \epsilon_{mn} [J_1(\alpha_{mn} r) + D_{mn} Y_1(\alpha_{mn} r)] \left( \sin(p_n z) \right) (\cos(p_n z)^\prime) \hat{\epsilon}_\phi, \tag{39}
\]

where \( \epsilon_{mn} , \alpha_{mn} , D_{mn} \), and \( p_n \) are as yet undetermined constants, with

\[
\lambda^2 = \lambda_{mn}^2 = \alpha_{mn}^2 + p_n^2. \tag{40}
\]

Here, \( J_1 \) and \( Y_1 \) are Bessel and Weber functions, respectively. The value of \( \epsilon_{mn} \) is chosen so that the two-dimensional integral of the square of the eigenfunctions is unity.

A solution of the Poisson equation (34) for \( \gamma \) is sought in the form of an expansion in scalar eigenfunctions of the Laplacian:

\[
\nabla^2 \gamma + \mu^2 \gamma = 0. \tag{41}
\]

The solution to this equation is any of the functions

\[
\gamma_{mn} = \delta_{mn} [J_0(\beta_{mn} r) + C_{mn} Y_0(\beta_{mn} r)] \left( \sin(q_n z) \right) (\cos(q_n z)^\prime), \tag{42}
\]

where again \( \delta_{mn} , \beta_{mn} , C_{mn} \), and \( q_n \) are as yet undetermined constants, with

\[
\mu^2 = \mu_{mn}^2 = \beta_{mn}^2 + q_n^2. \tag{43}
\]

The value of the coefficient \( \delta_{mn} \) is chosen so that the eigenfunctions are normalized to unity; upon integrating their absolute square over the cross section.

### IV. Numerical Solutions

We are now going to determine the toroidal vector fields \( B_1^{(1)} \) and \( u_1^{(1)} \) and the scalar potential \( \Phi^{(1)} \). In the following pages, as in the figures to be presented, it should be kept in mind that all quantities bearing the superscript "1" are formally of second order in \( M \), even when this dependence is not explicitly exhibited.

The toroidal, first-order magnetic field \( B_1^{(1)} \) is associated with the zeroth-order poloidal fluid velocity that has been computed in I, and the externally supported (zeroth-order) toroidal magnetic field that is proportional to \( 1/r \). As in I we consider a toroid with a rectangular cross section (see Fig. 1) that is bounded by the planes at \( z = \pm L \), and by the radii \( r = r_- , r = r_+ \), where \( r_- < r < r_+ \). In the present numerical calculations, all length scales are normalized to the major radius of the toroid, i.e., \( r_0 = (r_+ + r_-)/2 \). As mentioned before, the four boundaries are assumed to be perfectly smooth walls that cannot support any tangential stress. Furthermore, these walls are idealized as impenetrable, insulator-coated, perfect conductors. Hence the normal components of the velocity, current density, and magnetic field vanish at the wall of the toroid. We have greatly idealized the complex behav-
ior of the electric field near the wall of a real device, but our idealization is necessary to introduce a toroidal electric field that is curl-free inside the magnetofluid and also axisymmetric. The assumed thin layer of insulator permits a tangential current density just inside the wall, but no normal component to the current density.

The Poisson equation (36) for $B^{(1)}_\phi$ can be solved in terms of an expansion in eigenfunctions as given by Eq. (39). For the details of this procedure we refer to Appendix A. It is found that

$$B^{(1)}_\phi(r,z) = \sum_{mn} \Lambda_{mn} \frac{\epsilon_m}{\sqrt{L}} [J_1(\alpha_m r) + D_m Y_1(\alpha_m r)] \cos \left( \frac{(2n-1)\pi z}{2L} \right),$$

(44)

where $n = 1,2,3,...$, and $\Lambda_{mn}$, $\epsilon_m$, $\alpha_m$, and $D_m$ are specified in Appendix A.

