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Some Topological Aspects of the Convolution Algebra $E'(\mathbb{R})$ and its representing space of Fourier Transforms

by

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Some topological aspects of the convolution algebra $\mathcal{E}'(\mathbb{R})$ and its representing space of Fourier transforms

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Abstract

By describing the Fréchet topology on the space $\mathcal{E}(\mathbb{R})$ of all infinitely continuously differentiable functions as a projective limit of Banach spaces, its topological dual, the space $\mathcal{E}'(\mathbb{R})$ of all Schwartz distributions of compact support, is naturally equipped with an inductive limit topology of Banach spaces. Algebraically, $\mathcal{E}'(\mathbb{R})$ is isomorphic to an operator algebra of shift-invariant operators on $\mathcal{E}(\mathbb{R})$, and to the Paley-Wiener algebra $PW(\mathbb{C})$ of holomorphic functions of exponential type that are polynomially bounded on the real axis. In this paper, we study the topological relationship between $\mathcal{E}'(\mathbb{R})$ and $PW(\mathbb{C})$. The function space $PW(\mathbb{C})$ is equipped with a natural inductive limit topology, and the properties of this space are investigated. It is shown that the Fourier transformation from $\mathcal{E}'(\mathbb{R})$ to $PW(\mathbb{C})$ is a homeomorphism: it preserves both the algebraic and topological characteristics of its domain $\mathcal{E}'(\mathbb{R})$ in its codomain $PW(\mathbb{C})$.

1 The Heine-Borel property of the space $\mathcal{E}(\mathbb{R})$

Let $C(\mathbb{R})$ denote the vector space of all continuous functions from $\mathbb{R}$ into $\mathbb{C}$. The space $C(\mathbb{R})$, endowed with the seminorms

$$ q_n(f) = \max_{t \in [-n,n]} |f(t)|, $$

is a Fréchet space (see e.g. [7]).

Definition 1.1 A subset $S$ in $C(\mathbb{R})$ is said to be locally equicontinuous if for every compact subset $K \subset \mathbb{R}$, the set

$$ S|_K = \{ f|_K \mid f \in S \} $$

is equicontinuous in the Banach space $C(K)$ of continuous functions on $K$.

The theorem of Arzela-Ascoli states that if $K \subset \mathbb{R}$ is compact, a subset $S \subset C(K)$ is pre-compact if and only if it is bounded and equicontinuous. There is the following generalization.

Theorem 1.2 Let $S$ be a bounded, locally equicontinuous subset of the Fréchet space $C(\mathbb{R})$. Then $S$ is pre-compact.

Proof: Let $(f_j)$ denote a sequence in $S$. Then there are subsequences $(f_{n,j})$, $n \in \mathbb{N}$, such that
(i) \((f_{n+1,j})\) is a subsequence of \((f_{n,j})\),

(ii) \((f_{n,j})\) is convergent in \(C([-n,n])\).

It follows that for all \(n \in \mathbb{N}\) and all \(x \in [-n,n]\\):

\[
\lim_{j \to \infty} f_{n,j}(x) = \lim_{j \to \infty} f_{n+1,j}(x).
\]

Let \((g_j)\) denote the diagonal sequence \(g_j = f_{j,j}\\). Then for all \(n \in \mathbb{N}, (g_j)_{j \geq n} \) is a subsequence of \((f_{n,j})_{j \geq n}\\) and \((g_j)\) is a convergent subsequence in \(C(\mathbb{R})\) of the sequence \((f_j)\). 

Now for all \(k \in \mathbb{N}_0, \) with \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\\), let \(C^k(\mathbb{R})\) denote the vector space of all \(k\)-times continuously differentiable functions on \(\mathbb{R}\). Then \(C^k(\mathbb{R})\), endowed with the seminorms

\[
q_{k,n}(f) = \max_{t \in [-n,n]} \sum_{j=0}^{k} |f^{(j)}(t)|, \quad (n \in \mathbb{N}_0),
\]

is a Fréchet space. Obviously, \(C^{k+1}(\mathbb{R})\) is contained in \(C^k(\mathbb{R})\), and the canonical injection is continuous. By Theorem 1.2, this injection is compact:

**Lemma 1.3** Let \(S\) be a bounded subset in \(C^{k+1}(\mathbb{R})\). Then \(S\) is a pre-compact subset of \(C^k(\mathbb{R})\).

**Proof:** First we consider the case \(k = 0\). So let \(S\) be bounded in \(C^1(\mathbb{R})\). Then for \(f \in S\) and \(s, t \in [-n,n],\\)

\[
|f(t) - f(s)| = \left| \int_s^t f'(\tau) \, d\tau \right| \leq |t - s| \cdot \sup_{g \in S} q_{1,n}(g).
\]

So \(S\) is a locally equicontinuous, bounded subset of \(C(\mathbb{R})\), and therefore pre-compact.

Assume the assertion is true for all \(k\) with \(k < k_0\). Let \(S\) be a bounded subset of \(C^{k_0+1}(\mathbb{R})\). Then \(S' = \{f' \mid f \in S\}\) is bounded in \(C^{k_0}(\mathbb{R})\), and therefore compact in \(C^{k_0-1}(\mathbb{R})\). Let \((f_j)\) be a sequence in \(S\). Then the conclusion follows from the fact that

\[
f_j(t) = f_j(0) + \int_0^t f_j'(\tau) \, d\tau, \quad (t \in \mathbb{R}),
\]

where \((f_j(0))\) is a bounded sequence. 

Let \(\mathcal{E}(\mathbb{R})\) be the vector space of all infinitely continuously differentiable functions from \(\mathbb{R}\) into \(\mathbb{C}\), endowed with the seminorms \(q_{k,n}, (k, n \in \mathbb{N}_0)\). \(\mathcal{E}(\mathbb{R})\) is a Fréchet space. Since the collection of Fréchet spaces \(\{C^k(\mathbb{R}) \mid k \in \mathbb{N}_0\}\) is a projective system, with each canonical injection \(i_k : C^{k+1}(\mathbb{R}) \to C^k(\mathbb{R})\) continuous and compact, \(\mathcal{E}(\mathbb{R}) = \bigcap_{k=0}^{\infty} C^k(\mathbb{R})\), and its topology may be considered as the corresponding projective limit topology. Combining the results of this section, we obtain:

**Theorem 1.4** The Fréchet space \(\mathcal{E}(\mathbb{R})\) is a Montel space, in the sense that it satisfies the Heine-Borel property: every closed and bounded subset of \(\mathcal{E}(\mathbb{R})\) is compact.
2 Topologies on $\mathcal{E}(\mathbb{R})$ and $\mathcal{E}'(\mathbb{R})$

The topological dual $\mathcal{E}'(\mathbb{R})$ of $\mathcal{E}(\mathbb{R})$ is identified mostly as the space of Schwartz distributions of compact support. A natural topology for $\mathcal{E}'(\mathbb{R})$ is the weak-star-topology $\sigma(\mathcal{E}'(\mathbb{R}), \mathcal{E}(\mathbb{R}))$ (see e.g. [8, Chapter IV]). But for the topological considerations we have in mind, this topology is not appropriate. Therefore we present a collection of seminorms on $\mathcal{E}(\mathbb{R})$, that leads to a convenient description of $\mathcal{E}'(\mathbb{R})$, and a useful topology for this space.

Recalling (2), we define the seminorms $p_n$ on $\mathcal{E}(\mathbb{R})$ by

$$p_n = q_{n,n}, \quad (n \in \mathbb{N}_0).$$

Then the projective limit topology of $\mathcal{E}(\mathbb{R})$, as defined in Section 1, is brought about by these seminorms; observe that for all $n$ and $k$:

$$q_{k,n}(f) \leq p_{n}(f), \quad (f \in \mathcal{E}(\mathbb{R})), \quad (n \in \mathbb{N}_0),$$

with $m = \max(n,k)$. Consider for $n \in \mathbb{N}_0$ the quotient spaces $X_n = \mathcal{E}(\mathbb{R})/p_n^\perp \{0\}$. Then $X_n$ is a normed space with the quotient norm $\bar{p}_n$. Since $p_{n+1} \geq p_n$, the Banach completions $\bar{X}_n$ establish a projective system, and $\mathcal{E}(\mathbb{R})$ can be considered the projective limit of these Banach spaces. The canonical completion of $X_n$ is the Banach space $C^n([-n, n])$, consisting of all $n$-times continuously differentiable functions $f$ on $[-n, n]$ such that $f^{(n)} \in C([-n, n])$.

It follows that for $n \in \mathbb{N}_0$ the space

$$\mathcal{E}'(\mathbb{R}; n) := \{ G \in \mathcal{E}'(\mathbb{R}) \mid \exists C > 0 \ \forall f \in \mathcal{E}(\mathbb{R}) : |G(f)| \leq Cp_n(f) \}$$

is a Banach space with norm

$$\|G\|'_n = \sup\{|G(f)| \mid f \in \mathcal{E}(\mathbb{R}), p_n(f) \leq 1\}.$$  

(4)

Since $p_{n+1} \geq p_n$ for all $n \in \mathbb{N}_0$, we see that

$$\mathcal{E}'(\mathbb{R}; n) \subset \mathcal{E}'(\mathbb{R}; n+1),$$

with

$$\|G\|'_{n+1} \leq \|G\|'_n, \quad (G \in \mathcal{E}'(\mathbb{R}; n)).$$

(5)

So, the collection $(\mathcal{E}'(\mathbb{R}; n))_{n \in \mathbb{N}_0}$ is an inductive system of Banach spaces. We observe that the system is not strict. Since

$$\mathcal{E}'(\mathbb{R}) = \bigcup_{n=0}^{\infty} \mathcal{E}'(\mathbb{R}; n),$$

we endow $\mathcal{E}'(\mathbb{R})$ with the corresponding inductive limit topology. Thus $\mathcal{E}'(\mathbb{R})$ becomes an (LB)-space. It is the intention of this paper to prove some properties of the present inductive limit description of $\mathcal{E}'(\mathbb{R})$. Note that this inductive limit topology on $\mathcal{E}'(\mathbb{R})$ is stronger than the weak-star-topology $\sigma(\mathcal{E}'(\mathbb{R}), \mathcal{E}(\mathbb{R}))$.

Remark 2.1 If $\mathcal{E}(\mathbb{R})$ is considered as a projective limit of Fréchet spaces, as in Section 1, its dual space $\mathcal{E}'(\mathbb{R})$ may be characterized as an inductive limit of Fréchet spaces, endowed with a (non-strict) inductive limit topology. Although this approach leads to an interesting classification of the elements of $\mathcal{E}'(\mathbb{R})$, it is inconvenient for our topological considerations.
3 Linear translation invariant operators on $\mathcal{E}(\mathbb{R})$ and convolution

Let $(\sigma_t)_{t \in \mathbb{R}}$ denote the translation group on $\mathcal{E}(\mathbb{R})$, where

$$ (\sigma_t f)(\tau) = f(t + \tau), \quad t \in \mathbb{R}, \tau \in \mathbb{R}, f \in \mathcal{E}(\mathbb{R}). $$

Then $(\sigma_t)_{t \in \mathbb{R}}$ is a one parameter $c_\infty$-group on $\mathcal{E}(\mathbb{R})$, in the sense that for all $f \in \mathcal{E}(\mathbb{R})$, the $\mathcal{E}(\mathbb{R})$-valued function

$$ t \mapsto \sigma_t f, \quad t \in \mathbb{R}, $$

is infinitely differentiable as function from $\mathbb{R}$ to $\mathcal{E}(\mathbb{R})$, with

$$ \left( \frac{d}{dt} \right)^k (\sigma_t f) = \left\{ \sigma_t \frac{d^k f}{dt^k} \right\}. $$

In particular, $(\sigma_t)_{t \in \mathbb{R}}$ is a one parameter $c_0$-group on $\mathcal{E}(\mathbb{R})$ with everywhere defined infinitesimal generator $\frac{d}{dt}$.