A contour plot of $B^{(1)}_\phi$ appears in Fig. 2 for $r_-/r_0 = 0.6$, $r_+/r_0 = 1.4$, and $L/r_0 = 0.3$. The first-order toroidal magnetic field vanishes at the toroidal wall which is equivalent to the vanishing of the normal component of the first-order current density at the wall of the toroid. Contours of the function $\alpha = rB^{(1)}_\phi$ are shown in Fig. 3 for the same parameters. These contours are the streamlines for the first-order poloidal current density $J^{(1)}_p$, which can be computed from $J^{(1)}_p = \nabla \times \nabla \phi$.

Before we continue determining $u^{(1)}_\phi$, we revisit the computation of the zeroth-order magnetic flux function $\chi$. For computational reasons we will now reevaluate this flux function along a different path than it has been done in I. Although slightly more complex, the final expression for the flux function $\chi$ has the same structure as the expression for the function $\alpha$, i.e., Eq. (44) multiplied by $r$. This permits the evaluation of the source term in Eq. (37) for $u^{(1)}_\phi$ in a rather compact form. The flux function $\chi$ is related to the toroidal vector potential $A^{(0)} = A^{(0)}_\phi \hat{e}_\phi$ by $A^{(0)}_\phi = \chi/r$. Here $A^{(0)}$ obeys the Poisson equation

$$\nabla^2 (A^{(0)}_\phi \hat{e}_\phi) = - \frac{E_0 r_0}{\eta r} \hat{e}_\phi,$$

(45)

which can be solved along the same lines as $B^{(1)}_\phi$ has been evaluated. Without going through the details, we merely state the final result, which again is an expansion in vector eigenfunctions of the Laplacian. The final result is

$$A^{(0)}_\phi(r,z) = \frac{\chi(r,z)}{r} = \sum_{mn} \Gamma_{mn} \frac{\epsilon_m}{\sqrt{L}} [J_1(\alpha_m r) + D_m Y_1(\alpha_m r)] \cos \left( \frac{(2n-1)\pi z}{2L} \right),$$

(46)

where the expansion coefficients $\Gamma_{mn}$ are given by

$$\Gamma_{mn} = \frac{8 E_0 r_0}{\eta} \frac{4 L^2}{\eta} \frac{\epsilon_m \sqrt{L}}{4 \alpha_m^2 L^2 + (2n-1)^2 \pi^2 / \alpha_m^2} \frac{(-1)^{n-1}}{2n-1} \times \left[ \frac{1}{r_- Y_1(\alpha_m r_-)} - \frac{1}{r_+ Y_1(\alpha_m r_+)} \right].$$

(47)

We are now ready to evaluate the right-hand side of the Poisson equation for $u^{(1)}_\phi$, i.e., Eq. (37). Since there is no toroidal pressure gradient, the toroidal, first-order fluid velocity $u^{(1)}_\phi$ is associated with the first-order $J \times B$ force, where $J = J^{(1)}_p = \nabla \alpha \times \nabla \phi$ and $B = B^{(0)}_\phi = -\nabla \chi \times \nabla \phi$. Taking the expressions for the functions $\alpha$ and $\chi$, Eq. (44) multiplied by $r$ and Eq. (46) multiplied by $r$, respectively, to the right-hand side of the Poisson equation (37) results in

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FIG. 2. Contours of the toroidal, first-order magnetic field $B^{(1)}_\phi$ for $r_-/r_0 = 0.6$, $r_+/r_0 = 1.4$, and $L/r_0 = 0.3$.