It follows that for all $G \in \mathcal{E}'(\mathbb{R})$ and $f \in \mathcal{E}(\mathbb{R})$, the function $\sigma[G]f$ defined by

$$ (\sigma[G]f)(t) = G(\sigma_t f) $$

belongs to $\mathcal{E}(\mathbb{R})$, with $\left( \frac{d}{dt} \right)^k (\sigma[G]f) = \sigma[G] \frac{d^k f}{dt^k}$. The linear operator $\sigma[G] : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ is continuous, which can be proved, using the Closed-Graph-Theorem for Fréchet spaces (see e.g. [8, p. 78]). Clearly, for all $G \in \mathcal{E}'(\mathbb{R})$,

$$ \sigma_0 \sigma[G] = \sigma[G] \sigma_0, \quad t \in \mathbb{R}, $$

so $\sigma[G]$ is a translation invariant operator. If on the other hand $L : \mathcal{E}(\mathbb{R}) \to \mathcal{E}(\mathbb{R})$ is a continuous linear operator, satisfying $L \sigma_t = \sigma_t L$ for all $t \in \mathbb{R}$, then $L = \sigma[G]$, with $G \in \mathcal{E}'(\mathbb{R})$ defined by $G(f) = (Lf)(0)$.

**Theorem 3.1** Let $B_\sigma(\mathcal{E}(\mathbb{R}))$ denote the algebra of all continuous translation invariant linear operators from $\mathcal{E}(\mathbb{R})$ into $\mathcal{E}(\mathbb{R})$. Then $\sigma : \mathcal{E}'(\mathbb{R}) \to B_\sigma(\mathcal{E}(\mathbb{R}))$ is an algebra isomorphism, where the product $G_1 * G_2$ of $G_1, G_2$ in the vector space $\mathcal{E}'(\mathbb{R})$ is defined by

$$ \sigma(G_1 * G_2) = \sigma[G_1] \sigma[G_2]. $$

The product $\ast$ is the classical convolution product in $\mathcal{E}'(\mathbb{R})$, as introduced by Schwartz (see e.g. [9, Chapter VI]). Since this convolution product $\ast$ is known to be commutative, $B_\sigma(\mathcal{E}(\mathbb{R}))$ is a commutative algebra too. In this paper, we shall show that the convolution algebra $\mathcal{E}'(\mathbb{R})$ is algebra-homeomorphic to a topological algebra of holomorphic functions with (of course) an (LB)-structure.

4 Fourier transformation and the Paley-Wiener-Theorem

Let $e_z \in \mathcal{E}(\mathbb{R})$, with $z \in \mathbb{C}$, denote the exponential function

$$ e_z(t) = e^{-itz}, \quad t \in \mathbb{R}. $$

(8)
Then $z \mapsto e_z$ is an $\mathcal{E}(\mathbb{R})$-valued holomorphic function on $\mathbb{C}$. Indeed, for each $z_0 \in \mathbb{C}$,

$$e_z = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}e_{z_0,n},$$

where $e_{z_0,n}(t) = (-it)^ne^{-iz_0t}$, with convergence in $\mathcal{E}(\mathbb{R})$. So for each $G \in \mathcal{E}'(\mathbb{R})$, the Fourier transform $\mathcal{F}G$ defined by

$$\mathcal{F}G : \mathbb{C} \rightarrow \mathbb{C} : \mathcal{F}G(z) = G(e_z) \tag{9}$$

is a holomorphic function on $\mathbb{C}$. The mapping $G \mapsto \mathcal{F}G$ is called the Fourier transformation on $\mathcal{E}'(\mathbb{R})$, and is denoted by $\mathcal{F}$. The Paley-Wiener-Theorem characterizes the Fourier transforms $\mathcal{F}G$ for $G \in \mathcal{E}'(\mathbb{R})$ (see e.g. [9, p. 271] or [3, p. 156]).

**Definition 4.1** The **Paley-Wiener class** $PW(\mathbb{C})$ is the vector space of all holomorphic functions $\phi$ on $\mathbb{C}$, with the property that

$$\exists C > 0 \ \exists N \in \mathbb{N} \ \exists a > 0 \ \forall z \in \mathbb{C} : |\phi(z)| \leq C(1 + |z|)^Ne^{|\text{Im} \ z|}. \tag{10}$$

**Theorem 4.2** (Paley-Wiener) The Fourier transformation $\mathcal{F}$ defined on $\mathcal{E}'(\mathbb{R})$, is a vector space isomorphism from $\mathcal{E}'(\mathbb{R})$ onto $PW(\mathbb{C})$. So $\mathcal{F}$ is linear and

(i) $\forall G \in \mathcal{E}'(\mathbb{R}) : \mathcal{F}G \in PW(\mathbb{C}),$

(ii) $\forall \phi \in PW(\mathbb{C}) \ \exists H \in \mathcal{E}'(\mathbb{R}) : \mathcal{F}H = \phi.$ \hfill \blacksquare

The definition of $PW(\mathbb{C})$ implies that it is a subalgebra of the algebra $H(\mathbb{C})$ of all holomorphic functions on $\mathbb{C}$. Since for $G \in \mathcal{E}'(\mathbb{R})$ and $z \in \mathbb{C}$: $\sigma[G]e_z = \mathcal{F}G(z)e_z$, it follows that for $G_1, G_2 \in \mathcal{E}'(\mathbb{R})$:

$$\mathcal{F}(G_1 \ast G_2) = (\mathcal{F}G_1)(\mathcal{F}G_2).$$

Hence $\mathcal{F}$ is an algebra isomorphism from the commutative convolution algebra $\mathcal{E}'(\mathbb{R})$, onto the function algebra $PW(\mathbb{C})$.

So we presented three mutually isomorphic commutative algebras, namely the operator algebra $B_\sigma(\mathcal{E}(\mathbb{R}))$, the convolution algebra $\mathcal{E}'(\mathbb{R})$, and the function algebra $PW(\mathbb{C})$. Further, we presented an (LB)-topology for $\mathcal{E}'(\mathbb{R})$. In the next section we shall endow $PW(\mathbb{C})$ with an (LB)-topology, so that the Fourier transformation is a homeomorphism.

## 5 An inductive limit topology on $PW(\mathbb{C})$

We recall that $H(\mathbb{C})$, the vector space of all holomorphic functions, is a Fréchet space for the seminorms

$$s_k(\phi) = \sup\{|\phi(z)| \ | \ z \in \mathbb{C}, |z| \leq k\}. \tag{11}$$

These seminorms describe uniform convergence on compact subsets of $\mathbb{C}$.

For each $n \in \mathbb{N}_0$, let $PW(\mathbb{C}; n)$ denote the subspace of $PW(\mathbb{C})$, consisting of all $\phi \in PW(\mathbb{C})$ such that

$$\exists C_\phi > 0 \ \forall z \in \mathbb{C} : |\phi(z)| \leq C_\phi(A(z))^n, \tag{12}$$

such that
where $A$ is the function on $\mathbb{C}$ with $A(z) = (1 + |z|) \cdot \exp(|\text{Im } z|)$. Then $PW(\mathbb{C}; n)$ is a Banach space of holomorphic functions for the norm

$$\|\phi\|_{PW,n} = \sup_{z \in \mathbb{C}} (A(z)^{-n}|\phi(z)|). \quad (13)$$

We see that

$$PW(\mathbb{C}) = \bigcup_{n=0}^{\infty} PW(\mathbb{C}; n),$$

and the sequence of Banach spaces $(PW(\mathbb{C}; n))_{n \in \mathbb{N}_0}$ is a (non-strict) inductive system, i.e.

1. $PW(\mathbb{C}; n) \subset PW(\mathbb{C}; n + 1)$,
2. $\forall \phi \in PW(\mathbb{C}; n) : \|\phi\|_{PW,n+1} \leq \|\phi\|_{PW,n}$.

We endow $PW(\mathbb{C})$ with the corresponding inductive limit. Then the canonical injection $PW(\mathbb{C}) \hookrightarrow H(\mathbb{C})$ is continuous because for each $n \in \mathbb{N}_0$, the canonical injection $PW(\mathbb{C}; n) \hookrightarrow H(\mathbb{C})$ is continuous. Indeed, if $k > 0$ and $\varepsilon > 0$, then for every $\phi \in PW(\mathbb{C}; n)$, satisfying $\|\phi\|_{PW,n} < \frac{((1 + k)e^\varepsilon)^n}{(1 + k)}$, we have

$$s_k(\phi) = \sup \left\{ \frac{|\phi(z)|}{A(z)^n} A(z)^n \mid z \in \mathbb{C}, |z| < k \right\} \leq \|\phi\|_{PW,n}((1 + k)e^\varepsilon)^n < \varepsilon,$$

i.e. the seminorm $s_k$ is continuous on $PW(\mathbb{C}; n)$. It follows that the (LB)-space $PW(\mathbb{C})$ is a Hausdorff topological space. We shall prove that the (LB)-space $PW(\mathbb{C})$ exhibits two important properties:

(I) $PW(\mathbb{C})$ is regular, i.e. every bounded set in $PW(\mathbb{C})$ is bounded in some $PW(\mathbb{C}; n)$,

(II) $PW(\mathbb{C})$ is sequentially retractive, i.e. every convergent sequence in $PW(\mathbb{C})$ is contained in $PW(\mathbb{C}; n)$ for certain $n \in \mathbb{N}_0$, and converges with respect to the topology on $PW(\mathbb{C}; n)$.

For this terminology, see also [4].

6 A result on positive double sequences

In the proof of properties (I) and (II), we need a mechanism to keep track of the exponential growth of a $PW(\mathbb{C})$-function in the direction of the imaginary axis, and of its polynomial growth along the real axis. For this purpose we use a result from [2] (see also [1], [6]) on positive 2-D sequences. Let $\omega_+(\mathbb{N}_0^2)$ denote the collection of all positive functions from $\mathbb{N}_0^2$ into $\mathbb{R}^+$, partially ordered by the usual pointwise ordering. All operations such as scalar multiplication, addition, and multiplication are taken pointwise. Let $a_0 \in \omega_+(\mathbb{N}_0^2)$ be defined by

$$a_0(k, \ell) = (1 + \sqrt{k^2 + \ell^2})e^\ell, \quad (k, \ell) \in \mathbb{N}_0^2, \quad (14)$$

and $\rho \subset \omega_+(\mathbb{N}_0^2)$ by the countable collection

$$\rho = \{ a_0^n \mid n \in \mathbb{N}_0 \}. \quad (15)$$

Then it is obvious that $a_0(k, \ell) \geq 1$ for all $(k, \ell) \in \mathbb{N}_0^2$, and $a_0^0 \leq a_0^1 \leq a_0^2 \leq a_0^3 \leq \cdots$ with respect to the partial ordering on $\omega_+(\mathbb{N}_0^2)$.