FIG. 3. Contours of the first-order current density stream function $\alpha$ for $r_-/r_0 = 0.6$, $r_+/r_0 = 1.4$, and $L/r_0 = 0.3$. 
Appendix B. There it is demonstrated that

\[ \nabla^2 (u^{(1)}_\phi e_\phi) = \sum_{mn \, n'} Y_{mm'n'} \frac{(2n - 1) \pi}{2L^2} \alpha_{mn} \epsilon_m \epsilon_{m'} [J_1(\alpha_{m'n'}) + D_m Y_1(\alpha_{m'n'}) |J_0(\alpha_{m'n'})| + D_m Y_0(\alpha_{m'n'}) \sin \left(\frac{(2n - 1) \pi z}{2L}\right) \times \cos \left(\frac{(2n - 1) \pi z}{2L}\right), \]

where

\[ Y_{mm'n'} = \Lambda_{mn} \Gamma_{m'n'} - \Lambda_{m'n} \Gamma_{mn}. \]

Note that

\[ Y_{mm'n'} = -Y_{m'n'mn} \quad \text{and} \quad Y_{mmmn} = 0. \]

For the details of solving this Poisson equation in terms of an eigenfunction expansion as given by Eq. (39) we refer to Appendix B. There it is demonstrated that

\[ u^{(1)}_\phi = \sum_{mn} \Xi_{mn} \frac{\mu_m}{\sqrt{L}} [J_1(\beta_{m'r}) + C_m Y_1(\beta_{m'r})] \sin \left(\frac{(2n - 1) \pi z}{2L}\right), \]

where \( n = 1, 2, 3, \ldots \), and \( \Xi_{mn}, \mu_m, \beta_m, \) and \( C_m \) are specified in Appendix B.

A contour plot of \( u^{(1)}_\phi \) is given in Fig. 4 for the same parameters used in Figs. 2 and 3. Positive contours are denoted by a dashed line and negative contours by a solid line. It will be seen that the first-order toroidal velocity not only changes sign when going from the upper half of the toroid to the lower half, which is plausible, but also reverses direction when crossing a boundary that is located somewhere in the middle, in between the inner and outer radii of the toroid.

This is a consequence of the change in direction at these crossings of the toroidal component of the first-order Lorentz force \( J \times B \).

Finally we consider the function \( \gamma \) that determines the scalar potential \( \Phi^{(1)} \) according to Eq. (33). Once again we employ an expansion in terms of eigenfunctions that are now as given by Eq. (42). In Appendix C it is shown that

\[ \gamma(r, z) = \sum_{mn} \Theta_{mn} \frac{\delta_m}{\sqrt{L}} [J_0(\alpha_{m'r}) + D_m Y_0(\alpha_{m'r}) \sin \left(\frac{(2n - 1) \pi z}{2L}\right)], \]

where \( n = 1, 2, 3, \ldots \), and \( \Theta_{mn} \) and \( \delta_m \) are specified in Appendix C. Using the expression for the zeroth-order velocity streamfunction \( \psi \) as given in I, we can now evaluate \( \Phi^{(1)} \) with Eq. (33). We thus arrive at

\[ \Phi^{(1)}(r, z) = -\frac{B_\theta r_0}{\eta r} \sum_{mn} \frac{\delta_m \Omega_n L^2}{\alpha_m^2 L^2 + n \pi^2} \frac{\mu_m}{\sqrt{L}} [J_1(\alpha_{m'r}) + D_m Y_1(\alpha_{m'r}) \sin \left(\frac{n \pi z}{L}\right)] + \sum_{mn} \Theta_{mn} \frac{\delta_m}{\sqrt{L}} [J_0(\alpha_{m'r}) + D_m Y_0(\alpha_{m'r}) \sin \left(\frac{(2n - 1) \pi z}{2L}\right)]. \]

A plot of equipotential lines appears in Fig. 5 for the same parameters used in Figs. 2–4. Positive potentials are denoted by dashed lines and negative potentials by solid line.
FIG. 6. Contours of the zeroth-order velocity stream function \( \psi \) for \( r_0 = 0.6, r_1 = r_0 = 1.4 \), and \( \Omega / r_0 = 0.3 \). Negative contours are denoted by a solid line and positive contours by a dashed line. The rms value of the actual (zeroth-order) poloidal speed is \( 1.24 \rho / n \pi \times 10^{-4} \) (in “Alfvenic” units).