The next lemma is a restricted version of a far more general result proved in [2].
Lemma 6.1 ([1], [2], [6]) Define

\[ \rho^\# = \{ b \in \omega_+(N_0^2) \mid \forall n \in N_0 : \sup_{(k,\ell) \in N_0^2} (a_0(k,\ell)b(k,\ell)) < \infty \}, \] (16)

and

\[ \rho^{\#\#} = \{ a \in \omega_+(N_0^2) \mid \forall b \in \rho^\# : \sup_{(k,\ell) \in N_0^2} (a(k,\ell)b(k,\ell)) < \infty \}. \] (17)

Then

\[ \forall a \in \rho^{\#\#} \exists n \in N_0 \exists \gamma > 0 : a \leq \gamma a_0^n. \] (18)

Proof: (by contradiction)
Assume that (18) does not hold, and let \( a \in \rho^{\#\#} \) be such that for all \( n \in N_0 \) and \( \gamma > 0 \): \( -(a \leq \gamma a_0^n) \). For \( n \in N_0 \) we define the index set \( I_n \) by

\[ I_n = \{(k, \ell) \in N_0^2 \mid a(k, \ell) > n \cdot a_0(k, \ell)^n\}. \]

Then \( I_n \neq \emptyset \), \( I_{n+1} \subset I_n \), and since \( a_0(k, \ell) \geq 1 \) for all \( (k, \ell) \in N_0^2 \), also \( \cap_{n \in N_0} I_n = \emptyset \). This implies that every index set \( I_n \) is infinite.

Let \( (k_n, \ell_n)_{n \in N_0} \) be a sequence in \( N_0^2 \) such that \( (k_n, \ell_n) \in I_n \), and \( (k_n, \ell_n) \neq (k_n', \ell_n') \) for \( n \neq n' \). Define \( c \in \omega_+(N_0^2) \) by

\[ c(k, \ell) = \begin{cases} 0 & \text{if } (k, \ell) \notin \{(k_n, \ell_n) \mid n \in N_0\}, \\ \frac{1}{a_0(k, \ell)^n} & \text{if } (k, \ell) = (k_n, \ell_n). \end{cases} \]

Then \( c \in \rho^\# \). Indeed, if \( n_0 \in N_0 \) is fixed, then for all \( n > n_0 \):

\[ a_0(k_n, \ell_n)^{n_0}c(k_n, \ell_n) = a_0(k_n, \ell_n)^{n_0} \frac{1}{a_0(k_n, \ell_n)^n} \leq 1, \]

and thus

\[ \sup_{(k, \ell) \in N_0^2} (a_0(k, \ell)^{n_0}c(k, \ell)) < \infty. \]

On the other hand,

\[ a(k_n, \ell_n)c(k_n, \ell_n) > n \cdot a_0(k_n, \ell_n)^n \frac{1}{a_0(k_n, \ell_n)^n} = n, \]

and therefore \( a \notin \rho^{\#\#} \). This yields a contradiction. \( \blacksquare \)

7 Some properties of the (LB)-topology on \( PW(C) \)

In this section, we want to use Lemma 6.1 for our study of the topological properties of the (LB)-space \( PW(C) \). For this purpose, we define for \( (k, \ell) \in N_0^2 \) the subset \( V(k, \ell) \) of the complex plane by

\[ V(k, \ell) = \{ z \in C \mid k \leq |\text{Re } z| < k + 1, \ell \leq |\text{Im } z| < \ell + 1 \} \] (19)

(see Figure 1). It is obvious that \( C = \cup_{(k, \ell) \in N_0^2} V(k, \ell) \). Due to the structure of \( V(k, \ell) \), an upper
and lower bound of the function $A(z) = (1 + |z|) \exp(|\text{Im } z|)$ on $V(k, \ell)$ is easily obtained, using the definition of the double sequence $a_0$ in (14):

$$\forall z \in V(k, \ell) : a_0(k, \ell) \leq |A(z)| \leq (1 + \sqrt{2})e \cdot a_0(k, \ell).$$

(20)

In order to study the topologies on $PW(\mathbb{C})$ and $H(\mathbb{C})$, we define for every $(k, \ell) \in \mathbb{N}_0^2$ the seminorm $s(k, \ell)$ on $H(\mathbb{C})$ by

$$s(k, \ell)(\psi) := \sup\{|\psi(z)| \mid z \in V(k, \ell)\}, \quad \psi \in H(\mathbb{C}).$$

(21)

Let $n \in \mathbb{N}_0$, and $\phi \in PW(\mathbb{C}; n)$. Then

$$\|\phi\|_{PW,n} = \sup_{z \in \mathbb{C}} (A(z)^{-n}|\phi(z)|) \leq \sup_{(k, \ell) \in \mathbb{N}_0^2} (a_0(k, \ell)^{-n}s(k, \ell)(\phi)),$$

and also

$$\|\phi\|_{PW,n} = \sup_{z \in \mathbb{C}} (A(z)^{-n}|\phi(z)|) \geq \left(\frac{1}{(1 + \sqrt{2})e}\right)^n \sup_{(k, \ell) \in \mathbb{N}_0^2} (a_0(k, \ell)^{-n}s(k, \ell)(\phi)).$$

So, if we define for $\phi \in PW(\mathbb{C}; n)$ and $n \in \mathbb{N}_0$:

$$\|\phi\|_{\widetilde{PW},n} := \sup_{(k, \ell) \in \mathbb{N}_0^2} (a_0(k, \ell)^{-n}s(k, \ell)(\phi)),$$

(22)

then the norms $\|\cdot\|_{\widetilde{PW},n}$ and $\|\cdot\|_{PW,n}$ on $PW(\mathbb{C}; n)$ are equivalent:

$$\forall \phi \in PW(\mathbb{C}; n) : \left(\frac{1}{(1 + \sqrt{2})e}\right)^n \|\phi\|_{\widetilde{PW},n} \leq \|\phi\|_{PW,n} \leq \|\phi\|_{\widetilde{PW},n}.$$ 

Lemma 7.1 Let $b \in \rho^\#$, as defined in (16), and define $q_b$ on $PW(\mathbb{C})$ by

$$q_b(\phi) = \sup_{(k, \ell) \in \mathbb{N}_0^2} (b(k, \ell)s(k, \ell)(\phi)), \quad \phi \in PW(\mathbb{C}).$$

(23)

Then $q_b$ is a continuous seminorm on $PW(\mathbb{C})$. 

Figure 1: The set $V(k, \ell)$
Proof: We have to prove that $q_b |_{PW(C,n)}$ is continuous on the Banach space $PW(C;n)$ for each $n \in \mathbb{N}_0$. Now indeed for $\phi \in PW(C;n)$:

$$q_b(\phi) = \sup_{(k,\ell) \in \mathbb{N}_0^2} (b(k,\ell)s_{(k,\ell)}(\phi)) \leq \sup_{(k,\ell) \in \mathbb{N}_0^2} (a_0(k,\ell)^n b(k,\ell)) \cdot \sup_{(k,\ell) \in \mathbb{N}_0^2} (a_0(k,\ell)^{-n}s_{(k,\ell)}(\phi)) = \left( \sup_{(k,\ell) \in \mathbb{N}_0^2} (a_0(k,\ell)^n b(k,\ell)) \right) \cdot \|\phi\|_{PW,n}.$$\hfill\Box

Using the seminorms $q_b$ with $b \in \rho^#$, one may translate the growth conditions on a $PW(C)$-function $\phi$ (cf. (10)), to a boundedness condition on the corresponding double sequence $s_{(k,\ell)}(\phi)$.

Lemma 7.2 Let $\phi \in H(C)$. Then $\phi \in PW(C)$ if and only if for all $b \in \rho^#$

$$\sup_{(k,\ell) \in \mathbb{N}_0^2} (b(k,\ell)s_{(k,\ell)}(\phi)) < \infty.$$ \hfill (24)

Proof: We only need to prove necessity. So, let $\phi \in H(C)$ satisfy

$$\sup_{(k,\ell) \in \mathbb{N}_0^2} (b(k,\ell)s_{(k,\ell)}(\phi)) < \infty,$$

for all $b \in \rho^#$. Then $s_{(k,\ell)}(\phi) \in \rho^{##}$, and according to Lemma 6.1, there exist $n \in \mathbb{N}_0$ and $\gamma > 0$ such that

$$s_{(k,\ell)}(\phi) \leq \gamma \cdot a_0(k,\ell)^n, \quad ((k,\ell) \in \mathbb{N}_0^2).$$

Hence

$$\sup_{z \in C} (A(z)^{-n}|\phi(z)|) \leq \sup_{(k,\ell) \in \mathbb{N}_0^2} (a_0(k,\ell)^{-n}s_{(k,\ell)}(\phi)) \leq \gamma,$$

and $\phi \in PW(C;n)$ with $\|\phi\|_{PW,n} \leq \gamma$.\hfill\Box

In the next theorem we use the seminorms $\{q_b \mid b \in \rho^#\}$ to characterize the bounded subsets of $PW(C)$, thus proving that $PW(C)$ is a regular (LB)-space.

Theorem 7.3 Let $B \subset H(C)$. Then the following statements are equivalent:

(i) $B \subset PW(C)$ is bounded,

(ii) For every $b \in \rho^#$:

$$\sup_{\phi \in B} \left( \sup_{(k,\ell) \in \mathbb{N}_0^2} (b(k,\ell)s_{(k,\ell)}(\phi)) \right) < \infty,$$

(iii) $B$ is a bounded subset of $PW(C;n)$ for some $n \in \mathbb{N}_0$. 

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Proof: (i) $\implies$ (ii): According to Lemma 7.1, the seminorms $q_b$, with $b \in \rho^\#$, are continuous on $PW(C)$.

(ii) $\implies$ (iii): Combining (ii) and Lemma 7.2, we know that $B$ is a subset of $PW(C)$. The double sequence $a$ defined by

$$a(k, \ell) = \sup_{\phi \in B} s_{(k, \ell)}(\phi)$$

belongs to $\rho^{\#\#}$. So Lemma 6.1 implies that there exist $n \in \mathbb{N}_0$ and $\gamma > 0$ such that $a \leq \gamma \cdot a_0^n$. It follows that $B \subset PW(C; n)$, and $B$ is bounded: $\forall \phi \in B : ||\phi||_{PW,n} \leq \gamma$.

(iii) $\implies$ (i): This is obvious because $PW(C; n)$ is continuously embedded in $PW(C)$.

Next, we investigate the notion of convergence for sequences in the (LB)-space $PW(C)$.

**Theorem 7.4** Let $(\phi_m)_{m \in \mathbb{N}}$ denote a sequence in $PW(C)$. Then the following statements are equivalent:

(i) $(\phi_m)$ is a convergent sequence in the (LB)-space $PW(C)$;

(ii) $(\phi_m)$ is a Cauchy sequence in the (LB)-space $PW(C)$;

(iii) $(\phi_m)$ is a bounded sequence in the (LB)-space $PW(C)$ and $(\phi_m)$ is a convergent sequence in the Fréchet space $H(C)$;

(iv) There is an $n \in \mathbb{N}_0$ such that $(\phi_m)$ is a convergent sequence in the Banach space $PW(C; n)$.