V. DISCUSSION AND CONCLUSION

The principal results of the rather lengthy foregoing calculation are considered to be the toroidal flow patterns which have emerged as displayed in the figures. It is to be emphasized that these are toroidal velocity components that must be added vectorially to the (larger) poloidal flows found in I. For reference purposes we have reproduced in Fig. 6 the streamlines of the zeroth-order poloidal velocity as given in I. We do not specify the velocity magnitudes beyond our formal statement that they are of first and second order in the viscous Lundquist number. The situation for liquid conductors is in some cases slightly better; in I, some arguments were given that measurable velocity fields should be possible in attainable regimes. However, the qualitative features of the flow patterns shown in the figures may be expected to prevail rather generally. Thus the steady-state streamlines are topologically equivalent to helices, which circle the toroid in alternate senses, amounting to what are essentially four “convection cells” in all. These helical streamlines, it should be emphasized, are not parallel to the magnetic field \( B \). It seems not unlikely to us that there are nonlinear helical states with flow that lie still out of reach that must be considered as the true toroidal resistive steady states, and for which \( (\mathbf{v} \cdot \nabla) \mathbf{v} \) is not negligible.

A rather far-reaching revision of what is called “MHD equilibrium” would seem to be in order. It has been all too easy, in ideal MHD, to construct toroidal steady states when neither Ohm’s law nor Faraday’s law is brought into the picture. It is our viewpoint that the wealth of resulting solutions are likely no more physical or realizable than are ideal shear flows that can be written down in hydrodynamics if viscous stresses and boundary conditions are neglected.

Nor should one minimize the effects that are left out of the physical description used here, such as the tensor character of viscosity and resistivity as well as the compressibility in many laboratory magnetofluids of interest, in particular; the curvature of the toroidal boundaries; or the (here oversimplified) nature of the interaction of the wall with the magnetofluid. All these effects need to be considered, along with the eventual possible replacement of the perturbative procedure used here by a fully nonlinear treatment.

Two possible situations suggest themselves as ones in which it might be possible to illustrate the effects described here. The simplest might be in a toroid of aqueous glycerol solution of variable concentration. The viscous Lundquist number \( M \) depends, most importantly, on the ratio of electrical conductivity to viscosity, for a given toroidal electric field strength. At linear dimensions of the order of a few centimeters and magnetic field strengths of the order of a few Gauss, the viscosity of the solution can be raised by orders of magnitude at the same time that the electrical conductivity is lowered, by increasing the glycerol concentration, so that \( M \) may be tuned continuously from being \( \gg 1 \) for water with a high concentration of glycerol. Only in the latter limit would the present calculation be expected to apply, but in both limits mechanical motions would be expected to result.

The second case, and one of considerable complexity, is that of a hot plasma such as those used in tokamak confinement devices. For plasmas in current regimes, reliable estimates and measurements of viscous stress effects are in short supply. Convincing theoretical Chapman–Enskog calculations exist, but only in the short mean-free path limit, not strictly applicable to tokamak plasmas. These calculations have been carried out with and without including the effects of a strong magnetic field on the particle collisions. Only in the unmagnetized case does a Newtonian viscosity term of the type appearing in Eq. (1) result. For the strong magnetic field case, a complicated viscous stress tensor results, with different viscosity coefficients that span many orders of magnitude. If one takes the largest viscosity coefficient (the “ion parallel” viscosity) from this set, or chooses the unmagnetized result, the kinematic viscosity can be estimated as an ion mean-free path times an ion thermal speed. This is an extraordinarily large number for the current generation of tokamaks, so large that the inequality \( M \ll 1 \) would be satisfied all the time. However, we must regard the situation as an unsatisfactory one of considerable uncertainty which can be resolved only when much more reliable and detailed measurements of viscous effects in tokamaks have been made.

We should also note that we are well aware that for plasmas which are slightly nonelectrically neutral in the presence of a toroidal electric field, toroidal accelerations can result which might produce toroidal velocities in excess of the ones calculated here. To apprehend these, however, we would have to step outside the MHD framework to include charge separation, and that is beyond the scope of this article.