Proof: (i) $\implies$ (ii): Obvious, because every convergent sequence in the (LB)-space $PW(C)$ is a Cauchy sequence.

(ii) $\implies$ (iii): $(\phi_m)$ is a Cauchy sequence in $PW(C)$, hence $(\phi_m)$ is bounded in $PW(C)$. Since the canonical injection from $PW(C)$ into $H(C)$ is continuous, $(\phi_m)$ is a Cauchy sequence in the Fréchet space $H(C)$, and therefore convergent.

(iii) $\implies$ (iv): Since $(\phi_m)$ is a bounded sequence in $PW(C)$, Theorem 7.3 implies that there exists an $n \geq 1$, such that $(\phi_m)$ is a bounded sequence in the Banach space $PW(C; n - 1)$. We will show that $(\phi_m)$ is a Cauchy sequence in $PW(C; n)$, and therefore convergent.

Let $\varepsilon > 0$. Define $B := \sup_{n \in \mathbb{N}} \|\phi_m\|_{PW,n-1}$, and choose $R > 0$ so large that $\frac{1}{1+R} < \frac{\varepsilon}{4B}$. Since $(\phi_m)$ is a Cauchy sequence in $H(C)$, there exists an $m_0 \in \mathbb{N}$ such that for all $m, p > m_0$

$$\sup_{|z| \leq R} |\phi_m(z) - \phi_p(z)| < \frac{\varepsilon}{2}.$$ 

Then for all $m > m_0$

$$\|\phi_m - \phi_p\|_{PW,n} \leq \sup_{z \in C} (A(z)^{-n} |\phi_m(z) - \phi_p(z)|) \leq \sup_{|z| \leq R} |\phi_m(z) - \phi_p(z)| + \frac{1}{1+R} \|\phi_m - \phi_p\|_{PW,n-1} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4B} 2B = \varepsilon.$$ 

(iv) $\implies$ (i): By definition, the canonical injection from $PW(C; n)$ into $PW(C)$ is continuous.

**Corollary 7.5** The (LB)-space $PW(C)$ is sequentially retractive.

**Corollary 7.6** The (LB)-space $PW(C)$ is sequentially complete.
Note that the arguments of Theorem 7.4 cannot be used to prove that \( PW(\mathbb{C}) \) is complete, because a Cauchy net in \( PW(\mathbb{C}) \) is not necessarily bounded.

Finally, we want to show that \( PW(\mathbb{C}) \) is a Montel space, i.e. it satisfies the Heine-Borel property. In the proof, we need the following preliminary result:

**Lemma 7.7** Let \((\phi_m)_{m \in \mathbb{N}}\) be a bounded sequence in \( H(\mathbb{C}) \). Then the sequence has a convergent subsequence.

**Proof:** Let \( C(\mathbb{C}) \) denote the vector space of all continuous functions on \( \mathbb{C} \) with Fréchet topology. \( H(\mathbb{C}) \) is a closed subspace of \( C(\mathbb{C}) \) because the Fréchet topologies on both spaces are brought about by the same seminorms \((11)\). Therefore the sequence \((\phi_m)\) is also bounded in \( C(\mathbb{C}) \). We will show that the sequence \((\phi_m)\) is locally equicontinuous.

Let \( K \) be a compact subset of \( \mathbb{C} \), and define \( R := 1 + \max\{|z| \mid z \in K\} \). Then the circle \( \Gamma = \{z \in \mathbb{C} \mid |z| = R\} \) is a Jordan curve in \( \mathbb{C} \), encircling \( K \) at a distance of at least 1. Let \( m \in \mathbb{N} \) and \( z, w \in K \). Using Cauchy’s integral formula, we obtain

\[
|\phi_m(z) - \phi_m(w)| \leq \frac{1}{2\pi} \frac{2\pi R}{|z - w|} \cdot |\phi_m(\zeta)| \leq \frac{1}{2\pi} \sup_{\zeta \in \Gamma} (|\phi_m(\zeta)|) \cdot 2\pi R \cdot |z - w|.
\]

Since \( \{\phi_m \mid m \in \mathbb{N}\} \) is bounded in \( H(\mathbb{C}) \),

\[
C := \sup_{m \in \mathbb{N}} \sup_{\zeta \in \Gamma} (|\phi_m(\zeta)|),
\]

is well-defined and finite. So for all \( m \in \mathbb{N} \) and \( z, w \in K \),

\[
|\phi_m(z) - \phi_m(w)| \leq CR \cdot |z - w|,
\]

and the sequence \((\phi_m)\) is locally equicontinuous.

Eventually, application of the generalized version of the Arzela-Ascoli theorem, as stated in Theorem 1.2 (with \( \mathbb{R} \) replaced by \( \mathbb{C} \)), yields that \((\phi_m)\) has a convergent subsequence in \( C(\mathbb{C}) \). Since \( H(\mathbb{C}) \) is a closed subspace of \( C(\mathbb{C}) \), this subsequence is also convergent in \( H(\mathbb{C}) \).

**Corollary 7.8** The Fréchet space \( H(\mathbb{C}) \) is a Montel space.

**Theorem 7.9** The (LB)-space \( PW(\mathbb{C}) \) is a Montel space.

**Proof:** Let \( K \) be a closed and bounded subset of \( PW(\mathbb{C}) \). Then there is an \( n \geq 1 \), such that \( K \subset PW(\mathbb{C};n - 1) \). We prove that \( K \) is a compact subset of \( PW(\mathbb{C};n) \).

So, let \((\phi_m)\) be a sequence in \( K \). Then \((\phi_m)\) is a bounded sequence in \( H(\mathbb{C}) \), and according to Lemma 7.7 there is a convergent subsequence \((\phi_{m_k})\), with limit \( \phi \in H(\mathbb{C}) \). It follows as in the proof of Theorem 7.4, that \((\phi_{m_k})\) is a convergent sequence in the Banach space \( PW(\mathbb{C};n) \).

**8 Fourier transformation on \( \mathcal{E}'(\mathbb{R}) \) as a homeomorphism**

In this final section, we relate the commutative convolution algebra \( \mathcal{E}'(\mathbb{R}) \) and the function algebra \( PW(\mathbb{C}) \) topologically.

**Theorem 8.1** The Fourier transformation \( \mathcal{F} \) is an algebra homeomorphism from the (LB)-space \( \mathcal{E}'(\mathbb{R}) \) onto the (LB)-space \( PW(\mathbb{C}) \).
In the proof of this theorem, we need the following result, stated in [5, pp 12-13]:

**Proposition 8.2** Let $\psi \in H(\mathbb{C})$, and assume that there exist $a > 0$ and $C > 0$ such that for all $z \in \mathbb{C}$:

$$|\psi(z)| \leq C(1 + |z|)^{-2}e^{|z|} \Im z.$$ 

Then there is a continuous functional $F$ on the Fréchet space $C(\mathbb{R})$, with support contained in $[-a, a]$, such that

$$F(e^z) = \psi(z), \quad z \in \mathbb{C}.$$ 

**Proof of Theorem 8.1:** Let $G \in \mathcal{E}'(\mathbb{R}; n)$. Then for all $z \in \mathbb{C}$:

$$|y(z)| = |G(e^z)| \leq \|G\|'_n \cdot \max_{t \in [-n, n]} \sum_{j=0}^n |z|^j e^{|t|} \leq \|G\|'_n \cdot A(z)^n,$$

so that

$$\|FG\|_{PW,n} \leq \|G\|'_n.$$ 

Hence $F$ is a continuous operator from $\mathcal{E}'(\mathbb{R}; n)$ into $PW(\mathbb{C}; n)$, and $F : \mathcal{E}'(\mathbb{R}) \rightarrow PW(\mathbb{C})$ is continuous.

For the converse we apply Proposition 8.2. Let $\phi \in PW(\mathbb{C}; n)$, with $n \in \mathbb{N}_0$. Then there exists a $C > 0$ such that

$$|\phi(z)| \leq C(1 + |z|)^n e^{|z|} \Im z, \quad z \in \mathbb{C}.$$ 

If $\phi$ has less than $n + 2$ zeros, then

$$\phi(z) = p(z)e^{iaz},$$

for some polynomial $p$ of degree smaller than $n + 2$, and some $|a| \leq n$. Hence $\phi = FG$ with

$$G(f) = (p(i \frac{d}{dt}) f)(a),$$

i.e. $G \in \mathcal{E}'(\mathbb{R}; n + 1)$. In the other case, there is a polynomial $p$ of degree $n + 2$ such that

$$\psi(z) := \frac{p(z)}{p'(z)}$$

is holomorphic. Then

$$|\psi(z)| \leq \tilde{C}(1 + |z|)^{-2}e^{|z|} \Im z, \quad z \in \mathbb{C},$$

and so there is a continuous functional $F$ on $C(\mathbb{R})$ with support in $[-n, n]$ such that $F(e^z) = \psi(z)$. So, with $G = F \circ p(i \frac{d}{dt})$ we define an element of $\mathcal{E}'(\mathbb{R}; n + 2)$, satisfying $G(e^z) = \phi(z)$, $z \in \mathbb{C}$. We see that the inverse Fourier transformation maps $PW(\mathbb{C}; n)$ into $\mathcal{E}'(\mathbb{R}; n + 2)$.

Since $F$ is continuous from $\mathcal{E}'(\mathbb{R})$ into $PW(\mathbb{C})$, $F^{-1}$ is closed, and so $F^{-1}$ as an operator from $PW(\mathbb{C}; n)$ into $\mathcal{E}'(\mathbb{R}; n + 2)$ is closed, and therefore continuous. We conclude that $F^{-1}$ from $PW(\mathbb{C})$ onto $\mathcal{E}'(\mathbb{R})$ is continuous.

Since the Fourier transformation $F : \mathcal{E}'(\mathbb{R}) \rightarrow PW(\mathbb{C})$ is an algebra homeomorphism, all topological properties proved in Section 7 for the (LB)-space $PW(\mathbb{C})$ carry over to $\mathcal{E}'(\mathbb{R})$. 

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Theorem 8.3 The \((LB)\)-space \(\mathcal{E}'(\mathbb{R})\) has the following properties:

(i) \(\mathcal{E}'(\mathbb{R})\) is regular, i.e. every bounded set in \(\mathcal{E}'(\mathbb{R})\) is bounded in some \(\mathcal{E}'(\mathbb{R};n)\);

(ii) \(\mathcal{E}'(\mathbb{R})\) is sequentially retractive, i.e. every convergent sequence in \(\mathcal{E}'(\mathbb{R})\) is contained in \(\mathcal{E}'(\mathbb{R};n)\) for some \(n \in \mathbb{N}_0\), and converges with respect to the topology on \(\mathcal{E}'(\mathbb{R};n)\);

(iii) \(\mathcal{E}'(\mathbb{R})\) is sequentially complete;

(iv) \(\mathcal{E}'(\mathbb{R})\) is a Montel space.

References