It should be stressed that the limitations to rectangular cross sections and stress-free viscous boundary conditions can both be removed at the price of more elaborate analytical manipulations. It is clear, following the work of Bates...
et al., 22, 23 that the same calculation is possible in toroidal coordinates for circular cross sections and for no-slip boundaries.

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APPENDIX A: COMPUTATION OF $B_{\phi}^{(1)}$

A solution for $B_{\phi}^{(1)}$ that satisfies the Poisson equation (36) is sought in the form of an expansion in terms of the eigenfunctions as given by Eq. (39). Assuming that these eigenfunctions from a complete orthonormal set, and taking into account that the right-hand side of Eq. (36) has even parity in $z$, we can write

$$B_{\phi}^{(1)} = \sum_{mn} \lambda_{mn} \frac{1}{\sqrt{L}} \phi_1(r) \cos \left( \frac{(2n-1) \pi z}{2L} \right),$$

where $1/\sqrt{L}$ is a normalization factor. The functions $\phi_1(r)$ are defined by

$$\phi_1 = \epsilon_m[J_1(\alpha_m r) + D_m Y_1(\alpha_m r)],$$

where the normalization parameters $\epsilon_m$ are real constants determined by

$$\int_{r_-}^{r_+} r \phi_1(r) \phi_1(r) \, dr = \delta_{m+m'}.$$  

Here, $\delta_{m+m'}$ is the Kronecker delta. The "'m'" in $\alpha_m$, designates the $m$th zero of the function

$$\mathcal{D} = J_1(\alpha_m r_-) Y_1(\alpha_m r_+) - J_1(\alpha_m r_+) Y_1(\alpha_m r_-),$$

which is the determinant of the set of equations that result from the boundary condition

$$J_1^{(1)}(r, z) = -\frac{1}{r} \frac{\partial \alpha}{\partial z} = -\frac{\partial B_\phi^{(1)}}{\partial z} = 0, \text{ at } r = r_+, r_-.$$  

The form of the argument of the cosine term in Eq. (A1) ensures that also the other boundary condition, that is

$$J^{(1)}(r, z) = \frac{1}{r} \frac{\partial \alpha}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} (r B_\phi^{(1)}) = 0, \text{ at } z = L, -L,$$

is satisfied by the eigenfunction expansion. The coefficients

$$D_m$$

are determined by

$$D_m = -\frac{J_1(\alpha_m r_-)}{Y_1(\alpha_m r_-)} = -\frac{J_1(\alpha_m r_+)}{Y_1(\alpha_m r_+)}.$$  

Substituting Eq. (A1) into Eq. (36) gives

$$- \sum_{mn} \lambda_{mn}^2 \Lambda_{mn} \frac{1}{\sqrt{L}} \phi_1(r) \cos \left( \frac{(2n-1) \pi z}{2L} \right)$$

$$= 2H^2 B_0 \frac{r_0}{r} \frac{\partial \psi_f}{\partial z} = 2H^2 B_0 r_0$$

$$\times \sum_{ij} \frac{\nu \Omega_{ij}}{\alpha^2_l L^2 + j^2 \pi^2} \frac{j \pi \sqrt{L}}{r^2} \cos \left( \frac{j \pi z}{L} \right),$$  

(A8)

where

$$\lambda_{mn}^2 = \alpha^2_m + \frac{(2n-1)^2 \pi^2}{4 L^2},$$  

(A9)

and $\Omega_{ij}$ are expansion coefficients for the zeroth-order toroidal vorticity $\omega_0^{(0)}$. These coefficients $\Omega_{ij}$ are detailed in I. Multiplying Eq. (A8) by $(1/\sqrt{L}) \phi_1(r) \cos((2n - 1) \pi z/(2L) \, dr \, dz$ and integrating over the range $r_+ \leq r \leq r_+$ and $-L \leq z \leq L$ determines the expansion coefficients $\Lambda_{mn}$. The result is

$$\Lambda_{mn} = \frac{8B_0 r_0}{\eta} \frac{4L}{4 \alpha_m^2 L^2 + (2n-1)^2 \pi^2}$$

$$\times \sum_{ij} (-1)^{a+j} \frac{(2n-1)j}{(2n-1)^2 - 4 j^2} \frac{\Omega_{ij} L^2}{\alpha^2_l L^2 + j^2 \pi^2}$$

$$\times \int_{r_-}^{r_+} \frac{\phi_1(r)}{r} \phi_1(r) \, dr.$$  

(A10)

APPENDIX B: COMPUTATION OF $u_\phi^{(1)}$

Along the same lines as for $B_\phi^{(1)}$ we can expand $u_\phi^{(1)}$ in vector eigenfunctions of the Laplacian according to

$$u_\phi^{(1)}(r, z) = \sum_{mn} \frac{1}{\sqrt{L}} \psi_{1m}(\beta_m r) \sin \left( \frac{(2n-1) \pi z}{2L} \right),$$

where the functions $\psi_{1m}$ are defined by

$$\psi_{1m}(r) = \mu_m[J_1(\beta_m r) + C_m Y_1(\beta_m r)].$$

In Eq. (B1) we already have imposed the stress-free boundary condition at $z = L, -L$, that is

$$\frac{\partial u_\phi^{(1)}}{\partial z} = 0, \text{ at } z = L, -L.$$  

(B3)

The stress-free boundary condition at $r = r_+, r_-$ implies

$$\frac{\partial}{\partial r} \left( \frac{u_\phi^{(1)}}{r} \right) = 0, \text{ at } r = r_+, r_-.$$  

(B4)

This equation is satisfied by requiring that

$$J_2(\beta_m r_-) + C_m Y_2(\beta_m r_-) = 0,$$

(B5)

$$J_2(\beta_m r_+) + C_m Y_2(\beta_m r_+) = 0.$$  

(B6)

These two equations can only be solved consistently if the determinant

$$\mathcal{D} = J_2(\beta_m r_-) Y_2(\beta_m r_-) - J_2(\beta_m r_+) Y_2(\beta_m r_+)$$

(B7)
vanishes. There is an infinite sequence of \( \beta_m \)-values, with each \( \beta_m \) corresponding to a particular zero of \( \beta \) for given values of \( r_+ \) and \( r_- \). The coefficients \( C_m \) are given by

\[
C_m = -\frac{J_2(\beta_m r_-)}{Y_2(\beta_m r_-)} = -\frac{J_2(\beta_m r_+)}{Y_2(\beta_m r_+)}.
\]

The \( \mu_m \) are real constants chosen to normalize the \( \psi_{1m} \):

\[
\int_{r_-}^{r_+} \psi_{1m}(r) \psi_{1m}(r) r \, dr = \delta_{m,m'}.
\]

In Table I we have given the values of \( r_0 \beta_m \), \( C_m \), and \( r_0 \mu_m \) for \( m = 1.2, \ldots, 10 \) for \( r_+/r_0 = 0.6, r_-/r_0 = 1.4 \), and \( L/r_0 = 0.3 \). Substituting expansion (B1) into the Poisson equation (48) and multiplying both sides of the resulting equation by \( 1/\sqrt{L} \psi_{1m} \sin(2n' - 1)\pi z/(2L) \) and integrating over the rectangular cross section of the toroid determines the expansion coefficients \( \Xi_{mn} \). The result is

\[
\Xi_{mn} = \frac{\sqrt{L}}{4b_0^2 L^2 + (2n - 1)^2 \pi^2} \sum_{ij} (1)^{n+j'+j'} (j - 1) \times \left( \frac{1}{n+j'-j-1/2} - \frac{1}{n+j'-j+1/2} \right) \times \left( \frac{1}{n+j'+j-3/2} - \frac{1}{n+j'+j-1/2} \right) \times Y_{ij} Y_{i'j'} \int_{r_-}^{r_+} \frac{d}{dr} \left[ \psi_{1m}(r) \right] \psi_{1m}(r) \, dr.
\]

\( \Xi_{mn} \) is a normalization factor. The functions \( \phi_{0m} \) are defined by

\[
\phi_{0m} = \delta_m J_0(\alpha_m r) + D_m Y_0(\alpha_m r),
\]

where the normalization parameters \( \delta_m \) are real constants chosen to normalize \( \phi_{0m} \):

\[
\int_{r_-}^{r_+} r \phi_{0m}(r) \phi_{0m}(r) \, dr = \delta_{m,m'}.
\]

It is straightforward to demonstrate that these eigenfunctions make \( \gamma \) satisfy the boundary condition (35). Substituting expansion (C1) into the Poisson equation (34) and multiplying both sides of the resulting equation by \( 1/\sqrt{L} \phi_{0m} \sin(2n' - 1)\pi z/(2L) \) and integrating over the range \( r_- \leq r \leq r_+ \), \( -L \leq z \leq L \) determines the expansion coefficients \( \Theta_{mn} \). The result is

\[
\Theta_{mn} = \frac{64b_0^2 r_0}{\eta} \frac{L^2}{\alpha_m^2 L^2 + (2n - 1)^2 \pi^2} \frac{1}{\pi} \times \sum_{ij} (1)^{n+j} (j - 1) \Omega_{ij} \times \frac{1}{r} \int_{r_-}^{r_+} \phi_{0m}(r) \, dr.
\]

\( \gamma \) is assumed to be expandable in the scalar eigenfunctions of the Laplacian according to

\[
\gamma(r,z) = \sum_{mn} \Theta_{mn} \frac{1}{\sqrt{L}} \phi_{0m}(r) \sin \left( \frac{(2n - 1)\pi z}{2L} \right),
\]

\( n = 1,2,3, \ldots \).

APPENDIX C: COMPUTATION OF \( \gamma \)

The function \( \gamma \) is assumed to be expandable in the scalar eigenfunctions of the Laplacian according to

\[
\gamma(r,z) = \sum_{mn} \Theta_{mn} \frac{1}{\sqrt{L}} \phi_{0m}(r) \sin \left( \frac{(2n - 1)\pi z}{2L} \right),
\]

\( n = 1,2,3, \ldots \).

\[
\phi_{0m} = \delta_m J_0(\alpha_m r) + D_m Y_0(\alpha_m r),
\]

where \( \delta_m \) are real constants chosen to normalize \( \phi_{0m} \):

\[
\int_{r_-}^{r_+} r \phi_{0m}(r) \phi_{0m}(r) \, dr = \delta_{m,m'}.
\]

\( \Theta_{mn} \) are real constants chosen to normalize \( \phi_{0m} \):

\[
\int_{r_-}^{r_+} r \phi_{0m}(r) \phi_{0m}(r) \, dr = \delta_{m,m'}.
\]

\( \Theta_{mn} \) are real constants chosen to normalize \( \phi_{0m} \):

\[
\Theta_{mn} = \frac{64b_0^2 r_0}{\eta} \frac{L^2}{\alpha_m^2 L^2 + (2n - 1)^2 \pi^2} \frac{1}{\pi} \times \sum_{ij} (1)^{n+j} (j - 1) \Omega_{ij} \times \frac{1}{r} \int_{r_-}^{r_+} \phi_{0m}(r) \, dr.
\]

\( \gamma \) is assumed to be expandable in the scalar eigenfunctions of the Laplacian according to

\[
\gamma(r,z) = \sum_{mn} \Theta_{mn} \frac{1}{\sqrt{L}} \phi_{0m}(r) \sin \left( \frac{(2n - 1)\pi z}{2L} \right),
\]

\( n = 1,2,3, \ldots \).

\[
\phi_{0m} = \delta_m J_0(\alpha_m r) + D_m Y_0(\alpha_m r),
\]


